ELIMINATION OF DEFINITE FOLD. II

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Dedicated to Professor Takashi Nishimura on the occasion of his 60th birthday

Abstract. In this paper, we first give a new simple proof for the elimination theorem of definite folds by homotopy for generic smooth maps of manifolds of dimension strictly greater than two into the 2-sphere or into the real projective plane. Our new proof has the advantage that it is not only constructive, but also algorithmic: the procedure enables us to construct various explicit examples. We also study simple stable maps of 3-manifolds into the 2-sphere without definite folds. Furthermore, we prove the non-existence of singular Legendre fibrations on 3-manifolds, answering negatively a question posed in our previous paper.

1. Introduction

This is a continuation of our previous paper [20] in which we proved that for an arbitrary C^∞ stable (or excellent) map of a closed manifold of dimension $n > 2$ into the 2-sphere S^2 or into the real projective plane $\mathbf{R}P^2$, we can eliminate definite fold points by homotopy.

The present paper has basically four purposes. The first one is to give a new simple proof for the elimination theorem of definite folds in which we modify a given map in its homotopy class (see Section 2). Recall that in [20], we used surgery operations, which modified the source manifold of the given map to another manifold. This made an explicit construction very hard to realize. On the contrary, our new proof presented in this paper modifies the original map only by homotopy: furthermore, it is not only constructive, but also algorithmic, which is an important ingredient for the simplification process for maps on 4-manifolds proposed in [3] (see also [4]).

The second purpose of the present paper is to give several explicit examples of C^∞ stable maps without definite fold points (see Section 3). We explicitly construct such a map in each element of the homotopy groups $\pi_3(S^2) \cong \mathbf{Z}$ and $\pi_4(S^2) \cong \mathbf{Z}_2$. We also construct explicit C^∞ stable maps on homotopy n -spheres, $n \geq 5$, without definite fold points. Note that these examples are now easy consequences of our new constructive proof.

The third purpose of the present paper is to study simple stable maps of 3-manifolds into S^2 without definite folds (see Section 4). We show that a closed orientable 3-manifold admits a C^∞ stable map without definite folds into S^2 with embedded indefinite fold image if and

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only if it is a graph manifold, by combining our techniques with those developed in [19]. We also study the construction of such a C^∞ stable map in a given homotopy class.

The final purpose of the present paper is to show that there exist no singular Legendre fibrations on an arbitrary orientable 3-manifold, answering negatively a problem posed in [20, Section 3] (see Section 5). Recall that it had been known that such a singular Legendre fibration must be an excellent map without definite folds. We will show that, even locally, there exists no contact structure which makes fibers near an indefinite fold point all Legendrian. The result is somewhat surprising, but the proof is elementary.

Throughout the paper, manifolds and maps are differentiable of class C^∞ unless otherwise indicated. A stable map will always mean a C^∞ stable map (for stable maps, the reader is referred to [10, 14]).

2. A new proof

Let M be a closed n -dimensional manifold with $n \geq 2$ and N a surface. For a smooth map $f : M \rightarrow N$, we denote the set of singular points by

$$S(f) = \{x \in M \mid \text{rank } df_x < 2\}.$$

Definition 2.1. (1) A singular point $x \in S(f)$ of f is called a *fold point* if there exist local coordinates $(x_1, x_2, \dots, x_n)$ around x and (y_1, y_2) around $f(x)$ such that f has the form

$$y_1 \circ f = x_1, \quad y_2 \circ f = -x_2^2 - \dots - x_{1+\lambda}^2 + x_{2+\lambda}^2 + \dots + x_n^2$$

for some λ with $0 \leq \lambda \leq n-1$. The integer λ is called the *index* of the fold point with respect to the y_2 -direction: this means that x corresponds to a non-degenerate critical point of index λ for the function $y_2 \circ f$ restricted to the submanifold $y_1 = 0$. We say that x is a *definite fold point* if $\lambda = 0$ or $n-1$; otherwise, it is an *indefinite fold point*. We denote by $S_0(f)$ (respectively $S_1(f)$) the set of definite (respectively indefinite) fold points of f .

(2) A singular point $x \in S(f)$ of f is called a *cusp point* if there exist local coordinates $(x_1, x_2, \dots, x_n)$ around x and (y_1, y_2) around $f(x)$ such that f has the form

$$y_1 \circ f = x_1, \quad y_2 \circ f = x_1 x_2 + x_2^3 \pm x_3^2 \pm \dots \pm x_n^2.$$

(3) A smooth map $f : M \rightarrow N$ is called an *excellent map* if $S(f)$ consists only of fold and cusp points.

It is known that every smooth map $f : M \rightarrow N$ can be approximated by (and hence is homotopic to) an excellent map (or a stable map). Note that a smooth map $f : M \rightarrow N$ is stable if and only if it is excellent and $f|_{S(f)}$ satisfies certain normal crossing conditions (for details, see [10], for example).

The following theorem was announced and proved in [20, Theorem 2.6]. Here we give a much simpler proof without performing surgeries of manifolds.

THEOREM 2.2. *Let M be a closed manifold of dimension $n > 2$ and let N be the 2-sphere S^2 or the real projective plane $\mathbf{RP}^2$. Then every continuous map $f : M \rightarrow N$ is homotopic to a C^∞ stable map without definite fold points.*

FIGURE 1. Disk move. The curve α is replaced by the curve β up to slight smoothing.

Proof. We may assume that M is connected and that f is stable. By the swallow-tail move (or flip) as described in [18, Lemma 3.3] applied to $S_0(f)$, we see that f is homotopic to a stable map f_1 such that each connected component of $S_0(f_1)$ is an open arc. Note that the two cusps appearing at the ends of each component constitute a matching pair in the sense of [13, 18]. Then, since M is connected, we can connect the components by appropriate joining curves, and by the beak-to-beak move (or cusp merge) as described in [13] or [18, Lemma 3.7], we see that f_1 is homotopic to a stable map f_2 such that $S_0(f_2)$ is a connected open arc. Then, again by using the same move with a joining curve connecting the pair of cusps in its ends, we see that f_2 is homotopic to a stable map f_3 such that $S_0(f_3)$ is a circle. Note that, in this last move, we can choose the joining curve parallel to $S_0(f_2)$ so that $f_3|_{S_0(f_3)}$ is null-homotopic. Furthermore, by applying the move described in Remark 2.4 below if necessary, we may assume that the immersion $f_3|_{S_0(f_3)}$ is regularly homotopic to an embedding.

Let c be an immersed curve in the surface N with normal crossings. Let us consider a 2-disk D with two corner points embedded in N such that $c \cap \partial D$ consists of an embedded arc, say α , in c , which coincides with one of the two smooth boundary curves of D , possibly together with some transverse intersection points of c and β , where β is the closure of $\partial D \setminus \alpha$. We assume that the two corner points of ∂D are not double points of c and that β does not contain a double point of c . Then the *disk move* with respect to D transforms c to the closure of

$$(c \setminus \alpha) \cup \beta$$

with the corners smoothed (see Figure 1). It is clear that the resulting immersed curve c' with normal crossings is regularly homotopic to c . Furthermore, it is not difficult to show that any

regular homotopy is realized by a finite iteration of disk moves. (In fact, Reidemeister type moves II and III are so realized, and also any isotopy is also so realized.)

Now, let us consider our situation: $c = f_3(S_0(f_3))$ is the image of the definite fold curve immersed in S^2 (or $\mathbf{RP}^2$). If a small collar neighborhood of α in D is not contained in the image of a small tubular neighborhood of the definite fold in M , then it is easy to realize the disk move by a homotopy of the excellent map, even in the presence of an indefinite fold image. Otherwise, we can use the method described in [20, Case 2, p. 375] in order to realize the disk move, at the cost of creating two small embedded circles of indefinite fold image (see the procedures described by Figures 6–8 of [20]). Repeating this procedure finitely many times, we get a stable map f_4 homotopic to f_3 such that $f_4|_{S_0(f_4)}$ is an embedding.

Moreover, we may assume that $f_4(S_0(f_4))$ bounds a 2-disk Δ in N in such a way that the image of a small tubular neighborhood of $S_0(f_4)$ in M is disjoint from $\text{Int } \Delta$. This is possible, since $f_3|_{S_0(f_3)}$ and such a curve are homotopic to each other. Then, by further modifying f_4 , we may assume that $N(\Delta) \cap f_4(S(f_4)) = f_4(S_0(f_4))$, where $N(\Delta)$ is a small neighborhood of Δ in N .

Finally, we can use a set of moves as explained in [7, Figure 7]. We apply two flips, which create two pairs of cusp points; a Reidemeister II type move applied to the definite fold image that is realized by a disk move; a Reidemeister II type move applied to the indefinite fold image, which is seen to be realizable by an index argument; and then unflips (see [8, Lemma 4.8]), which eliminate cusps and definite fold points.

This completes the proof. $\square$

Remark 2.3. In [20], regular homotopy was decomposed into Reidemeister type moves II and III, and each such move was realized by a homotopy of maps. However, strictly speaking, it is not enough: one needs to use isotopies as well. Usually, this causes no problem: however, in our situation, it does, since the image of the definite fold component may intersect with the image of the indefinite fold components. Unfortunately, this was not thoroughly explained in [20].

Remark 2.4. In the course of the proof in [20], we have used the connected sum operations with the map $\tau : S^n \rightarrow \mathbf{R}^2$, constructed in [20, Example 2.2]. In fact, this is also realized as a composition of the homotopy moves as depicted in Figure 2. In the figure, the dotted curve represents the image of the joining curve used for the cusp merge, and the image of the definite (or indefinite) fold is depicted by thick (respectively thin) curves.

Remark 2.5. Note that our new proof has the advantage that the modifications are performed in the same homotopy class. Furthermore, the procedures can be realized algorithmically: we start with the procedures for making the definite fold a circle, then use disk moves and the moves described in Remark 2.4 to get an embedded definite fold image, and finally use the procedure described in [7, Figure 7].

Note that such an algorithmic construction is an important ingredient in [3, 4] for constructing simplified broken Lefschetz fibrations and simplified trisections on four-dimensional manifolds.

FIGURE 2. Moves that realize the connected sum with τ . In this and the following figure, thick lines depict definite fold images, while thin lines depict indefinite fold images. The labels attached to small arrows indicate the index in the designated direction: e.g. label 0 corresponds to a definite fold image and the fiber over the region the arrow points into has one additional $(n - 2)$ -sphere component when compared with the fiber over the region the arrow starts from.

3. Examples

As has been pointed out, our new proof is constructive, which enables us to give explicit examples as follows. Let us start with examples in dimension three.

Example 3.1. Let us consider the positive Hopf fibration $S^3 \rightarrow S^2$. This is a non-singular map and hence is a stable map. By applying the birth (see [18, Lemma 3.1 and Remark 3.2]) and then a cusp merge as depicted in Figure 3(a), we get a stable map with one definite fold circle whose image is embedded. Then, we can apply the moves as in [7, Figure 7] to get a stable map without definite folds. Note that the resulting stable map has two indefinite fold circles that are disjointly embedded into S^2 .

For a positive integer n , by taking the connected sum of n copies of the above stable map, we get a stable map $S^3 \rightarrow S^2$ representing $n \in \mathbf{Z} \cong \pi_3(S^2)$ as follows. For $n = 2$, let f_1 and $f_2 : S^3 \rightarrow S^2$ be stable maps each of which has one definite fold circle whose image is embedded as in the right picture of Figure 3(a). By composing diffeomorphisms isotopic to the identity to f_1 and f_2 if necessary, we may assume that $f_1(S_0(f_1))$ (respectively $f_2(S_0(f_2))$) lies in the northern (respectively southern) hemisphere of S^2 in such a way that they form circles parallel to the equator of S^2 and that the images of the definite fold circles are both adjacent to the equator. Let Δ_1 (respectively Δ_2) be the 2-disk bounded by $f_1(S_0(f_1))$ (respectively $f_2(S_0(f_2))$) lying in the northern (respectively southern) hemisphere. Then, Δ_j contains the image of a small regular neighborhood of the definite fold, $j = 1, 2$, and we may assume that $f_1(p_1)$ and $f_2(p_2)$ are close to each other for some definite fold points $p_j \in S_0(f_j)$, $j = 1, 2$. Let D_j be a small closed 2-disk neighborhood of $f_j(p_j)$ in S^2 , $j = 1, 2$, such that $D_j \cap f_j(S(f_j))$ is diffeomorphic to a line segment consisting of definite fold images and that ∂D_j intersects $f_j(S(f_j))$ transversely. Then, $f_j^{-1}(D_j)$ contains a connected component B_j diffeomorphic to the 3-ball that contains p_j in its interior. Now, we can construct a stable map $f_1 \sharp f_2$ into S^2 by gluing f_j restricted to $S^3 \setminus \text{Int } B_j$, $j = 1, 2$, in an appropriate way. (As to the gluing procedure, refer to [17]. See also Figure 3(b).) For a general positive integer n , we repeat the same procedure. Note that in the gluing process, we need to make sure that the orientations of the source manifolds are consistent so that we get a map representing $n \in \mathbf{Z} \cong \pi_3(S^2)$.

FIGURE 3. (a) A stable map $S^3 \rightarrow S^2$ homotopic to the positive Hopf fibration. (b) A stable map $S^3 \rightarrow S^2$ corresponding to a positive integer $n \in \mathbf{Z} \cong \pi_3(S^2)$, where we take the connected sum of n copies of the stable map homotopic to the positive Hopf fibration. Thick (or thin) curves indicate the image of definite (respectively indefinite) fold points.

Then, by applying the moves as in [7, Figure 7], we get a stable map without definite folds. Note that the resulting stable map has $n + 1$ indefinite fold circles that are disjointly embedded.

For non-positive integers n , we can also use stable maps that are homotopic to the negative Hopf fibration. In this way, for every homotopy class $S^3 \rightarrow S^2$, we get an explicit stable map without definite folds.

Note that in the gluing process above, if we use k positive Hopf fibrations and ℓ negative ones for some non-negative integers k and ℓ with $k + \ell > 0$, then their connected sum yields a stable map without definite folds such that it has $k + \ell + 1$ indefinite fold circles that are disjointly embedded into S^2 and which represents $k - \ell \in \mathbf{Z} \cong \pi_3(S^2)$. This observation shows that there exist pairs of stable maps $S^3 \rightarrow S^2$ without definite folds which have exactly the same indefinite fold images but which are not homotopic to each other.

Problem 3.2. (This problem is originally due to Masamichi Takase.) For an integer $n \in \mathbf{Z} \cong \pi_3(S^2)$, let us consider stable maps $f : S^3 \rightarrow S^2$ without definite folds which represent the associated homotopy class and which satisfies $S(f) \neq \emptyset$ and $f|_{S(f)}$ is an embedding. Then, is the number of components of $S(f)$ congruent modulo two to $n + 1$? Furthermore, is the minimum number of components of $S(f)$ over all such f equal to $|n| + 1$?

Now, let us consider the case of dimension four.

Example 3.3. Some examples of stable maps without definite fold points are given in [1, Section 8.2]. The map $S^4 \rightarrow S^2$ given there corresponds, in fact, to the one obtained in our simple proof starting from the standard stable map $S^4 \rightarrow \mathbf{R}^2 \rightarrow S^2$ with only definite fold points as its singularities, where the first map is the standard projection $\mathbf{R}^5 \rightarrow \mathbf{R}^2$ restricted to the unit 4-sphere and the second map is an embedding.

Example 3.4. Matsumoto [15] constructed a Lefschetz fibration $S^4 \rightarrow S^2$ of genus one with a single ‘twin’ singular fiber, as the composition $h \circ \Sigma h$, where $\Sigma h : S^4 = \Sigma S^3 \rightarrow \Sigma S^2 = S^3$ is the suspension of the Hopf fibration $h : S^3 \rightarrow S^2$. As is well known, this represents the non-trivial element of $\pi_4(S^2) \cong \mathbf{Z}_2$. Note that the twin singular fiber has two Lefschetz critical points. It is known that, by deforming the fibration slightly, the twin singular fiber splits into two singular fibers (of type $I_1^\pm$ in the notation in [16]). By the wrinkling moves (see, for example, [12]), we can transform the two Lefschetz critical points to two circles of indefinite fold and cusp points each of which contains exactly three cusp points. Then, by applying the cusp merges three times, we get three circles of indefinite fold embedded by the projection to S^2 , where the image of one circle component encircles the other two. The result is an example of a stable map without definite folds whose homotopy class represents the non-trivial element of $\pi_4(S^2)$. Note that the stable map $S^4 \rightarrow S^2$ constructed in Example 3.3 represents the neutral element of $\pi_4(S^2)$.

Example 3.5. As has been shown in [17], for every homotopy n -sphere Σ^n with $n \geq 5$, which may possibly be an exotic n -sphere [11], there exists a stable map $g : \Sigma^n \rightarrow \mathbf{R}^2$ with only definite fold as its singularities. In fact, the image is diffeomorphic to the 2-disk. By embedding the 2-disk into S^2 , and by applying the procedure as depicted in [7, Figure 7], we get a stable map $f : \Sigma^n \rightarrow S^2$ with only one circle of indefinite fold points.

In the above examples, the map f restricted to $S(f)$ is an embedding. If we impose this condition for stable maps without definite folds, then we do not know if the number of components of $S(f) = S_1(f)$ is minimal in the given homotopy class, unless it is connected (see also Problem 3.2). Note that if we allow $f|_{S(f)}$ to have self-intersections, then we can always arrange so that $S(f_1) = S_1(f_1)$ is connected by using the techniques developed in [18].

Remark 3.6. As has been shown in [20, Proposition 2.11], for maps $f : M \rightarrow N$ into a general surface N , a similar result does not hold. In fact, the finiteness of the index $[\pi_1(N); f_*\pi_1(M)]$ is a necessary condition for the existence of a stable map without definite folds homotopic to f . Later, Gay and Kirby [8] showed that this is, in fact, sufficient, answering the author’s question posed in [20].

Remark 3.7. After [20, Theorem 2.6] (or Theorem 2.2 of the present paper) was proved, it was used for proving the existence of a broken Lefschetz fibration on an arbitrary closed orientable 4-manifold [2]. (In [2] Baykur also gave a topological proof of the existence of broken Lefschetz pencils on near-symplectic 4-manifolds, using the theorem.) Furthermore, Gay and Kirby [8] extensively generalized the theorem in more general settings. For example, they showed that for a closed connected manifold M of dimension $n > 2$ and a closed connected surface N , a continuous map $f : M \rightarrow N$ is homotopic to a stable map without definite fold points if and only if the index $[\pi_1(N) : f_*\pi_1(M)]$ is finite.

4. Simple stable maps without definite folds

In [19], simple stable maps of closed orientable 3-manifolds into $\mathbf{R}^2$ or S^2 were studied. In this section, combining the techniques developed there with those in this paper, we study simple stable maps without definite folds. Recall that a stable map is *simple* if it does not have cusp points and every component of the inverse image of a point in the target always contains at most one singular point. For example, if a stable map f satisfies that $f|_{S(f)}$ is an embedding, then f is necessarily simple.

We first prove the following.

PROPOSITION 4.1. *Let M be a closed orientable 3-manifold. Then, the following conditions are equivalent.*

- (i) *M is a graph manifold, i.e. it is the union of finitely many S^1 -bundles over compact surfaces attached along their torus boundaries.*
- (ii) *There exists a stable map $f : M \rightarrow S^2$ without definite folds such that $f|_{S(f)}$ is an embedding.*
- (iii) *There exists a simple stable map $f : M \rightarrow S^2$ without definite folds.*

Proof. If M is a graph manifold, then by [19], there exists a stable map $g : M \rightarrow S^2$ without cusps such that $g|_{S(g)}$ is an embedding. Then, by using the method described in the last step of proof of Theorem 2.2, we can replace each definite fold circle with an indefinite fold circle by homotopy. Furthermore, we can arrange that the restriction to the singular point set of the resulting map is still an embedding. Thus, (i) implies (ii). If (ii) is satisfied, then the same stable map satisfies condition (iii). If (iii) is satisfied, then (i) holds, as is proved in [19]. This completes the proof. $\square$

Let M be a closed oriented graph 3-manifold. Yano [21] introduced a subgroup $G(M)$ of $H_1(M; \mathbf{Z})$ and proved that a homology class α of $H_1(M; \mathbf{Z})$ can be represented by a graph link if and only if $\alpha \in G(M)$. Here, a link L in M is a *graph link* if its exterior $M \setminus \text{Int } N(L)$ is a graph manifold, where $N(L)$ is a tubular neighborhood of L in M . On the other hand, let $[M, S^2]$ be the set of homotopy classes of continuous maps $M \rightarrow S^2$, and let $\text{deg} : [M, S^2] \rightarrow H_1(M; \mathbf{Z})$ be the map which associates to an element represented by a smooth map $M \rightarrow S^2$ the homology class represented by a regular fiber (see [5]). Here, we fix an orientation of S^2 , and each regular fiber is oriented in accordance with the orientations of M and S^2 . Note that it is known that deg is a well-defined surjective map, which can be seen by the Pontrjagin–Thom construction. Then, for graph 3-manifolds, we have the following.

PROPOSITION 4.2. *Let M be a closed oriented graph 3-manifold. A continuous map $g : M \rightarrow S^2$ is homotopic to a stable map $f : M \rightarrow S^2$ without definite folds such that $f|_{S(f)}$ is an embedding if and only if the homotopy class of g lies in $\text{deg}^{-1}(G(M))$.*

In particular, if the Jaco–Shalen–Johannson complex of M is a tree (see [21]), then every continuous map $M \rightarrow S^2$ is homotopic to a stable map $f : M \rightarrow S^2$ without definite folds such that $f|_{S(f)}$ is an embedding. For example, if $H_1(M; \mathbf{Z})$ is finite, then this always holds.

Proof of Proposition 4.2 Let $f : M \rightarrow S^2$ be a stable map without definite folds such that $f|_{S(f)}$ is an embedding. Then, by [19], every regular fiber is a graph link. Hence, the homotopy class of f must necessarily lie in $\text{deg}^{-1}(G(M))$.

Conversely, let $g : M \rightarrow S^2$ be a continuous map whose homotopy class lies in $\deg^{-1}(G(M))$. By [21], there exists an oriented graph link L in M that represents a homology class associated with the homotopy class of g . Then, there is a decomposition of M into a finite number of S^1 -bundles over compact surfaces attached along their torus boundaries such that each component of L is an S^1 -fiber of some bundle piece. Then, we mostly follow the procedures described in [19, Section 4] as follows. We first decompose the S^1 -bundle pieces in such a way that each base surface has genus zero. Then, we embed each base surface into S^2 . At this stage, in [19], we embedded the surfaces so that their images were disjoint. However, in our present situation, we embed the base surfaces so that the union of all the boundary curves are embedded, that the base points over which lies a component of L are mapped to the same point, say $x \in S^2$, and that no other points are mapped to x . Furthermore, we arrange the embeddings so that the S^1 -fibers have the correct orientations. Then, we follow the procedures as described in [19, Section 4] to get a stable map $f_1 : M \rightarrow S^2$ such that $f_1|_{S(f_1)}$ is an embedding and that $(f_1)^{-1}(x)$ contains L as an oriented link.

Note that f_1 may have definite fold points. However, each definite fold circle is embedded: therefore, by the final procedure as described in the proof of Theorem 2.2, we can replace each definite fold component with an indefinite fold component, one by one by homotopy. Thus, we get a stable map $f_2 : M \rightarrow S^2$ without definite folds such that $f_2|_{S(f_2)}$ is an embedding and that $(f_2)^{-1}(x)$ contains L as an oriented link.

Then, by performing certain surgery operations to f_2 on a neighborhood of $(f_2)^{-1}(x) \setminus L$ as described in [19, Proof of Lemma 3.6], we get a stable map $f_3 : M \rightarrow S^2$ such that $(f_3)^{-1}(x)$ coincides with L . However, during the surgery operation, we need to create definite folds and $S_0(f_3)$ is not empty in general. As each component of $S_0(f_3)$ is embedded by f_3 , we can eliminate each such component by homotopy to get a stable map $f_4 : M \rightarrow S^2$ without definite folds. After each such homotopy, we can observe that a circle component is added to the inverse image of the point x . However, each such component is a fiber situated near a definite fold, and we see that the union of all such components constitutes a trivial link. In particular, $(f_4)^{-1}(x)$ is an oriented link which is $\mathbf{Z}$ -homologous to L .

Now, the homotopy classes of f_4 and g have the same image by $\deg$. Then, according to [5], we see that by taking the connected sum of f_4 with finitely many positive (or negative) Hopf fibration maps $S^3 \rightarrow S^2$, we can arrange so that the resulting map is homotopic to g . On the other hand, such connected sum operations can be performed as described in Example 3.1: we first create a definite fold circle for each of the maps, we take the connected sum along definite fold circles, and then we eliminate the definite fold. In this way, we get a stable map $f_5 : M \rightarrow S^2$ without definite folds homotopic to g such that $f_5|_{S(f_5)}$ is an embedding. This completes the proof. $\square$

5. Non-existence of singular Legendre fibrations

Let M be a closed orientable 3-manifold endowed with a contact structure and $f : M \rightarrow N$ a stable map of M into a surface N without definite folds or cusp points. Note that then, for every $y \in N$, the fiber $f^{-1}(y)$ is a union of immersed circles in M . Such a map f is called a *singular Legendre fibration* if each fiber $f^{-1}(y)$, $y \in N$, is a union of Legendre curves: i.e. if each fiber is tangent to the plane field (or the two-dimensional distribution) given by the

contact structure. In [20] we proposed the following problem, which was originally due to Goo Ishikawa.

Problem 5.1. Determine those C^∞ stable maps $f : M \rightarrow N$ which are singular Legendre fibrations for some contact structure on M .

In this section, we solve the above problem negatively.

PROPOSITION 5.2. *If a C^∞ stable map $f : M \rightarrow N$ has non-empty singular point set, then it cannot be a singular Legendre fibration.*

Proof. Suppose f is a singular Legendre fibration with respect to a contact structure ξ on M . Let $p \in S(f)$ be a singular point, which is necessarily an indefinite fold point as has been pointed out in [20]. There exist local coordinates (x, y, z) around p , and (X, Y) around $f(p)$ such that f has the form

$$X \circ f(x, y, z) = x^2 - y^2, \quad Y \circ f(x, y, z) = z.$$

Suppose that the contact structure ξ is locally given by a non-degenerate 1-form α of the form

$$\alpha = \varphi_1 dx + \varphi_2 dy + \varphi_3 dz$$

for some C^∞ functions φ_i , $i = 1, 2, 3$, defined locally around p .

First note that the fiber over $f(p)$ contains the set

$$\{(x, y, z) \mid x^2 - y^2 = 0, z = 0\},$$

and hence the tangent vectors for the two crossing curve segments at p span the vector space defined by $dz = 0$ at $p = 0$. This implies that

$$\varphi_1(0) = \varphi_2(0) = 0, \quad \varphi_3(0) \neq 0.$$

Thus, at p , we have

$$\alpha \wedge d\alpha|_p = \varphi_3(0) \left(\frac{\partial \varphi_2}{\partial x}(0) - \frac{\partial \varphi_1}{\partial y}(0) \right) dx \wedge dy \wedge dz|_p.$$

Since α is non-degenerate, we have

$$\frac{\partial \varphi_2}{\partial x}(0) - \frac{\partial \varphi_1}{\partial y}(0) \neq 0.$$

On the other hand, the local vector field v defined around p by $v = (y, x, 0)$ is tangent to the fibers. Therefore, v should lie on the planes defined by $\alpha = 0$. This implies that $y\varphi_1 + x\varphi_2$ constantly vanishes. Since $\varphi_1(0) = \varphi_2(0) = 0$, by the Hadamard lemma, we have

$$\varphi_1 = xg_1 + yh_1 + zk_1, \quad \varphi_2 = xg_2 + yh_2 + zk_2$$

for some C^∞ functions g_1, h_1, k_1, g_2, h_2 and k_2 defined near p . In this case, we have

$$\frac{\partial \varphi_1}{\partial y}(0) = h_1(0), \quad \frac{\partial \varphi_2}{\partial x}(0) = g_2(0).$$

As

$$y\varphi_1 + x\varphi_2 = xyg_1 + y^2h_1 + yzk_1 + x^2g_2 + xyh_2 + xzk_2$$

constantly vanishes, by differentiating the above function with respect to y twice and substituting $(x, y, z) = (0, 0, 0)$, we see that $h_1(0) = 0$. Similarly, we have $g_2(0) = 0$. Therefore, we have

$$\frac{\partial \varphi_2}{\partial x}(0) - \frac{\partial \varphi_1}{\partial y}(0) = g_2(0) - h_1(0) = 0,$$

which is a contradiction. This completes the proof. $\square$

The above proof shows, in fact, that there is no contact structure which makes local fibers near an indefinite fold all Legendrian.

Recall that if an empty singular point set is allowed, then there do exist (non-singular) Legendre fibrations (see [9] and [6, Proposition 1.1.7]).

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