ON RIEMANNIAN MANIFOLDS WITH POSITIVE WEIGHTED RICCI CURVATURE OF NEGATIVE EFFECTIVE DIMENSION

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(Received 20 February 2018 and revised 10 July 2018)

Abstract. In this paper, we investigate complete Riemannian manifolds satisfying the lower weighted Ricci curvature bound $\text{Ric}_N \geq K$ with $K > 0$ for the negative effective dimension $N < 0$. We analyze two one-dimensional examples of constant curvature $\text{Ric}_N \equiv K$ with finite and infinite total volumes. We also discuss when the first non-zero eigenvalue of the Laplacian takes its minimum under the same condition $\text{Ric}_N \geq K > 0$, as a counterpart to the classical Obata rigidity theorem. Our main theorem shows that, if $N < -1$ and the minimum is attained, then the manifold splits off the real line as a warped product of hyperbolic nature.

1. Introduction

Riemannian manifolds of Ricci curvature bounded below are classical research subjects in comparison geometry and geometric analysis. Recently, diverse developments on the *curvature-dimension condition* in the sense of Lott, Sturm and Villani have shed new light on this theory. The curvature-dimension condition $\text{CD}(K, N)$ is a synthetic notion of lower Ricci curvature bounds for metric measure spaces. The parameters K and N are usually regarded as ‘a lower bound of the Ricci curvature’ and ‘an upper bound of the dimension’, respectively. Thus N is sometimes called the *effective dimension*. The roles of K and N are better understood when we consider a weighted Riemannian manifold (M, g, m) , a Riemannian manifold of dimension n equipped with an arbitrary smooth measure m . Then the Ricci curvature is modified into the *weighted Ricci curvature* Ric_N involving a parameter N , and $\text{CD}(K, N)$ is equivalent to $\text{Ric}_N \geq K$ (see [vRS, St1, St2, LV] as well as [Vi] for $N \in [n, \infty]$, [Oh2, Oh3] for $N \leq 0$, and also [Oh1] for the Finsler analogue).

For $N \in [n, \infty]$, the weighted Ricci curvature Ric_N (also called the *Bakry–Émery–Ricci curvature*) has been intensively studied by, for instance, Bakry and his collaborators in the framework of the Γ -calculus (see [BGL]). Recently it turned out that there is a rich theory also for $N \in (-\infty, 1]$, though this range seems strange due to the above interpretation of N as an upper dimension bound. Among others, various Poincaré-type inequalities [KM], the curvature-dimension condition [Oh2, Oh3], isoperimetric inequalities [Mi, Kl, Oh3], and the splitting theorem [Wy] were studied for $N < 0$ or $N \leq 1$.

2010 Mathematics Subject Classification: Primary 53C24.

Keywords: spectral gap; weighted Ricci curvature; negative effective dimensions.

The aim of this article is to contribute to the study of the structure of Riemannian manifolds with $\text{Ric}_N \geq K$ for $K > 0$ and $N < 0$. In Section 3 we analyze two one-dimensional examples M_1, M_2 of $\text{Ric}_N \equiv K$ (appearing in [KM, Mi]) having different natures. The first example M_1 has finite volume and enjoys only the exponential concentration; in particular, M_1 does not satisfy the logarithmic Sobolev inequality. For $N < -1$, this space attains the minimum of the first non-zero eigenvalue of the (weighted) Laplacian (derived from the Bochner inequality, see Proposition 2.5). The second example M_2 has infinite volume and reveals the difficulty in obtaining a volume comparison under $\text{Ric}_N \geq K$.

In Section 4, we investigate when the minimum of the first non-zero eigenvalue is attained under the condition $\text{Ric}_N \geq K > 0$ with $N < -1$, as a counterpart to the Obata rigidity theorem [Ob]. Our main theorem (Theorem 4.5) asserts that, when $\dim M \geq 2$, we have a warped product splitting $M \cong \mathbb{R} \times_{\cosh(\sqrt{K/(1-N)}t)} \Sigma^{n-1}$ of hyperbolic nature. Moreover, Σ enjoys $\text{Ric}_{N-1} \geq K(2-N)/(1-N)$. This warped product splitting should be compared with Cheng–Zhou’s theorem in the case of $N = \infty$, where the sharp spectral gap forces the space to isometrically split off a one-dimensional Gaussian space (see [CZ] and Theorem 4.3 for details). Our proof considers the equality case in the Bochner inequality, which is related to the dimension-free version in [CZ] but requires a further discussion due to the fact that the Hessian of the eigenfunction does not vanish (Lemma 4.4). Therefore we have only the warped product rather than the isometric product in [CZ]. We stress that in our setting of $N < 0$, it is interesting to have a hyperbolic structure even when the curvature is positive.

2. Preliminaries

2.1. Weighted Riemannian manifolds

A weighted Riemannian manifold (M, g, m) will be a pair of a complete, connected, boundaryless manifold M equipped with a Riemannian metric g and a measure $m = e^{-\psi} \text{vol}_g$, where $\psi \in C^\infty(M)$ and vol_g is the standard volume measure on (M, g) . On (M, g, m) , we define the weighted Ricci curvature as follows.

Definition 2.1. (Weighted Ricci curvature) Given a unit vector $v \in U_x M$ and $N \in (-\infty, 0) \cup [n, \infty]$, the *weighted Ricci curvature* $\text{Ric}_N(v)$ is defined by:

- (1) $\text{Ric}_N(v) := \text{Ric}_g(v) + \text{Hess } \psi(v, v) - \langle \nabla \psi(x), v \rangle^2 / (N - n)$ for $N \in (-\infty, 0) \cup (n, \infty)$;
- (2) $\text{Ric}_n(v) := \text{Ric}_g(v) + \text{Hess } \psi(v, v)$ if $\langle \nabla \psi(x), v \rangle = 0$, and $\text{Ric}_n(v) := -\infty$ otherwise;
- (3) $\text{Ric}_\infty(v) := \text{Ric}_g(v) + \text{Hess } \psi(v, v)$,

where $n = \dim M$ and Ric_g denotes the Ricci curvature of (M, g) . The parameter N is sometimes called the *effective dimension*. We also define $\text{Ric}_N(cv) := c^2 \text{Ric}_N(v)$ for $c \geq 0$.

Note that if ψ is constant then the weighted Ricci curvature coincides with $\text{Ric}_g(v)$ for all N . When $\text{Ric}_N(v) \geq K$ holds for some $K \in \mathbb{R}$ and all unit vectors $v \in TM$, we will write $\text{Ric}_N \geq K$. By definition,

$$\text{Ric}_n(v) \leq \text{Ric}_N(v) \leq \text{Ric}_\infty(v) \leq \text{Ric}_{N'}(v)$$

holds for $n \leq N < \infty$ and $-\infty < N' < 0$, and $\text{Ric}_N(v)$ is non-decreasing in N in the ranges $(-\infty, 0)$ and $[n, \infty]$.

Remark 2.2. The weighted Ricci curvature for $N \in [n, \infty]$ has been intensively and extensively investigated, see [Qi, BGL] for instance. The study for the negative effective dimension $N < 0$ is more recent. One can find in [Mi, Kl, Oh3] the isoperimetric inequality, in [Wy] the Cheeger–Gromoll type splitting theorem, and in [Oh2, Oh3] the curvature-dimension condition in this context.

We also define the weighted Laplacian with respect to m .

Definition 2.3. (Weighted Laplacian) The *weighted Laplacian* (also called the *Witten Laplacian*) of $u \in C^\infty(M)$ is defined as follows:

$$\Delta_m u := \Delta u - \langle \nabla u, \nabla \psi \rangle.$$

Notice that the Green formula (the integration by parts formula)

$$\int_M u \Delta_m v \, dm = - \int_M \langle \nabla u, \nabla v \rangle \, dm = \int_M v \Delta_m u \, dm$$

holds provided u or v belongs to $C_c^\infty(M)$ (smooth functions with compact supports) or $H_0^1(M)$.

2.2. Bochner inequality and eigenvalues of the Laplacian

Recall that the *Bochner–Weitzenböck formula* associated with the weighted Ricci curvature Ric_∞ and the weighted Laplacian Δ_m holds as follows. For $u \in C^\infty(M)$, we have

$$\Delta_m \left(\frac{|\nabla u|^2}{2} \right) - \langle \nabla \Delta_m u, \nabla u \rangle = \text{Ric}_\infty(\nabla u) + \|\text{Hess } u\|_{HS}^2, \quad (1)$$

where $\|\cdot\|_{HS}$ stands for the Hilbert–Schmidt norm (with respect to g). As a corollary we obtain the *Bochner inequality* for $N \in (-\infty, 0) \cup [n, \infty]$ (the case of negative effective dimension was studied independently in [KM, Oh2]). We give an outline of the proof for later use.

THEOREM 2.4. (Bochner inequality) *For $N \in (-\infty, 0) \cup [n, \infty]$ and any $u \in C^\infty(M)$, we have*

$$\Delta_m \left(\frac{|\nabla u|^2}{2} \right) - \langle \nabla \Delta_m u, \nabla u \rangle \geq \text{Ric}_N(\nabla u) + \frac{(\Delta_m u)^2}{N}.$$

Proof. Fix $x \in M$ and let B be the matrix representation of $\text{Hess } u(x)$ for some orthonormal basis. Then we find

$$\|\text{Hess } u\|_{HS}^2 = \text{tr}(B^2) \geq \frac{(\text{tr } B)^2}{n} = \frac{(\Delta u)^2}{n}.$$

Putting $p = \Delta_m u$ and $q = \langle \nabla u, \nabla \psi \rangle$, we have

$$\begin{aligned} \frac{(\Delta u)^2}{n} &= \frac{(p+q)^2}{n} = \frac{p^2}{N} - \frac{q^2}{N-n} + \frac{N(N-n)}{n} \left(\frac{p}{N} + \frac{q}{N-n} \right)^2 \\ &\geq \frac{p^2}{N} - \frac{q^2}{N-n} = \frac{(\Delta_m u)^2}{N} - \frac{\langle \nabla u, \nabla \psi \rangle^2}{N-n}. \end{aligned}$$

Combining these with the Bochner–Weitzenböck formula (1) we obtain the Bochner inequality. $\square$

We remark that $N(N - n) \geq 0$ used in the proof fails for $N \in (0, n)$. The Bochner inequality has quite rich applications in geometric analysis. In this article we are interested in the first non-zero eigenvalue of the weighted Laplacian, a generalization of the *Lichnerowicz inequality*. The case of the negative effective dimension $N < 0$ was studied in [KM, Oh2]. Here we give a proof for completeness.

PROPOSITION 2.5. (First non-zero eigenvalue) *Let (M, g, m) be a complete weighted Riemannian manifold satisfying $\text{Ric}_N \geq K$ for some $K > 0$ and $N < 0$, and assume $m(M) < \infty$. Then the first non-zero eigenvalue of the non-negative operator $-\Delta_m$ is bounded from below by $KN/(N - 1)$.*

Proof. Let $u \in H_0^1 \cap C^\infty(M)$ be a non-constant eigenfunction of Δ_m with $\Delta_m u = -\lambda u$, $\lambda > 0$. We deduce from the Bochner inequality (Theorem 2.4) and the condition $\text{Ric}_N \geq K$ that

$$\Delta_m \left(\frac{|\nabla u|^2}{2} \right) \geq \langle \nabla \Delta_m u, \nabla u \rangle + K |\nabla u|^2 + \frac{(\Delta_m u)^2}{N}.$$

Since constant functions belong to $H_0^1(M)$ thanks to the hypothesis $m(M) < \infty$, the integration of the above inequality yields

$$0 \geq \left(\frac{1}{N} - 1 \right) \int_M (\Delta_m u)^2 dm + K \int_M |\nabla u|^2 dm.$$

We again use the integration by parts to see

$$\int_M (\Delta_m u)^2 dm = -\lambda \int_M u \Delta_m u dm = \lambda \int_M |\nabla u|^2 dm.$$

Therefore we have

$$\frac{1 - N}{N} \lambda + K \leq 0,$$

which shows the claim $\lambda \geq KN/(N - 1)$. $\square$

It was observed in [KM, §3.2] and [Mi, Theorem 6.1] that the estimate $\lambda \geq KN/(N - 1)$ is sharp for $N \in (-\infty, -1]$ and, somewhat surprisingly, *not sharp* for $N < 0$ close to 0. See Section 4 for a further discussion, where we discuss the rigidity for the constant $KN/(N - 1)$ in the case $N < -1$. The first non-zero eigenvalue of the weighted Laplacian is related to the concentration of measures. When $m(M) = 1$, we define the *concentration function* of (M, g, m) by

$$\alpha(r) := \sup\{1 - m(B(A, r)) \mid A \subset M : \text{Borel}, m(A) \geq 1/2\}.$$

See [Le, Theorem 3.1] for the following corollary.

COROLLARY 2.6. (Exponential concentration) *Let (M, g, m) be a compact weighted Riemannian manifold satisfying $\text{Ric}_N \geq K$ for some $K > 0$ and $N < 0$, and assume $m(M) = 1$. Then (M, g, m) satisfies the exponential concentration $\alpha(r) \leq e^{-\sqrt{KN/(N-1)}r/3}$.*

3. Two examples of one-dimensional spaces of constant curvature

In this section we analyze two examples of one-dimensional spaces such that $\text{Ric}_N \equiv K$ for some $K > 0$ and $N < 0$. These examples will be helpful to understand the general picture of the spaces satisfying $\text{Ric}_N \geq K > 0$. In particular, we shall give negative answers to some naive guesses.

The first example is the following.

Example 3.1. For $K > 0$ and $N < 0$, the space

$$M_1 := \left(\mathbb{R}, |\cdot|, \cosh\left(\sqrt{\frac{K}{1-N}}x\right)^{N-1} dx \right),$$

where $|\cdot|$ denotes the canonical distance structure, satisfies $\text{Ric}_N \equiv K$.

This is a model space in Milman's isoperimetric inequality (see Case 1 of [Mi, Corollary 1.4]). Notice that the total volume is finite since

$$\begin{aligned} \int_{\mathbb{R}} \cosh\left(\sqrt{\frac{K}{1-N}}x\right)^{N-1} dx &\leq 2 \int_0^\infty \left(e^{\sqrt{K/(1-N)}x}\right)^{N-1} dx \\ &= 2 \int_0^\infty e^{-\sqrt{K(1-N)}x} dx < \infty. \end{aligned}$$

In order to see the curvature identity, we observe that the weight function is given by

$$\psi(x) = (1-N) \log\left(\cosh\sqrt{\frac{K}{1-N}}x\right),$$

and

$$\begin{aligned} \psi'(x) &= \sqrt{K(1-N)} \tanh\left(\sqrt{\frac{K}{1-N}}x\right), \\ \psi''(x) &= K - K \tanh\left(\sqrt{\frac{K}{1-N}}x\right)^2. \end{aligned}$$

Since the Ricci curvature vanishes in the one-dimensional case, we have the curvature identity

$$\begin{aligned} \text{Ric}_N\left(\frac{\partial}{\partial x}\Big|_x\right) &= \psi''(x) + \frac{\psi'(x)^2}{1-N} \\ &= K - K \tanh\left(\sqrt{\frac{K}{1-N}}x\right)^2 + \frac{K(1-N)}{1-N} \tanh\left(\sqrt{\frac{K}{1-N}}x\right)^2 \\ &= K. \end{aligned}$$

In [KM] it was observed that this space attains the minimum value of the first non-zero eigenvalue in respect of Proposition 2.5.

LEMMA 3.2. (Minimal first non-zero eigenvalue) *For $N < -1$, the first non-zero eigenvalue of $-\Delta_m$ in the space M_1 in Example 3.1 coincides with $KN/(N-1)$.*

Proof. We shall show that the function

$$u(x) = \sinh\left(\sqrt{\frac{K}{1-N}}x\right)$$

gives the minimal first non-zero eigenvalue. Notice that $u \in L^2(M_1)$ by the assumption $N < -1$, and

$$u'(x) = \sqrt{\frac{K}{1-N}} \cosh\left(\sqrt{\frac{K}{1-N}}x\right), \quad u''(x) = \frac{K}{1-N}u.$$

Thus the weighted Laplacian of u is calculated as

$$\begin{aligned} \Delta_m u(x) &= u''(x) - u'(x)\psi'(x) = \frac{K}{1-N}u(x) - K \sinh\left(\sqrt{\frac{K}{1-N}}x\right) \\ &= \frac{K}{1-N}u(x) - Ku(x) = -\frac{KN}{N-1}u(x). \end{aligned} \quad \square$$

Remark 3.3. (Failure of log-Sobolev inequality) For $N \in [n, \infty]$, the bound $\text{Ric}_N \geq K > 0$ is known to imply the *logarithmic Sobolev inequality* (see, for example, [BGL] for proofs based on the Γ -calculus):

$$\int_{\{f>0\}} f \log f \, dm \leq \frac{N-1}{2KN} \int_{\{f>0\}} \frac{|\nabla f|^2}{f} \, dm \quad (2)$$

for non-negative functions $f \in H^1(M)$ with $\int_M f \, dm = 1$. It is well known that the logarithmic Sobolev inequality implies the *normal concentration*, see [Le, Theorem 5.3]. One might expect that the logarithmic Sobolev inequality (2) has a counterpart for $N < 0$ similarly to the spectral gap. This is, however, not the case since M_1 (normalized as $m(M_1) = 1$) given in Example 3.1 enjoys only the exponential concentration (consider $A = [0, \infty)$). See [Mi, Proposition 6.4] for a detailed estimate of the concentration function.

Our second example is as follows.

Example 3.4. For $K > 0$ and $N < 0$, the space

$$M_2 := (\mathbb{R}, |\cdot|, e^{-\sqrt{K(1-N)}x} \, dx)$$

satisfies $\text{Ric}_N \equiv K$.

This space also appeared in Case 2 of [Mi, Corollary 1.4]. The weight function is the linear function $\psi(x) = \sqrt{K(1-N)}x$ and the curvature identity is straightforward:

$$\text{Ric}_N\left(\frac{\partial}{\partial x}\Big|_x\right) = \frac{\psi'(x)^2}{1-N} = K.$$

We stress that M_2 has very different natures from M_1 in Example 3.1. Recall that, when $N = \infty$, $\text{Ric}_\infty \geq K > 0$ implies that the measure has Gaussian decay and the total volume is finite [St1, Theorem 4.26]. From Example 3.1 one may expect that $\text{Ric}_N \geq K > 0$ for $N < 0$ still implies the exponential decay; however, it is not the case by the second example M_2 above. We refer to [Sa2] for a related work on weighted Riemannian manifolds

with boundaries, where a volume comparison for regions spreading from the boundary was established for $N \leq 1$.

We also remark that, different from the case of $N = \infty$, taking products will destroy the curvature bound for $N < 0$ due to the nonlinearity of the term $\langle \nabla \psi, v \rangle^2$ in ψ in the definition of $\text{Ric}_N(v)$. For instance,

$$(\mathbb{R}^n, |\cdot|, e^{-x^1-x^2-\dots-x^n} dx^1 dx^2 \dots dx^n)$$

satisfies only $\text{Ric}_N \geq 0$ (consider v in the kernel of $d\psi$).

4. First non-zero eigenvalue of the Laplacian

As we saw in Proposition 2.5, the first non-zero eigenvalue of $-\Delta_m$ on a weighted Riemannian manifold satisfying $\text{Ric}_N \geq K > 0$ is bounded from below by $KN/(N-1)$ (or K if $N = \infty$), which is sharp for $N \leq -1$. In this section we consider the rigidity problem when the minimum value is attained. In the unweighted ($N = n$) case, the classical Obata theorem [Ob] asserts the following.

THEOREM 4.1. (Obata rigidity theorem) *Let (M, g) be an n -dimensional Riemannian manifold with $\text{Ric}_g \geq K > 0$. Then the first non-zero eigenvalue of $-\Delta$ coincides with $Kn/(n-1)$ if and only if M is isometric to the n -dimensional sphere $\mathbb{S}^n$ with radius $\sqrt{(n-1)/K}$.*

The case of $N \in (n, \infty)$ turns out void as follows.

THEOREM 4.2. (Ketterer, Kuwada) *Let (M, g, m) be a weighted Riemannian manifold of dimension n with $\text{Ric}_N \geq K > 0$ for some $N \in [n, \infty)$. Then the first non-zero eigenvalue of $-\Delta_m$ coincides with $KN/(N-1)$ if and only if $N = n$, $m = c \text{vol}_g$ for some constant $c > 0$ and M is isometric to the n -dimensional sphere $\mathbb{S}^n$ with radius $\sqrt{(n-1)/K}$.*

Precisely, it is shown in [Ke] in the general framework of RCD-spaces (metric measure spaces satisfying the *Riemannian curvature-dimension condition*) that the existence of an eigenfunction for the eigenvalue $KN/(N-1)$ implies the maximal diameter $\text{diam}(M) = \pi\sqrt{(N-1)/K}$ in the Bonnet–Myers type theorem. This is, however, achieved only when $N = n$ by Kuwada’s result in [Ku]. If we admit singularities, then the sharp spectral gap is attained for spherical suspensions as discussed in [Ke] for RCD-spaces (see also [KM, §3.2]).

Next, in the case of $N = \infty$, an interesting rigidity theorem was established in [CZ].

THEOREM 4.3. (Cheng–Zhou) *Let (M, g, m) be an n -dimensional weighted Riemannian manifold with $\text{Ric}_\infty \geq K > 0$. If the first non-zero eigenvalue of $-\Delta_m$ coincides with K , then M is isometric to the product space $\Sigma^{n-1} \times \mathbb{R}$ as weighted Riemannian manifolds, where Σ^{n-1} is an $(n-1)$ -dimensional manifold with $\text{Ric}_\infty \geq K$ and $\mathbb{R}$ is equipped with the Gaussian measure $e^{-Kx^2/2} dx$.*

The proof of Theorem 4.3 has a certain similarity with the *Cheeger–Gromoll splitting theorem*. The role of the Busemann function in [CG] is replaced with the eigenfunction in [CZ]. Theorem 4.3 was recently generalized to RCD-spaces in [GKKO].

Now we consider the case of $N < 0$. We begin with an important equation for an eigenfunction.

LEMMA 4.4. *Let (M, g, m) be a complete weighted Riemannian manifold satisfying $\text{Ric}_N \geq K$ for some $N < 0$ and $K > 0$, and $m(M) < \infty$. Suppose that the first non-zero eigenvalue of $-\Delta_m$ coincides with $KN/(N-1)$. Then the eigenfunction u of $KN/(N-1)$ necessarily satisfies*

$$\text{Hess } u = -\frac{Ku}{N-1}g \quad (3)$$

as operators $TM \times TM \longrightarrow \mathbb{R}$.

Proof. Let u be an eigenfunction of the eigenvalue $KN/(N-1)$, namely

$$\Delta_m u = -\frac{KN}{N-1}u.$$

Tracing back to the proof of the lower bound of the first non-zero eigenvalue, the Bochner inequality for u necessarily becomes an equality. Thus we find, from the proof of Theorem 2.4, that

$$\text{Hess } u = f \cdot g \quad \text{for some function } f : M \longrightarrow \mathbb{R}; \quad (4)$$

$$\frac{\Delta_m u}{N} + \frac{\langle \nabla u, \nabla \psi \rangle}{N-n} \equiv 0. \quad (5)$$

On the one hand, we observe from the former equation (4) that $\Delta u = nf$. On the other hand, the latter equation (5) yields

$$\Delta u = \Delta_m u + \langle \nabla u, \nabla \psi \rangle = \frac{n}{N} \Delta_m u = -\frac{Kn}{N-1}u.$$

Therefore we have $f = -Ku/(N-1)$ and complete the proof. $\square$

THEOREM 4.5. *Let (M, g, m) be a complete weighted Riemannian manifold of dimension n satisfying $\text{Ric}_N \geq K$ for some $N < -1$ and $K > 0$, and $m(M) < \infty$. Assume that the first non-zero eigenvalue of $-\Delta_m$ coincides with $KN/(N-1)$.*

(i) *If $n \geq 2$, then M is isometric to the warped product*

$$\mathbb{R} \times_{\cosh(\sqrt{K/(1-N)}t)} \Sigma = \left(\mathbb{R} \times \Sigma, dt^2 + \cosh^2\left(\sqrt{\frac{K}{1-N}}t\right) \cdot g_\Sigma \right)$$

and the measure m is written through the isometry as

$$m(dt \, dx) = \cosh^{N-1}\left(\sqrt{\frac{K}{1-N}}t\right) dt \, m_\Sigma(dx),$$

where $(\Sigma, g_\Sigma, m_\Sigma)$ is an $(n-1)$ -dimensional weighted Riemannian manifold satisfying $\text{Ric}_{N-1} \geq K(2-N)/(1-N)$.

(ii) *If $n = 1$, then (M, g, m) is isometric to the model space M_1 in Example 3.1 up to a constant multiplication of the measure.*

Proof. (i) By rescaling the Riemannian metric, we can assume that $K = 1 - N$. Let $u \in H_0^1 \cap C^\infty(M)$ be an eigenfunction satisfying $\Delta_m u = Nu$. Along any geodesic $\gamma : I \longrightarrow M$ for an interval $I \subset \mathbb{R}$, it follows from Lemma 4.4 that $(u \circ \gamma)'' = (u \circ \gamma)$. This implies that

$$u \circ \gamma(t) = u \circ \gamma(0) \cdot \cosh t + (u \circ \gamma)'(0) \cdot \sinh t. \quad (6)$$

Put $\Sigma := u^{-1}(0)$ (which is non-empty since $\int_M u \, dm = 0$). We deduce from (6) that, for $x, y \in \Sigma$, every geodesic connecting them is included in Σ . Hence Σ is totally geodesic and the mean curvature of Σ is identically zero. Lemma 4.4 also shows that, for any smooth vector field V along a geodesic γ included in Σ ,

$$\langle \nabla_{\dot{\gamma}}(\nabla u \circ \gamma), V \rangle = \text{Hess } u(\dot{\gamma}, V) = u \cdot \langle \dot{\gamma}, V \rangle \equiv 0.$$

Hence $\nabla u \circ \gamma$ is a parallel vector field and $|\nabla u|$ is constant on Σ . We will normalize u so as to satisfy $|\nabla u| \equiv 1$ on Σ .

Fix $x \in \Sigma$ and consider the geodesic $\gamma_x : [0, \infty) \rightarrow M$ such that $\dot{\gamma}_x(0) = \nabla u(x)$. Then (6) and our normalization $|\nabla u|(x) = 1$ yield

$$(u \circ \gamma_x)(t) = \sinh t, \quad t \in \mathbb{R}. \quad (7)$$

For any unit speed geodesic $\eta : [0, s] \rightarrow M$ from x to $\gamma_x(t)$, (7) and (6) imply $\sinh t = u(\eta(s)) \leq \sinh s$. Since $\sinh t$ is monotone increasing, we have $s \geq t$ and γ_x is globally minimizing. Denoting the distance function from Σ by ρ_Σ , we obtain from (7) that $\rho_\Sigma = |\text{arcsinh}(u)|$ and, on $u^{-1}((0, \infty))$,

$$\Delta u = \Delta(\sinh \rho_\Sigma) = \cosh \rho_\Sigma \cdot \Delta \rho_\Sigma + \sinh \rho_\Sigma \cdot |\nabla \rho_\Sigma|^2.$$

Since $|\nabla \rho_\Sigma| = 1$ and $\Delta u = nu = n \sinh \rho_\Sigma$, we have for $t \geq 0$

$$\Delta \rho_\Sigma(\gamma_x(t)) = (n - 1) \tanh \rho_\Sigma(\gamma_x(t)) = (n - 1) \tanh t. \quad (8)$$

We shall compare the Laplacian comparison inequality (8) with the (unweighted) Ricci curvature along γ_x . Using (7) and (5), we have

$$\begin{aligned} (\psi \circ \gamma_x)'(t) &= \langle \nabla \psi \circ \gamma_x, \dot{\gamma}_x \rangle(t) = \frac{1}{\cosh t} \langle \nabla \psi, \nabla u \rangle(\gamma_x(t)) \\ &= \frac{n - N}{\cosh t} u(\gamma_x(t)) = (n - N) \tanh t. \end{aligned}$$

Hence

$$\psi \circ \gamma_x(t) = (n - N) \log(\cosh t) + \psi(x) \quad (9)$$

and

$$\begin{aligned} \text{Ric}(\dot{\gamma}_x(t)) &= \text{Ric}_N(\dot{\gamma}_x(t)) - (\psi \circ \gamma_x)''(t) - \frac{(\psi \circ \gamma_x)'(t)^2}{n - N} \\ &= \text{Ric}_N(\dot{\gamma}_x(t)) - \frac{n - N}{\cosh^2 t} - (n - N) \tanh^2 t \\ &= \text{Ric}_N(\dot{\gamma}_x(t)) - (n - N) \geq 1 - n. \end{aligned}$$

This curvature bound together with the minimality of Σ implies that

$$\Delta \rho_\Sigma(\gamma_x(t)) \leq (n - 1) \tanh t \quad (10)$$

by [Ka, Corollary 2.44] (see also [HK] and [Sa1, Theorem 4.3]). Then, since we have equality (8), the rigidity theorem for the Laplacian comparison shows that $u^{-1}([0, \infty))$ is

isometric to the warped product

$$[0, \infty) \times_{\cosh} \Sigma := ([0, \infty) \times \Sigma, dt^2 + \cosh^2 t \cdot g_\Sigma),$$

where g_Σ is the Riemannian metric of Σ induced from g of M (see the proof of [Sa1, Theorem 1.8] for details). Here we give a sketch of the proof for the isometry along [Sa1].

For a regular point $x \in \Sigma$ of u , let $T_x^\perp \Sigma$ be the orthogonal complement of $T_x \Sigma$. Choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x \Sigma$ and for $i = 1, \dots, n-1$, let $Y_{x,i}$ be the Σ -Jacobi field along γ_x such that $Y(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{\nabla u(x)} e_{x,i}$, where $A_v : T_x \Sigma \rightarrow T_x \Sigma$ is the shape operator of the tangent vector v . Since the mean curvature on Σ is 0 and $\text{Ric}_N(\dot{\gamma}_x(t)) \geq 1 - n$, we have $\Delta \rho_\Sigma(\gamma_x(t)) \leq (n-1) \tanh t$ as in (10) and equality holds if and only if $Y_{x,i}(t) = \cosh(t) E_{x,i}(t)$ where $E_{x,i}$ are the parallel vector fields along γ_x with $E_{x,i}(0) = e_{x,i}$. Then the map $\phi : [0, \infty) \times \Sigma \rightarrow M$ defined by $\phi(t, x) := \gamma_x(t)$ provides the desired isometry between $[0, \infty) \times_{\cosh} \Sigma$ and M .

By a similar discussion on $u^{-1}((-\infty, 0])$ we can extend the isometry to

$$M \cong \mathbb{R} \times_{\cosh} \Sigma = (\mathbb{R} \times \Sigma, dt^2 + \cosh^2 t \cdot g_\Sigma).$$

Concerning the splitting of the measure, let Σ be equipped with the measure $m_\Sigma := e^{-\psi|_\Sigma} \text{vol}_{g_\Sigma}$. By (9) and the structure of the warped product, we see that the measure m is written through the isometry as

$$\begin{aligned} m(dt \, dx) &= \cosh^{N-n} t \cdot e^{-\psi(x)} \cdot (\cosh^{n-1} t \, dt \, \text{vol}_{g_\Sigma}(dx)) \\ &= \cosh^{N-1} t \, dt \, m_\Sigma(dx). \end{aligned}$$

With this expression one can see that the eigenfunction u is in $L^2(M)$ when $N < -1$ (but not in $L^2(M)$ for $N \in [-1, 0)$).

Finally, for any unit vector $v \in T_x \Sigma$ at $x \in \Sigma$, the sectional curvature of the plane $\dot{\gamma}_x(0) \wedge v$ coincides with -1 (see [ON, Proposition 42(2) in §7]). Therefore we have, on the weighted Riemannian manifold $(\Sigma, g_\Sigma, m_\Sigma)$,

$$\begin{aligned} \text{Ric}_{N-1}^\Sigma(v) &= \text{Ric}^\Sigma(v) + \text{Hess}^\Sigma \psi(v, v) - \frac{\langle \nabla^\Sigma \psi, v \rangle^2}{(N-1) - (n-1)} \\ &= \text{Ric}^M(v) + 1 + \text{Hess}^M \psi(v, v) - \frac{\langle \nabla^M \psi, v \rangle^2}{N-n} \\ &= \text{Ric}_N^M(v) + 1 \geq 2 - N. \end{aligned}$$

(ii) Fix an eigenfunction u of the eigenvalue $KN/(N-1)$, take $x \in M$ with $u(x) = 0$ and choose a unit speed geodesic $\gamma : \mathbb{R} \rightarrow M$ with $\gamma(0) = x$. It follows from Lemma 4.4 that

$$u \circ \gamma(t) = (u \circ \gamma)'(0) \cdot \sinh\left(\sqrt{\frac{K}{1-N}} t\right).$$

Since u is not constant, we have $(u \circ \gamma)'(0) \neq 0$ and hence u is injective. Therefore M is isometric to $\mathbb{R}$ and we will identify them via γ as $\gamma(t) = t$.

Denote by $\psi : \mathbb{R} \rightarrow \mathbb{R}$ the weight function, namely $m = e^{-\psi} dt$. Then it follows from

$$-\frac{KN}{N-1}u = \Delta_m u = u'' - u'\psi'$$

that

$$\begin{aligned}\psi'(t) &= \frac{1}{u'(t)} \left(u''(t) + \frac{KN}{N-1} u(t) \right) \\ &= \sqrt{\frac{1-N}{K}} \left(\frac{K}{1-N} + \frac{KN}{N-1} \right) \tanh \left(\sqrt{\frac{K}{1-N}} t \right) \\ &= \sqrt{K(1-N)} \tanh \left(\sqrt{\frac{K}{1-N}} t \right).\end{aligned}$$

Integrating this we have

$$\psi(t) = (1-N) \log \left(\cosh \left(\sqrt{\frac{K}{1-N}} t \right) \right) + \psi(0),$$

which implies $m = e^{-\psi(0)} \cosh(\sqrt{K/(1-N)}t)^{N-1} dt$ as desired. $\square$

The following corollary is a byproduct of the proof above (see [KM, Mi] for the one-dimensional case).

COROLLARY 4.6. *Let (M, g, m) be a complete weighted Riemannian manifold of dimension n satisfying $\text{Ric}_N \geq K$ for some $N \in [-1, 0)$ and $K > 0$, and $m(M) < \infty$. Then $KN/(N-1)$ cannot be an eigenvalue of $-\Delta_m$.*

Remark 4.7. The curvature bound of Σ implies the estimate of the first non-zero eigenvalue $\lambda_1(\Sigma)$ as

$$\lambda_1(\Sigma) \geq \frac{K(2-N)}{1-N} \frac{N-1}{N-2} = K.$$

This is better than the estimate $\lambda_1(M) \geq KN/(N-1)$ for M .

We close the article with a concrete example of the splitting phenomenon described in Theorem 4.5(i) (with $K = 1 - N$).

Example 4.8. Let $(\mathbb{H}^2, g)$ be the hyperbolic plane of constant sectional curvature -1 . There are smooth functions u and ψ satisfying

$$\Delta_m u = Nu, \quad \text{Ric}_N \equiv 1 - N,$$

where $m = e^{-\psi} \text{vol}_g$.

Let us give the precise expressions of u and ψ in the upper half-plane model:

$$(\mathbb{H}^2, g) = \left(\mathbb{R} \times (0, \infty), \frac{dx^2 + dy^2}{y^2} \right).$$

By the consideration in the proof of Theorem 4.5(i), we can expect that the function u is written by using the distance function from the y -axis ($\Sigma = u^{-1}(0)$ coincides with the

y -axis). One can explicitly calculate it as

$$\begin{aligned} u(x, y) &= \pm \sinh\left(d\left((x, y), \left(0, \sqrt{x^2 + y^2}\right)\right)\right) \\ &= \pm \sinh\left(\operatorname{arccosh}\left(1 + \frac{x^2 + (y - \sqrt{x^2 + y^2})^2}{2y\sqrt{x^2 + y^2}}\right)\right) \\ &= \pm \sqrt{\frac{x^2 + y^2}{y^2} - 1} = \frac{x}{y}, \end{aligned}$$

where we choose the sign $+$ for $x > 0$ and $-$ for $x < 0$, and we used the equation $\sinh(\operatorname{arccosh} t) = \sqrt{t^2 - 1}$ for $t \geq 0$. The Christoffel symbols are readily calculated as

$$\Gamma_{11}^1 = \Gamma_{22}^1 = \Gamma_{12}^2 \equiv 0, \quad \Gamma_{12}^1(x, y) = \Gamma_{22}^2(x, y) = -\frac{1}{y}, \quad \Gamma_{11}^2(x, y) = \frac{1}{y}.$$

Using these we find

$$\begin{aligned} (\operatorname{Hess} u)_{(x,y)} &= \begin{pmatrix} 0 & -y^{-2} \\ -y^{-2} & 2xy^{-3} \end{pmatrix} - \frac{1}{y} \begin{pmatrix} 0 & -y^{-1} \\ -y^{-1} & 0 \end{pmatrix} + \frac{x}{y^2} \begin{pmatrix} y^{-1} & 0 \\ 0 & -y^{-1} \end{pmatrix} \\ &= \frac{x}{y} \cdot \frac{1}{y^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = u(x, y) \cdot g_{(x,y)}. \end{aligned}$$

Next we consider the weight function. Let

$$\begin{aligned} \psi_1(x, y) &:= (2 - N) \log\left[\cosh\left(d\left((x, y), \left(0, \sqrt{x^2 + y^2}\right)\right)\right)\right] \\ &= (2 - N) \log\left(\frac{\sqrt{x^2 + y^2}}{y}\right), \end{aligned}$$

and

$$\psi_2(x, y) := -(2 - N) \log\left(\sqrt{x^2 + y^2}\right).$$

Notice that ψ_1 naturally appears in respect of (5), and ψ_2 is employed to improve the convexity in the directions perpendicular to ∇u . Indeed, we have $\langle \nabla u, \nabla \psi_2 \rangle = 0$ and

$$\begin{aligned} \langle \nabla u, \nabla \psi_1 \rangle(x, y) &= (2 - N)y^2 \left(\frac{1}{y} \cdot \frac{x}{x^2 + y^2} - \frac{x}{y^2} \cdot \left(\frac{y}{x^2 + y^2} - \frac{1}{y} \right) \right) \\ &= (2 - N) \frac{x}{y} = (2 - N)u(x, y). \end{aligned}$$

Our weighted function will be $\psi = \psi_1 + \psi_2$, namely

$$\psi(x, y) = -(2 - N) \log y.$$

The above calculations yield

$$\Delta_m u = \Delta u - \langle \nabla u, \nabla \psi \rangle = 2u - (2 - N)u = Nu.$$

We finally calculate Ric_N of this example. To this end, we observe that

$$\frac{(\operatorname{Hess} \psi)_{(x,y)}}{2 - N} = \begin{pmatrix} y^{-2} & 0 \\ 0 & 0 \end{pmatrix}, \quad \frac{(d\psi \otimes d\psi)_{(x,y)}}{(2 - N)^2} = \begin{pmatrix} 0 & 0 \\ 0 & y^{-2} \end{pmatrix}.$$

Therefore we obtain

$$\begin{aligned}(\operatorname{Ric}_N)_{(x,y)} &= -\frac{1}{y^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (\operatorname{Hess} \psi)_{(x,y)} + \frac{(d\psi \otimes d\psi)_{(x,y)}}{2-N} \\ &= \frac{1-N}{y^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

Acknowledgements. I would like to thank my supervisor, Professor Shin-ichi Ohta, for the kind guidance, encouragement and advice he has provided throughout my time working on this paper. I also would like to thank Professor Emanuel Milman for valuable comments on the earlier version of the paper, especially on the case of $N \in [-1, 0)$.

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