A FUNCTION FIELD ANALOGUE OF THE RASMUSSEN–TAMAGAWA CONJECTURE: THE DRINFELD MODULE CASE

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Abstract. In the arithmetic of function fields, Drinfeld modules play the role that elliptic curves play in the arithmetic of number fields. The aim of this paper is to study a non-existence problem of Drinfeld modules with constraints on torsion points at places with large degree. This is motivated by a conjecture of Christopher Rasmussen and Akio Tamagawa on the non-existence of abelian varieties over number fields with some arithmetic constraints. We prove the non-existence of Drinfeld modules satisfying Rasmussen–Tamagawa type conditions in the case where the inseparable degree of the base field is not divisible by the rank of Drinfeld modules. Conversely if the rank divides the inseparable degree, then we prove the existence of a Drinfeld module satisfying Rasmussen–Tamagawa type conditions.

1. Introduction

Let p be a prime number and fix some p -power $q = p^m$. Write $A := \mathbb{F}_q[t]$ for the polynomial ring in one variable t over $\mathbb{F}_q$ and set $F := \mathbb{F}_q(t)$. Let K be a finite extension of F . In this article, we identify every monic irreducible element π of A with the corresponding finite place of F . Write $\mathbb{F}_\pi = A/\pi A$ for the residue field at π and set $q_\pi := \#\mathbb{F}_\pi = q^{\deg(\pi)}$.

Let r be a positive integer and π a monic irreducible element of A . Define $\mathcal{D}(K, r, \pi)$ to be the set of isomorphism classes $[\phi]$ of rank- r Drinfeld modules over K which satisfy the following two conditions:

- (D1) ϕ has good reduction at any finite place of K not lying above π ;
- (D2) the mod π representation $\bar{\rho}_{\phi, \pi} : G_K \rightarrow \mathrm{GL}_r(\mathbb{F}_\pi)$ attached to ϕ is of the form

$$\bar{\rho}_{\phi, \pi} \simeq \begin{pmatrix} \chi_\pi^{i_1} & * & \cdots & * \\ & \chi_\pi^{i_2} & \ddots & \vdots \\ & & \ddots & * \\ & & & \chi_\pi^{i_r} \end{pmatrix},$$

where χ_π is the mod π Carlitz character (see Example 2.7) and $0 \leq i_1, \dots, i_r \leq q_\pi - 1$ are integers.

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Consider the following question.

Question 1.1. Does there exist a positive constant $C > 0$ depending only on K and r which satisfies the following: if $\deg(\pi) > C$, then the set $\mathcal{D}(K, r, \pi)$ is empty?

This question is motivated by a non-existence conjecture on abelian varieties stated by Rasmussen and Tamagawa [RT1]. Let k be a finite extension of $\mathbb{Q}$ and g a positive integer. For a prime number ℓ , denote by $\tilde{k}_\ell$ the maximal pro- ℓ extension of $k(\mu_\ell)$ which is unramified outside ℓ , where $\mu_\ell = \mu_\ell(\tilde{k})$ is the set of ℓ th roots of unity in $\tilde{k}$. For an abelian variety X over k , write $k(X[\ell^\infty]) := k(\bigcup_{n \geq 1} X[\ell^n])$ for the field generated by all ℓ -power torsion points of X . Define $\mathcal{A}(k, g, \ell)$ to be the set of isomorphism classes $[X]$ of g -dimensional abelian varieties over k which satisfy the following equivalent conditions:

- (RT-1) $k(X[\ell^\infty]) \subseteq \tilde{k}_\ell$;
- (RT-2) X has good reduction at any finite place of k not lying above ℓ and $k(X[\ell])/k(\mu_\ell)$ is an ℓ -extension;
- (RT-3) X has good reduction at any finite place of k not lying above ℓ and the mod ℓ representation $\bar{\rho}_{X,\ell} : G_k \rightarrow \mathrm{GL}_{\mathbb{F}_\ell}(X[\ell]) \simeq \mathrm{GL}_{2g}(\mathbb{F}_\ell)$ is of the form

$$\bar{\rho}_{X,\ell} \simeq \begin{pmatrix} \chi_\ell^{i_1} & * & \cdots & * \\ & \chi_\ell^{i_2} & \ddots & \vdots \\ & & \ddots & * \\ & & & \chi_\ell^{i_{2g}} \end{pmatrix},$$

where χ_ℓ is the mod ℓ cyclotomic character.

These conditions come from the study of a question of Ihara [Ih] related with the kernel of the canonical outer Galois representation of the pro- ℓ fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$; see [RT1]. Rasmussen and Tamagawa conjectured the following.

CONJECTURE 1.2. (Rasmussen and Tamagawa [RT1, Conjecture 1]) *The set $\mathcal{A}(k, g, \ell)$ is empty for any ℓ large enough.*

Since the set $\mathcal{A}(k, g, \ell)$ is always finite (see Section 5, or [RT1]), the conjecture is equivalent to saying that the union $\bigcup_\ell \mathcal{A}(k, g, \ell)$ is also finite. For example, the following cases are known:

- $k = \mathbb{Q}$ and $g = 1$ [RT1, Theorem 2];
- $k = \mathbb{Q}$ and $g = 2, 3$ [RT2, Theorem 7.1 and Theorem 7.2];
- for abelian varieties with everywhere semistable reduction [Oz1, Corollary 4.5] and [RT2, Theorem 3.6];
- for abelian varieties with abelian Galois representations [Oz2, Corollary 1.3];
- for QM abelian surfaces over certain imaginary quadratic fields [Ar, Theorem 9.3].

We notice that, under the assumption of the generalized Riemann hypothesis (GRH) for Dedekind zeta functions of number fields, the conjecture is true in general [RT2, Theorem 5.1]. The key tool of this proof is the effective version of the Chebotarev density theorem for number fields, which holds under GRH. Rasmussen and Tamagawa also state the ‘uniform version’ of the conjecture [RT2, Conjecture 2], which says that one can take a lower bound of ℓ satisfying $\mathcal{A}(k, g, \ell) = \emptyset$ depending only on the degree $[k : \mathbb{Q}]$ and g . For instance,

the uniform version of the conjecture for CM abelian varieties is proved by Bourdon [Bo, Corollary 1] and Lombardo [Lo, Theorem 1.3]. Under GRH, the uniform version of the conjecture is true if $[k : \mathbb{Q}]$ is odd [RT2, Theorem 5.2].

The arithmetic properties of Drinfeld modules are similar to those of elliptic curves over number fields. Under this analogy, the condition (D1)+(D2) can be regarded as a natural translation of the condition (RT-3). In fact, we have also Drinfeld module versions of (RT-1) and (RT-2); we will show that the set $\mathcal{D}(K, r, \pi)$ is characterized by three equivalent conditions as in the abelian variety case (Proposition 5.3).

The main purpose of this paper is to give a complete answer to Question 1.1.

THEOREM 1.3. (Theorem 3.7 and Theorem 4.9) *If r does not divide the inseparable degree $[K : F]_i$ of K/F , then the set $\mathcal{D}(K, r, \pi)$ is empty for any π whose degree is large enough.*

THEOREM 1.4. (Theorem 5.6) *If r divides $[K : F]_i$, then the set $\mathcal{D}(K, r, \pi)$ is never empty for any π .*

The proof of Theorem 1.3 consists of the two cases: (i) $r = p^v$, and (ii) $r = r_0 \cdot p^v$ for some $r_0 > 1$ which is prime to p . In the case (ii), we employ the strategy in [RT2] and use the effective version of the Chebotarev density theorem for function fields proved by Kumer and Scherk [KS]. In this case, the uniform version, which is an analogue of [RT2, Theorem 5.2], is also shown (Theorem 4.10). However, the same argument does not work well in the case (i). The proof in the case (i) is provided by observations about the tame inertia weights of $\bar{\rho}_{\phi, \pi}$ for any $[\phi] \in \mathcal{D}(K, r, \pi)$. This technique is used in [Oz1] and [RT2].

There are differences between the number field setting and the function field setting. Indeed, if r divides $[K : F]_i$ (for which there is no number field setting), then we construct a rank- r Drinfeld module Φ over K satisfying (D1) and (D2) for any π and prove Theorem 1.4. This means that the Drinfeld module analogue of the Rasmussen–Tamagawa conjecture is not true in this case although the original conjecture is generally true under GRH.

The organization of the paper is as follows. In Section 2, after reviewing several basic facts on Drinfeld modules, we study the ramification of Galois representations coming from Drinfeld modules. The results are needed in the next section. In Section 3, for any $[\phi] \in \mathcal{D}(K, r, \pi)$, an important integer e_ϕ is introduced and we prove some non-trivial properties of it. This implies the result in the case (i). The aim of Section 4 is to give the proof in the case (ii). For any $[\phi] \in \mathcal{D}(K, r, \pi)$, we introduce a character $\chi(m_\phi)$ and show the property that $\chi(m_\phi)$ never vanishes on the Frobenius elements of places with some conditions. It contradicts a consequence of the effective Chebotarev density theorem if $\deg(\pi)$ is sufficiently large. Finally, in Section 5, we construct a Drinfeld module satisfying both (D1) and (D2) for any π in the case where r divides $[K : F]_i$. We also show that the set $\mathcal{D}(K, r, \pi)$ is infinite if $\pi = t$ and $r \geq 2$.

Notation

Let p, q, A, F , and K be as above. Set $n_K := [K : F]$ and write K_s for the separable closure of F in K . For a finite place u of K above π , let K_u be the completion of K at u , $\mathcal{O}_{K_u}$ its valuation ring, and $\mathbb{F}_u$ its residue field. We use the same symbol u for the normalized valuation of K_u . Set $q_u := \#\mathbb{F}_u$. Identify G_{K_u} with the decomposition group of G_K at u and regard it

as a subgroup of G_K . Denote by I_{K_u} the inertia subgroup of G_{K_u} at u and choose a lift $\text{Frob}_u \in G_{K_u}$ of the Frobenius element of G_{K_u}/I_{K_u} . Denote by $e_{u|\pi}$ the absolute ramification index of u and set $f_{u|\pi} := [\mathbb{F}_u : \mathbb{F}_\pi]$.

Let $F_\infty := \mathbb{F}_q((1/t))$ be the completion of F at the infinite place ∞ of F and $\mathbb{C}_\infty$ the completion of a fixed algebraic closure of F_∞ . Every algebraic extension of F is always regarded as a subfield of $\mathbb{C}_\infty$. Let $|\cdot|$ be the absolute value of F_∞ attached to the normalized valuation of F_∞ . We also denote by $|\cdot|$ the unique extension of it to $\mathbb{C}_\infty$ and its restriction to each algebraic extension of F . For any non-zero $a \in A$, we see that $|a| = \#(A/aA) = q^{\deg(a)}$.

For any field L , denote by $G_L := \text{Gal}(L^{\text{sep}}/L)$ the absolute Galois group of L . The notation $C = C(x, y, \dots, z)$ indicates a constant C depending only on $x, y, \dots$, and z . We use the notation ρ^{ss} for the semisimplification of a representation ρ .

2. Drinfeld modules

2.1. Basic definitions

Let L be a field equipped with an $\mathbb{F}_q$ -algebra homomorphism $\iota : A \rightarrow L$. Such a pair (L, ι) is called an *A-field*. Let $\mathbb{G}_{a,L}$ be the additive group scheme defined over L . Denote by $\text{End}_{\mathbb{F}_q}(\mathbb{G}_{a,L})$ the ring of $\mathbb{F}_q$ -linear endomorphisms of $\mathbb{G}_{a,L}$. It is isomorphic to the non-commutative polynomial ring $L\{\tau\}$ in one variable τ satisfying $\tau c = c^q \tau$ for any $c \in L$, where τ is the q -power Frobenius map. Let r be a positive integer.

Definition 2.1. A Drinfeld module ϕ of rank r defined over the A -field L is an $\mathbb{F}_q$ -algebra homomorphism

$$\phi : A \rightarrow L\{\tau\}; \quad a \mapsto \phi_a$$

such that $\phi_t = \iota(t) + c_1 \tau + \dots + c_r \tau^r$, where $c_1, \dots, c_r \in L$ and $c_r \neq 0$.

Note that ϕ is completely determined by the image ϕ_t of t . For two Drinfeld modules ϕ and ψ over L , a *morphism* from ϕ to ψ is an element $f \in L\{\tau\}$ such that $f \phi_a = \psi_a f$ for any $a \in A$. We say that f is an *isomorphism* if there exists a morphism g from ψ to ϕ such that $fg = gf = 1$. It is easy to check that f is an isomorphism if and only if $f \in L^\times$.

For any $a \in A$, its image ϕ_a is an endomorphism of $\mathbb{G}_{a,L}$, so that ϕ endows the additive group $\mathbb{G}_{a,L}(L^{\text{sep}}) = L^{\text{sep}}$ with a new A -module structure defined by $a \cdot \lambda := \phi_a(\lambda)$. Denote this A -module by ${}_\phi L^{\text{sep}}$. For any non-zero element $a \in A$, the set of *a-torsion points*

$$\phi[a] = \{\lambda \in {}_\phi L^{\text{sep}}; a \cdot \lambda = \phi_a(\lambda) = 0\}$$

of ϕ is an A -submodule of ${}_\phi L^{\text{sep}}$ on which G_L acts. If a is not contained in the kernel of ι , then $\phi[a]$ is a free A/aA -module of rank r .

Let K be a finite extension of F . From now on, unless otherwise stated, we regard K as an A -field via the inclusion $A \hookrightarrow F \subset K$. Let ϕ be a rank- r Drinfeld module over K . For any finite place v of K , we can regard ϕ as a Drinfeld module over K_v via the canonical inclusion $K\{\tau\} \hookrightarrow K_v\{\tau\}$.

Definition 2.2. (1) We say that ϕ has *stable reduction* at v if there exists a Drinfeld module ψ over K_v such that ψ is isomorphic to ϕ over K_v and

$$\psi_t = t + c'_1 \tau + \dots + c'_r \tau^r$$

such that $c'_1, \dots, c'_r \in \mathcal{O}_{K_v}$ and $c'_{r'} \in \mathcal{O}_{K_v}^\times$ for some $1 \leq r' \leq r$.

(2) We say that ϕ has *good reduction* at v if it has stable reduction at v and $c'_r \in \mathcal{O}_{K_v}^\times$.

PROPOSITION 2.3. (Drinfeld [Dr, Proposition 7.1]) *Every Drinfeld module ϕ over K has potentially stable reduction at any finite place v of K .*

Remark 2.4. Write $\phi_t = t + c_1\tau + \dots + c_r\tau^r$ and set $R := \min_{1 \leq s \leq r} \{v(c_s)/(q^s - 1)\}$. Let K'_v be a finite extension of K_v . If the ramification index $e(K'_v/K_v)$ satisfies $e(K'_v/K_v) \cdot R \in \mathbb{Z}$, then ϕ has stable reduction over K'_v . In particular, we can take K'_v/K_v as a tamely ramified finite separable extension whose ramification index $e(K'_v/K_v)$ divides $\prod_{s=1}^r (q^s - 1)$. Every rank-one Drinfeld module clearly has potentially good reduction at any finite place.

For any monic irreducible element $\pi \in A$, the set of π -torsion points $\phi[\pi]$ is a G_K -stable r -dimensional $\mathbb{F}_\pi$ -vector space, so that the *mod π representation*

$$\bar{\rho}_{\phi, \pi} : G_K \rightarrow \mathrm{GL}_{\mathbb{F}_\pi}(\phi[\pi]) \simeq \mathrm{GL}_r(\mathbb{F}_\pi)$$

attached to ϕ can be defined. Let $A_\pi := \varprojlim A/\pi^n A$ be the π -adic completion of A . Considering the maps $\phi[\pi^{n+1}] \rightarrow \phi[\pi^n]$ defined by $x \mapsto \pi \cdot x$, one can define the *π -adic Tate module* $T_\pi(\phi) := \varprojlim \phi[\pi^n]$. It is a free A_π -module of rank r with continuous G_K -action. Write $\rho_{\phi, \pi} : G_K \rightarrow \mathrm{GL}_r(A_\pi)$ for the representation attached to $T_\pi(\phi)$.

Let π_0 be a monic irreducible element of A with $\pi_0 \neq \pi$ and let v be a finite place of K above π_0 . The next proposition is an analogue of the Néron–Ogg–Shafarevich criterion for good reduction of abelian varieties (cf. [ST, Theorem 1]).

PROPOSITION 2.5. (Takahashi [Ta, Theorem 1]) *A Drinfeld module ϕ over K has good reduction at v if and only if $T_\pi(\phi)$ is unramified at v .*

Suppose that ϕ has good reduction at v . Then $\rho_{\phi, \pi}$ is unramified at v , and so $\rho_{\phi, \pi}(\mathrm{Frob}_v) \in \rho_{\phi, \pi}(G_K)$ is independent of the choice of Frob_v . Denote by

$$P_v(T) := \det(T - \rho_{\phi, \pi}(\mathrm{Frob}_v) | T_\pi(\phi)) \in A_\pi[T]$$

the characteristic polynomial of Frob_v . Then we have the following proposition.

PROPOSITION 2.6. (Takahashi [Ta, Proposition 3(ii)]) *The polynomial $P_v(T)$ has coefficients in A and is independent of π . Any root α of $P_v(T)$ satisfies $|\alpha| = q_v^{1/r}$.*

The following example gives a function field analogue of cyclotomic extensions of number fields.

Example 2.7. (Cf. [Ro, Ch. 12]) The rank-one Drinfeld module $\mathcal{C} : A \rightarrow F\{\tau\}$ determined by $\mathcal{C}_t = t + \tau$ is called the *Carlitz module*. For any monic irreducible element $\pi \in A$, define

$$\chi_\pi : G_F \rightarrow \mathrm{GL}_{\mathbb{F}_\pi}(\mathcal{C}[\pi]) \simeq \mathbb{F}_\pi^\times.$$

It is called the *mod π Carlitz character*. Since $\mathcal{C}$ has good reduction at any finite place π_0 of F , the character χ_π is unramified at π_0 if $\pi_0 \neq \pi$. For any finite place v of K above $\pi_0 \neq \pi$, it is known that χ_π satisfies

$$\chi_\pi(\mathrm{Frob}_v) \equiv \pi_0^{f_v/\pi_0} \pmod{\pi}.$$

The mod π Carlitz character induces an isomorphism $\mathrm{Gal}(F(\mathcal{C}[\pi])/F) \xrightarrow{\sim} \mathbb{F}_\pi^\times$, so that $F(\mathcal{C}[\pi])/F$ is a cyclic extension which is unramified outside π and ∞ . Moreover, it is known that π is totally ramified in $F(\mathcal{C}[\pi])$ and the ramification of the infinite place ∞ is as follows: there exists a subfield $F(\mathcal{C}[\pi])_+$ with degree $[F(\mathcal{C}[\pi]) : F(\mathcal{C}[\pi])_+] = q - 1$ such that ∞ is totally split in $F(\mathcal{C}[\pi])_+$ and any place of $F(\mathcal{C}[\pi])_+$ above ∞ is totally ramified in $F(\mathcal{C}[\pi])$.

2.2. Tate uniformization

Let u be a finite place of K above π and ϕ a rank- r Drinfeld module over K . Suppose that ϕ has stable reduction at u . Then Drinfeld's result on Tate uniformization gives an analytic description of ϕ as a Drinfeld module over K_u .

PROPOSITION 2.8. (Tate uniformization; Drinfeld [Dr, Section 7]) *There exist a unique Drinfeld module ψ over K_u with good reduction and a unique entire analytic surjective morphism $e : \psi \rightarrow \phi$ defined over K_u such that e is the identity on $\mathrm{Lie}(\mathbb{G}_{a, K_u})$.*

It is known that the rank r' of ψ satisfies $r' \leq r$ and the kernel $H := \mathrm{Ker}(e)(K_u^{\mathrm{sep}})$ is an A -lattice of rank $h := r - r'$ in ${}_\psi K_u^{\mathrm{sep}}$, endowed with an action of a finite quotient of G_{K_u} . For any monic irreducible element $\pi_0 \in A$, the analytic morphism e induces the short exact sequence

$$0 \rightarrow T_{\pi_0}(\psi) \rightarrow T_{\pi_0}(\phi) \rightarrow H \otimes_A A_{\pi_0} \rightarrow 0 \quad (2.1)$$

of $A_{\pi_0}[G_{K_u}]$ -modules. In the case where $\pi_0 \neq \pi$, the I_{K_u} -action on $T_{\pi_0}(\phi)$ is potentially unipotent since both $T_{\pi_0}(\psi)$ and $H \otimes_A A_{\pi_0}$ are potentially unramified at u .

Remark 2.9. By the theory of ‘analytic τ -sheaves’ (see [Ga1], [Ga2] and [Ga3]), the sequence (2.1) can be interpreted as follows. For any Drinfeld module ϕ over K_u , one can construct an analytic τ -sheaf $\tilde{M}(\phi)$ associated with ϕ . It is a locally free $\mathcal{O}_{\tilde{\mathbb{A}}_{K_u}^1}$ -module of finite rank on $\tilde{\mathbb{A}}_{K_u}^1$ with some additional structures, where $\tilde{\mathbb{A}}_{K_u}^1$ is the rigid analytic space associated with the affine line $\mathbb{A}_{K_u}^1 = \mathrm{Spec} A \times_{\mathrm{Spec} \mathbb{F}_q} \mathrm{Spec} K_u$. Then the π_0 -adic Tate module $T_{\pi_0}(\tilde{M}(\phi))$ of $\tilde{M}(\phi)$ can be defined and it is canonically isomorphic to $T_{\pi_0}(\phi)$. The Tate uniformization implies that there exist an analytic τ -sheaf $\tilde{N}$ which is potentially trivial and an exact sequence

$$0 \rightarrow \tilde{N} \rightarrow \tilde{M}(\phi) \rightarrow \tilde{M}(\psi) \rightarrow 0.$$

Since $\tilde{M} \mapsto T_{\pi_0}(\tilde{M})$ is a contravariant exact functor, we obtain

$$0 \rightarrow T_{\pi_0}(\tilde{M}(\psi)) \rightarrow T_{\pi_0}(\tilde{M}(\phi)) \rightarrow T_{\pi_0}(\tilde{N}) \rightarrow 0,$$

which coincides with the sequence (2.1) (for example, see [Ga4, Example 7.1]).

We would like to estimate the tame ramification of the lattice H . Suppose that $H \neq 0$ and consider the representation

$$\rho : I_{K_u} \rightarrow \mathrm{GL}_A(H) \simeq \mathrm{GL}_h(A).$$

Then we have the following proposition.

PROPOSITION 2.10. *There exists a finite separable extension L/K_u such that:*

- (1) *the action of I_L on H is trivial;*
- (2) *the ramification index $e(L_0/K_u)$ divides $\prod_{s=1}^{r-1}(q^s - 1)$, where L_0 is the maximal tamely ramified extension of K_u in L .*

Proof. Let E/K_u be a finite Galois extension such that the action of G_{K_u} on H factors through $\text{Gal}(E/K_u)$. Now the image I of I_{K_u} by the canonical projection map $G_{K_u} \rightarrow \text{Gal}(E/K_u)$ is the inertia subgroup of $\text{Gal}(E/K_u)$. For the representation

$$\rho : I \rightarrow \text{GL}_A(H) \simeq \text{GL}_h(A),$$

denote by L the fixed subfield of E by $J := \text{Ker}(\rho)$. By construction, the action of the inertia subgroup I_L of G_L on H is trivial. Now the image $\text{Im}(\rho)$ is a finite subgroup of $\text{GL}_h(A)$ of order $\#I/J = e(L/K_u)$ and $h \leq r - 1$. Applying Lemma 2.12 below to $\text{Im}(\rho)$, we see that $e(L_0/K_u)$ divides $\prod_{s=1}^{r-1}(q^s - 1)$. $\square$

Remark 2.11. Proposition 2.10 means that the Drinfeld module ϕ is ‘semistable’ over L in the following sense. By Remark 2.9, the analytic τ -sheaf $\tilde{M}(\phi)$ is the extension of $\tilde{M}(\psi)$ by $\tilde{N}$ and both $\tilde{M}(\psi)$ and $\tilde{N}$ are ‘good’ over L . Hence the analytic τ -sheaf $\tilde{M}(\phi)$ is *strongly semistable* over L in the sense of [Ga3, Definition 4.6].

LEMMA 2.12. *Let n be a positive integer and let G be a finite subgroup of $\text{GL}_n(A)$. Then the maximal prime-to- p divisor of $\#G$ is a factor of $\prod_{s=1}^n(q^s - 1)$.*

Proof. Consider the t -adic completion $\mathbb{F}_q[[t]]$ of A and regard G as a finite subgroup of $\text{GL}_n(\mathbb{F}_q[[t]])$. Let Γ_n be the kernel of the map $\text{GL}_n(\mathbb{F}_q[[t]]) \rightarrow \text{GL}_n(\mathbb{F}_q)$ induced by the reduction map $\mathbb{F}_q[[t]] \rightarrow \mathbb{F}_q$. Since $\mathbb{F}_q[[t]]$ is a complete noetherian local ring whose residue field is finite of characteristic p , Γ_n is a pro- p group. Hence the short exact sequence $1 \rightarrow \Gamma_n \rightarrow \text{GL}_n(\mathbb{F}_q[[t]]) \rightarrow \text{GL}_n(\mathbb{F}_q) \rightarrow 1$ shows that the maximal prime-to- p divisor of $\#G$ is a factor of $\#\text{GL}_n(\mathbb{F}_q) = q^{n(n-1)/2} \prod_{s=1}^n(q^s - 1)$. $\square$

3. Inertia action on torsion points

Let π be a monic irreducible element of A . In this section, studying the ramification of mod π representations attached to Drinfeld modules, we show the non-existence result (Theorem 3.7) in the case where r is a p -power and does not divide $[K : F]_i$.

3.1. Tame inertia weights

Let u be a finite place of K above π . For a fixed separable closure K_u^{sep} of K_u with residue field $\bar{\mathbb{F}}_u$, denote by K_u^{ur} (respectively K_u^t) the maximal unramified (respectively maximal tamely ramified) extension of K_u in K_u^{sep} , so that I_{K_u} is isomorphic to $\text{Gal}(K_u^{\text{sep}}/K_u^{\text{ur}})$. Denote by $I^t := \text{Gal}(K_u^t/K_u^{\text{ur}})$ the tame inertia subgroup of I_{K_u} . Let d be a positive integer and $\mathbb{F}$ the finite field with q_π^d elements in $\bar{\mathbb{F}}_u$. Then $\mathbb{F}$ is the finite extension of $\mathbb{F}_\pi$ of degree d . Write $\mu_{q_\pi^d-1}(K_u^{\text{sep}})$ for the set of $(q_\pi^d - 1)$ th roots of unity in K_u^{sep} and fix the isomorphism $\mu_{q_\pi^d-1}(K_u^{\text{sep}}) \xrightarrow{\sim} \mathbb{F}^\times$ coming from the reduction map $\mathcal{O}_{K_u^{\text{sep}}} \rightarrow \bar{\mathbb{F}}_u$. For a uniformizer ϖ of

K_u , choose a solution $\eta \in K_u^{\text{sep}}$ to the equation $X^{q_\pi^d-1} - \varpi = 0$ and define

$$\omega_{d,K_u} : I_{K_u} \rightarrow \mu_{q_\pi^d-1}(K_u^{\text{sep}}) \xrightarrow{\sim} \mathbb{F}^\times; \quad \sigma \mapsto \frac{\eta^\sigma}{\eta}.$$

It is independent of the choices of ϖ and η . The character ω_{d,K_u} factors through $I^\dagger$ (cf. [Se1]). We call the $\text{Gal}(\mathbb{F}/\mathbb{F}_\pi)$ -conjugates $(\omega_{d,K_u})^{q_\pi^i}$ for $0 \leq i \leq d-1$ of ω_{d,K_u} the *fundamental characters* of level d . It is easy to check that

$$(\omega_{d,K_u})^{1+q_\pi+\dots+q_\pi^{d-1}} = \omega_{1,K_u}$$

and $(\omega_{d,K_u})^{q_\pi^d-1} = 1$. For any finite extension L of K_u , we see that $(\omega_{d,K_u})|_{I_L} = (\omega_{d,L})^{e(L/K_u)}$ by definition.

As an analogue of Serre's classical result on the mod ℓ cyclotomic character [Se1, Proposition 8], the following fact is known.

PROPOSITION 3.1. (Kim [Ki, Proposition 9.4.3.(2)]) *The character $(\omega_{1,K_u})^{e_{u|\pi}}$ coincides with the mod π Lubin–Tate character restricted to I_{K_u} .*

Remark 3.2. The *mod π Lubin–Tate character* is the character describing the G_{K_u} action on the π -torsion points of the Lubin–Tate formal group over $\mathcal{O}_{K_u}$ associated with π . It coincides with the mod π Carlitz character χ_π restricted to G_{K_u} , so that $\chi_\pi = (\omega_{1,K_u})^{e_{u|\pi}}$ on I_{K_u} .

Let V be a d -dimensional irreducible $\mathbb{F}_\pi$ -representation of I_{K_u} . Then the action of I_{K_u} on V factors through $I^\dagger$, so that V can be regarded as a representation of $I^\dagger$. Using Schur's lemma, we see that $\text{End}_{I^\dagger}(V)$ is a finite field of order q_π^d . Fix an isomorphism $f : \text{End}_{I^\dagger}(V) \xrightarrow{\sim} \mathbb{F}$ and regard V as a one-dimensional $\mathbb{F}$ -representation

$$\rho : I^\dagger \rightarrow \text{End}_{I^\dagger}(V)^\times \xrightarrow{\sim} \mathbb{F}^\times$$

of $I^\dagger$. Since $I^\dagger$ is pro-cyclic and ω_{d,K_u} is surjective, there exists an integer $0 \leq j \leq q_\pi^d - 2$ such that $\rho = (\omega_{d,K_u})^j$. If we decompose $j = n_0 + n_1 q_\pi + \dots + n_{d-1} q_\pi^{d-1}$ with integers $0 \leq n_s \leq q_\pi - 1$, then the set $\{n_0, n_1, \dots, n_{d-1}\}$ is independent of the choice of f .

Definition 3.3. For a d -dimensional irreducible $\mathbb{F}_\pi$ -representation V of I_{K_u} , the integers $n_0, n_1, \dots, n_{d-1}$ as above are called the *tame inertia weights* of V . For any $\mathbb{F}_\pi$ -representation $\rho : G_{K_u} \rightarrow \text{GL}_{\mathbb{F}_\pi}(V)$, the tame inertia weights of ρ are the tame inertia weights of all the Jordan–Hölder quotients of $V|_{I_{K_u}}$.

Denote by $\text{TI}_{K_u}(\rho)$ the set of tame inertia weights of $\rho : G_{K_u} \rightarrow \text{GL}_d(\mathbb{F}_\pi)$.

3.2. Ramification of constrained torsion points

Let ϕ be a Drinfeld module over K satisfying $[\phi] \in \mathcal{D}(K, r, \pi)$ and u a finite place of K above π . By Remark 2.4, we can take a finite separable extension K'_u of K_u such that ϕ has stable reduction and $e(K'_u/K_u)$ divides $\prod_{s=1}^r (q^s - 1)$. By Tate uniformization, we obtain the exact sequence

$$0 \rightarrow \psi[\pi] \rightarrow \phi[\pi] \rightarrow H \otimes_A \mathbb{F}_\pi \rightarrow 0 \quad (3.1)$$

of $\mathbb{F}_\pi[G_{K'_u}]$ -modules. We also take a finite separable extension L of K'_u as in Proposition 2.10 and denote by L_0 the maximal tamely ramified extension of K'_u in L . Set

$$C_1 = C_1(q, r) := (q^r - 1) \prod_{s=1}^{r-1} (q^s - 1)^2$$

and $e_u := e(L_0/K'_u) \cdot e(K'_u/F_\pi)$. Then e_u divides $e_{u|\pi} C_1(q, r)$.

PROPOSITION 3.4. *Every tame inertia weight of $\bar{\rho}_{\phi, \pi}|_{G_{L_0}}$ is between 0 and e_u .*

Proof. By (D2), the restriction $\bar{\rho}_{\phi, \pi}^{\text{ss}}|_{L_{L_0}}$ is isomorphic to $(\omega_{1, L_0})^{j_1} \oplus \cdots \oplus (\omega_{1, L_0})^{j_r}$, where $\{j_1, \dots, j_r\} = \text{TI}_{L_0}(\bar{\rho}_{\phi, \pi})$. Write $\bar{\rho}: G_{L_0} \rightarrow \text{GL}_h(\mathbb{F}_\pi)$ for the representation arising from $H \otimes_A \mathbb{F}_\pi$. Then the sequence (3.1) implies $\bar{\rho}_{\phi, \pi}^{\text{ss}} = \bar{\rho}_{\psi, \pi}^{\text{ss}} \oplus \bar{\rho}^{\text{ss}}$ on G_{L_0} , so that $\text{TI}_{L_0}(\bar{\rho}_{\phi, \pi}) = \text{TI}_{L_0}(\bar{\rho}_{\psi, \pi}) \cup \text{TI}_{L_0}(\bar{\rho})$. Let $\tilde{M}(\psi)$ and $\tilde{N}$ be analytic τ -sheaves on $\tilde{\mathbb{A}}_{L_0}^1$ attached to ψ and H , respectively. Since $\tilde{M}(\psi)$ is good over L_0 , we see that $\text{TI}_{L_0}(\bar{\rho}_{\psi, \pi}) \subset [0, e_u]$ by [Ga1, Theorem 2.14]. On the other hand, the analytic τ -sheaf $\tilde{N}$ is of dimension zero and good over L , so that every tame inertia weight of $\bar{\rho}|_{G_L}$ is zero by [Ga1, Theorem 2.14]. This means that $\bar{\rho}^{\text{ss}}|_{L_L} = 1$. Hence we see that

$$(\omega_{1, L_0})^j|_{L_L} = (\omega_{1, L})^{e(L/L_0) \cdot j} = 1$$

for any $j \in \text{TI}_{L_0}(\bar{\rho})$, so that $e(L/L_0) \cdot j \equiv 0 \pmod{q_\pi - 1}$ holds. Since $e(L/L_0)$ is a p -power and $0 \leq j \leq q_\pi - 2$, we see that $j = 0$. $\square$

The condition (D2) means that $\bar{\rho}_{\phi, \pi}^{\text{ss}}$ is isomorphic to $\chi_\pi^{i_1} \oplus \cdots \oplus \chi_\pi^{i_r}$ for $0 \leq i_1, \dots, i_r \leq q_\pi - 1$. By renumbering $\{j_1, \dots, j_r\}$ if necessary, Propositions 3.1 and 3.4 mean that $\chi_\pi^{i_s}|_{L_{L_0}} = (\omega_{1, L_0})^{i_s \cdot e_u} = (\omega_{1, L_0})^{j_s}$ for any $1 \leq s \leq r$. Thus we obtain

$$i_s \cdot e_u \equiv j_s \pmod{q_\pi - 1} \quad (3.2)$$

for any $1 \leq s \leq r$.

For any finite place v of K not lying above π and any integer m , denote by $P_{v, m}(T) = \det(T - \rho_{\phi, \pi}(\text{Frob}_v^m)|T_\pi(\phi)) \in A[T]$ the characteristic polynomial of Frob_v^m . Set

$$C_2 = C_2(n_K, q, r) := r \cdot n_K^2 \cdot C_1(q, r).$$

Then we obtain the following important proposition.

PROPOSITION 3.5. *If $\deg(\pi) > C_2$, then r divides e_u and the congruence*

$$i_s \cdot e_u \equiv \frac{e_u}{r} \pmod{q_\pi - 1}$$

holds for any $1 \leq s \leq r$.

Proof. Suppose that $\deg(\pi) > C_2$. Take a monic irreducible element $\pi_0 \in A$ with $\deg(\pi_0) = 1$ and a finite place v of K above π_0 . Since ϕ has good reduction at v by (D1), the polynomial $P_{v, e_u}(T)$ is well defined. Now the roots of $P_{v, e_u}(T)$ are given by $\{\alpha_s^{e_u}\}_{s=1}^r$, where $\{\alpha_s\}_{s=1}^r$ are the roots of $P_v(T) = P_{v, 1}(T)$. On the other hand, the condition (D2) implies that the roots of the polynomial $P_{v, e_u}(T) \pmod{\pi}$ in $\mathbb{F}_\pi[T]$ are given by $\{\chi_\pi(\text{Frob}_v)^{i_s \cdot e_u}\}_{s=1}^r$. Set $\pi_v := \pi_0^{f_v|\pi_0}$. By the above relation (3.2), we see that $\chi_\pi(\text{Frob}_v)^{i_s \cdot e_u} = \chi_\pi(\text{Frob}_v)^{j_s}$ for

any $1 \leq s \leq r$. Since $\chi_\pi(\text{Frob}_v)^{j_s} \equiv \pi_v^{j_s} \pmod{\pi}$ holds by Example 2.7, we obtain

$$P_{v,e_u}(T) \equiv \prod_{s=1}^r (T - \chi_\pi(\text{Frob}_v)^{j_s}) \equiv \prod_{s=1}^r (T - \pi_v^{j_s}) \pmod{\pi}. \quad (3.3)$$

Denote by $S_k(x_1, \dots, x_r)$ the fundamental symmetric polynomial of degree k with r variables $x_1, \dots, x_r$ for $0 \leq k \leq r$. Then

$$\prod_{s=1}^r (T - x_s) = \sum_{k=0}^r (-1)^k S_k(x_1, \dots, x_r) T^{r-k}.$$

Now $|\alpha_s^{e_u}| = q_v^{e_u/r}$ for any $1 \leq s \leq r$ by Proposition 2.6. For any $0 \leq k \leq r$, we obtain

$$\begin{aligned} |S_k(\alpha_1^{e_u}, \dots, \alpha_r^{e_u}) - S_k(\pi_v^{j_1}, \dots, \pi_v^{j_r})| &\leq \max_{1 \leq s_1 < \dots < s_k \leq r} \{q_v^{k \cdot e_u/r}, q_v^{j_{s_1} + \dots + j_{s_k}}\} \\ &\leq q_v^{k \cdot e_u} \\ &\leq q_v^{r \cdot e_u} = q^{r \cdot e_u \cdot f_v|\pi_0} \end{aligned}$$

since $j_s \leq e_u$ for each s by Proposition 3.4. Since e_u divides $e_{u|\pi} \cdot C_1(q, r)$ and both $e_{u|\pi}$ and $f_v|\pi_0$ are less than or equal to $n_K = [K : F]$, we see that

$$q^{r \cdot e_u \cdot f_v|\pi_0} \leq q^{C_2} < q^{\deg(\pi)} = |\pi|,$$

which means that all absolute values of coefficients of $P_{v,e_u}(T) - \prod_{s=1}^r (T - \pi_v^{j_s})$ are smaller than $|\pi|$. Therefore the congruence (3.3) implies $P_{v,e_u}(T) = \prod_{s=1}^r (T - \pi_v^{j_s})$. Comparing the absolute values of the roots of $P_{v,e_u}(T)$ and $\prod_{s=1}^r (T - \pi_v^{j_s})$, we see that $e_u/r = j_s$ for any $1 \leq s \leq r$, which implies the conclusion. $\square$

Set $e_\phi := \gcd\{e_u; u|\pi\}$ and $\mathbb{S}_r := \{\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{Z}^r; 1 \leq s_k \leq r\}$.

LEMMA 3.6. *Suppose that $\deg(\pi) > C_2$.*

- (1) *$e_\phi | n_K \cdot C_1(q, r)$. If π is unramified in K_s , then $e_\phi | [K : F]_i \cdot C_1(q, r)$.*
- (2) *For any $(s_1, \dots, s_r) \in \mathbb{S}_r$, the relation $e_\phi \cdot (i_{s_1} + \dots + i_{s_r} - 1) \equiv 0 \pmod{q_\pi - 1}$ holds.*

Proof. Let u be a finite place of K above π and u_0 the place of K_s below u . Then $e_{u|\pi} = e_{u_0|\pi} [K : F]_i$ since u_0 is totally ramified in K if $K \neq K_s$. Hence (1) follows from the relation $e_u | e_{u|\pi} C_1(q, r)$. By Proposition 3.5, we see that $i_s \cdot e_\phi \equiv e_\phi / r \pmod{q_\pi - 1}$. Adding these congruences for $s_1, \dots, s_r$ gives

$$e_\phi \cdot (i_{s_1} + \dots + i_{s_r}) \equiv e_\phi \pmod{q_\pi - 1},$$

which proves (2). $\square$

There exist only finitely many places of F which are ramified in K_s . Define $C_3 = C_3(K_s)$ to be the maximal degree of such places and set

$$C_4 = C_4(n_K, q, r, K_s) := \max\{C_2(n_K, q, r), C_3(K_s)\}.$$

THEOREM 3.7. *Suppose that $r = p^v > 1$ and r does not divide $[K : F]_i$. If $\deg(\pi) > C_4$, then the set $\mathcal{D}(K, r, \pi)$ is empty.*

Proof. Assume that $[\phi] \in \mathcal{D}(K, r, \pi)$ and $\deg(\pi) > C_4$. Then π is unramified in K_s by $\deg(\pi) > C_3$. Proposition 3.5 and Lemma 3.6 imply that r divides $[K : F]_i \cdot C_1(q, r)$. The integer $C_1(q, r)$ is prime to p and so r must divide $[K : F]_i$, which contradicts the assumption on r . $\square$

Remark 3.8. In Section 5, we see that, if r divides $[K : F]_i$, then $\mathcal{D}(K, r, \pi)$ is not empty.

4. Observations at places with small degree

In this section, using Propositions 4.6 and 4.8, we prove Theorem 4.9 on the emptiness of $\mathcal{D}(K, r, \pi)$ when r is not a p -power. In this case, we also prove its uniform version (Theorem 4.10).

4.1. The effective Chebotarev density theorem

Recall some basic facts on function field arithmetic. Let L be an algebraic extension of K . The *constant field* $\mathbb{F}_L$ of L is the algebraic closure of $\mathbb{F}_q$ in L . If $L = \mathbb{F}_L K$, then L is called a *constant field extension* of K , which is unramified at any places [Ro, Proposition 8.5]. If $\mathbb{F}_L = \mathbb{F}_K$, then L is called a *geometric extension* of K . In general, the field $\mathbb{F}_L K$ is the maximal constant extension of K in L and the extension $L/\mathbb{F}_L K$ is geometric. Set $[L : K]_g := [L : \mathbb{F}_L K]$ if L/K is finite, which is called the *geometric extension degree* of L/K . For example, for any $a \in A$, the field $F(\mathcal{C}[a])$ arising from the Carlitz module is a geometric extension of F .

Denote by $\text{Div}(K)$ the divisor group of K , that is, the free abelian group generated by all places of K . We write divisors additively, so that a typical divisor is of the form $D = \sum_v n_v v$. The notation $v \notin D$ means that $n_v = 0$. The *degree* of a place v of K is defined by $\deg_K v := [\mathbb{F}_v : \mathbb{F}_K]$ and it is extended to any divisor $D = \sum_v n_v v$ by $\deg_K D = \sum_v n_v \deg_K v$. The degree $\deg_F \pi$ of a finite place π of F is exactly the degree $\deg(\pi)$ as a polynomial.

Suppose that L is a finite separable extension of K . Then the *conorm map* $i_{L/K} : \text{Div}(K) \rightarrow \text{Div}(L)$ is defined to be the linear extension of

$$i_{L/K} v = \sum_{w|v} e_{w|v} w,$$

where v is a place of K . The following is known (cf. [Ro, Proposition 7.7]).

LEMMA 4.1. *Let w be a place of L above a place v of K and $D \in \text{Div}(K)$. Then*

$$\deg_L i_{L/K} D = [L : K]_g \deg_K D \quad \text{and} \quad \deg_L w = \frac{f_{w|v}}{[\mathbb{F}_L : \mathbb{F}_K]} \deg_K v.$$

For any place w of L above a place v of K , denote by $\mathfrak{p}_w$ the maximal ideal of $\mathcal{O}_{L_w}$ and let δ_w be the exact power of $\mathfrak{p}_w$ dividing the different of $\mathcal{O}_{L_w}$ over $\mathcal{O}_{K_v}$. Then it satisfies $\delta_w \geq e_{w|v} - 1$ with equality holding if and only if p does not divide $e_{w|v}$. Define the *ramification divisor* of L/K by $\mathcal{D}_{L/K} = \sum_w \delta_w w$. For any intermediate field K' of L/K , we see that

$$\mathcal{D}_{L/K} = \mathcal{D}_{L/K'} + i_{L/K'} \mathcal{D}_{K'/K}$$

(for example, see [Se2, Ch. III 4]). Hence $\mathcal{D}_{L/K'} \leq \mathcal{D}_{L/K}$ holds. In addition, the following holds (cf. [CL, Lemma 2.6]).

LEMMA 4.2. *Let L/K and L'/K be finite separable extensions. Then*

$$\mathcal{D}_{LL'/K} \leq i_{LL'/L} \mathcal{D}_{L/K} + i_{LL'/L'} \mathcal{D}_{L'/K}.$$

Now let E be a finite Galois extension of K and v a place of K unramified in E . For any place w of E above v , denote by $\text{Fr}_{w|v}$ the Frobenius element in $\text{Gal}(E/K)$. Then the conjugacy class

$$\left[\frac{E/K}{v} \right] := \{\text{Fr}_{w|v}; w|v\}$$

in $\text{Gal}(E/K)$ consists of these elements. Define $\Sigma_{E/K}$ to be the divisor of K that is the sum of all ramified places of K in E . As a consequence of the effective version of the Chebotarev density theorem [KS, Theorem 1], the following holds.

PROPOSITION 4.3. (Chen and Lee [CL, Corollary 3.4]) *Let E/K be a finite Galois extension and Σ a divisor of K such that $\Sigma \geq \Sigma_{E/K}$. Set $d_0 := [\mathbb{F}_K : \mathbb{F}_q]$ and $d := [\mathbb{F}_E : \mathbb{F}_K]$. Define the constant $B = B(E/K, \Sigma)$ by*

$$B = \max\{\deg_K \Sigma, \deg_E \mathcal{D}_{E/\mathbb{F}_E K}, 2[E : \mathbb{F}_E K] - 2, 1\}.$$

Then for any non-empty conjugacy class $\mathcal{C}$ in $\text{Gal}(E/K)$, there exists a place v of K with $v \notin \Sigma$ such that:

- $\mathcal{C} = \left[\frac{E/K}{v} \right];$
- $\deg_K v \leq \frac{4}{d_0} \log_q \frac{4}{3} (B + 3g_K + 3) + d,$

where g_K is the genus of K .

Let π be a monic irreducible element of A and $m \geq 1$ an integer which divides $\#\mathbb{F}_\pi^\times = q_\pi - 1$. A monic irreducible element π_0 distinct from π is called an *m th power residue modulo π* if $(\pi_0 \bmod \pi) \in (\mathbb{F}_\pi^\times)^m$. As an application of Proposition 4.3, we show that one can find an m th power residue modulo π whose degree is smaller than $\deg(\pi)$ if $\deg(\pi)$ is sufficiently large. Denote by F_m the unique subfield of $F(\mathcal{C}[\pi])$ with $[F_m : F] = m$ and consider the character $\chi(m) : G_F \xrightarrow{\chi_\pi} \mathbb{F}_\pi^\times \rightarrow \mathbb{F}_\pi^\times / (\mathbb{F}_\pi^\times)^m$.

LEMMA 4.4. *The following are equivalent:*

- (1) π_0 is an m th power residue modulo π ;
- (2) $\chi(m)(\text{Frob}_{\pi_0}) = 1$;
- (3) $\text{Frob}_{\pi_0}|_{F_m} = \text{id}$.

Proof. It is trivial when $m = 1$. If not, then this lemma immediately follows from the facts that $\chi_\pi(\text{Frob}_{\pi_0}) \equiv \pi_0 \pmod{\pi}$ and F_m is the fixed subfield of F^{sep} by $\text{Ker}(\chi(m))$. $\square$

Denote by $\tilde{K}$ the Galois closure of K_s over F . Set $E := \tilde{K}F_m$, which is also a Galois extension of F . Consider the divisor $\Sigma := \Sigma_{E/F} + \pi + \infty \in \text{Div}(F)$. For the constant $B = B(E/F, \Sigma) = \max\{\deg_F \Sigma, \deg_E \mathcal{D}_{E/\mathbb{F}_E F}, 2[E : \mathbb{F}_E F] - 2, 1\}$, we obtain the following estimate.

LEMMA 4.5. *Let n be a positive integer. Then there exists a constant $C_5 = C_5(K_s, n_K, q, m, n) > 0$ such that for any π satisfying $\deg(\pi) > C_5$, the inequality*

$$4 \log_q \frac{4}{3}(B+3) + [\mathbb{F}_{\tilde{K}} : \mathbb{F}_q] < \frac{1}{n} \deg(\pi)$$

holds.

Proof. We first compute an upper bound of B . We may assume that π is unramified in K_s . Since the degree $[\tilde{K} : F]$ is less than or equal to $n_K!$, we see that $[\mathbb{F}_{\tilde{K}} : \mathbb{F}_q] \leq n_K!$ and $[E : \mathbb{F}_E F] \leq m \cdot n_K!$. By Example 2.7, the infinite place ∞ of F is split into at most m places in F_m whose ramification indices divide $q-1$ and π is totally ramified or unramified (if $m=1$) in F_m . Thus we see that

$$\deg_F \Sigma \leq \deg_F(\Sigma_{F_m/F} + \Sigma_{\tilde{K}/F} + \pi + \infty) \leq 2 \deg(\pi) + 2 + \deg_F \Sigma_{\tilde{K}/F}.$$

Now $\mathcal{D}_{E/\mathbb{F}_E F} = \mathcal{D}_{E/F}$ holds. Lemmas 4.1 and 4.2 imply that

$$\begin{aligned} \deg_E \mathcal{D}_{E/F} &\leq \deg_E i_{E/\tilde{K}} \mathcal{D}_{\tilde{K}/F} + \deg_E i_{E/F_m} \mathcal{D}_{F_m/F} \\ &\leq m \cdot \deg_{\tilde{K}} \mathcal{D}_{\tilde{K}/F} + [E : F_m]_g \cdot \deg_{F_m} \left(\sum_{v|\infty} (q-2)v + m\pi \right) \\ &\leq m \cdot \deg_{\tilde{K}} \mathcal{D}_{\tilde{K}/F} + n_K! \cdot m(q-2 + \deg(\pi)). \end{aligned}$$

Hence there exist positive constants B_1 and B_2 depending only on K_s, n_K, q and m such that $B \leq B_1 \cdot \deg(\pi) + B_2$ holds. Therefore if $\deg(\pi)$ is sufficiently large, then $4 \log_q \frac{4}{3}(B+3) + [\mathbb{F}_{\tilde{K}} : \mathbb{F}_q] < (1/n) \deg(\pi)$ holds. $\square$

Proposition 4.3 and Lemma 4.5 imply the following.

PROPOSITION 4.6. *Let n be a positive integer. If $\deg(\pi) > C_5$, then there exist a monic irreducible element $\pi_0 \in A$ and a place v of K above π_0 such that:*

- (1) π_0 is an m th power residue modulo π ;
- (2) $\deg(\pi_0) < (1/n) \deg(\pi)$;
- (3) $f_{v|\pi_0} = 1$.

Proof. We may assume that $K = K_s$ since the extension K/K_s is totally ramified at any place if $K \neq K_s$. Let $\tilde{K}$ and $E = \tilde{K}F_m$ be as above and fix an element $\sigma \in \text{Gal}(E/F)$ such that $\sigma|_{KF_m} = \text{id}$. For the conjugacy class $\mathcal{C}$ of σ in $\text{Gal}(E/F)$, by Proposition 4.3 and Lemma 4.5, there exists a place π_0 of F with $\pi_0 \notin \Sigma$ (hence it is a finite place) such that $[(E/F)/\pi_0] = \mathcal{C}$ and $\deg(\pi_0) < (1/n) \deg(\pi)$, so that $\sigma = \text{Fr}_{w|\pi_0}$ for some place w of E . Then the decomposition group Z_w of w over π_0 is generated by σ and it is a subgroup of $\text{Gal}(E/KF_m)$. Denote by K' the fixed subfield of E by Z_w . Then the place v' of K' below w satisfies $e_{v'|\pi_0} = 1$ and $f_{v'|\pi_0} = 1$. Hence $f_{v|\pi_0} = 1$, where v is the place of K below v' . By construction, we see that $\text{Frob}_v|_{F_m} = \text{id}$. Lemma 4.4 means that π_0 is an m th power residue modulo π . $\square$

4.2. Non- p -power rank case

Let ϕ be a rank- r Drinfeld module over K satisfying $[\phi] \in \mathcal{D}(K, r, \pi)$. In this subsection, we always assume that $r = r_0 \cdot p^\nu$ for some $r_0 > 1$ which is prime to p . Now let $i_1, \dots, i_r$ be the positive integers satisfying $\bar{\rho}_{\phi, \pi}^{\text{ss}} \simeq \chi_\pi^{i_1} \oplus \dots \oplus \chi_\pi^{i_r}$ by (D2). For any $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{S}_r$, set $\varepsilon_{\mathbf{s}} := \chi_\pi^{i_{s_1} + \dots + i_{s_r} - 1}$ and define

$$\epsilon := (\varepsilon_{\mathbf{s}})_{\mathbf{s} \in \mathbb{S}_r} : G_F \rightarrow (\mathbb{F}_\pi^\times)^{\oplus r}.$$

Set $m_\phi := \#\epsilon(G_F)$, which is the least common multiple of the orders of $\varepsilon_{\mathbf{s}}$. Since ϵ factors through $\mathbb{F}_\pi^\times$, the image $\epsilon(G_F)$ is cyclic and $m_\phi | (q_\pi - 1)$. Then we obtain the following commutative diagram.

$$\begin{array}{ccccc} & & \epsilon(G_F) & & \\ & \nearrow \epsilon & \uparrow & \nwarrow \simeq & \\ G_F & \xrightarrow{\chi_\pi} & \mathbb{F}_\pi^\times & \xrightarrow{\quad} & \mathbb{F}_\pi^\times / (\mathbb{F}_\pi^\times)^{m_\phi} \\ & \searrow \chi(m_\phi) & & & \end{array}$$

Hence a monic irreducible element π_0 is an m_ϕ th power residue modulo π if and only if $\varepsilon_{\mathbf{s}}(\text{Frob}_{\pi_0}) = 1$ for any $\mathbf{s} \in \mathbb{S}_r$.

LEMMA 4.7. *If $\deg(\pi) > C_2(n_K, q, r)$, then m_ϕ divides the greatest common divisor $(e_\phi, q_\pi - 1)$. In particular, it divides $n_K C_1(q, r)$.*

Proof. It follows from Lemma 3.6(2). $\square$

PROPOSITION 4.8. *If there exist a monic irreducible element π_0 and a finite place v of K above π_0 such that $\deg(\pi) > f_{v|\pi_0} \deg(\pi_0)$ and r_0 does not divide $f_{v|\pi_0}$, then $m_\phi > 1$ and $\chi(m_\phi)(\text{Frob}_v) \neq 1$.*

Proof. Assume that either $m_\phi = 1$ or $\chi(m_\phi)(\text{Frob}_v) = 1$ holds. Then $\varepsilon_{\mathbf{s}}(\text{Frob}_v) = 1$ for any $\mathbf{s} \in \mathbb{S}_r$. Denote by $a_{v, p^\nu} \in A$ the coefficient of T^{r-p^ν} in the characteristic polynomial $P_v(T)$ of Frob_v on $T_\pi(\phi)$. It is given by $a_{v, p^\nu} = (-1)^{p^\nu} S_{p^\nu}(\alpha_1, \dots, \alpha_r)$, where $\alpha_1, \dots, \alpha_r$ are the roots of $P_v(T)$ and $S_{p^\nu}(x_1, \dots, x_r)$ is the fundamental symmetric polynomial of degree p^ν with r variables. We regard the product $\mathbb{S}_{p^\nu}^{r_0}$ of $\mathbb{S}_{p^\nu}$ as a subset of $\mathbb{S}_r$. Consider the subset $\mathbb{S}_{r, p^\nu} := \{(s_1, \dots, s_{p^\nu}); 1 \leq s_1 < \dots < s_{p^\nu} \leq r\}$ of $\mathbb{Z}^{p^\nu}$. Since $S_{p^\nu}(x_1, \dots, x_r)$ is the sum of $\binom{r}{p^\nu}$ monomials of degree p^ν , we obtain that

$$\begin{aligned} (a_{v, p^\nu})^{r_0} &= (-1)^r S_{p^\nu}(\alpha_1, \dots, \alpha_r)^{r_0} \\ &\equiv (-1)^r \left(\sum_{(s_1, \dots, s_{p^\nu}) \in \mathbb{S}_{r, p^\nu}} \chi_\pi^{i_{s_1} + \dots + i_{s_{p^\nu}}}(\text{Frob}_v) \right)^{r_0} \\ &\equiv (-1)^r \sum_{\mathbf{s} \in \mathbb{S}_{p^\nu}^{r_0}} \varepsilon_{\mathbf{s}}(\text{Frob}_v) \chi_\pi(\text{Frob}_v) \end{aligned}$$

$$\begin{aligned}
&\equiv (-1)^r \sum_{\mathbf{s} \in \mathbb{S}_{p^v}^{r_0}} \chi_{\pi}(\text{Frob}_v) \\
&\equiv (-1)^r \binom{r}{p^v}^{r_0} \pi_0^{f_{v|\pi_0}} \pmod{\pi}.
\end{aligned}$$

Since $\binom{r}{p^v}$ is not divisible by p , we see that $(-1)^r \binom{r}{p^v}^{r_0} \pi_0^{f_{v|\pi_0}} \not\equiv 0 \pmod{\pi}$. Here

$$|(a_{v,p^v})^{r_0}| \leq q_v = q^{f_{v|\pi_0} \deg(\pi_0)} < |\pi| \quad \text{and} \quad \left| (-1)^r \binom{r}{p^v}^{r_0} \pi_0^{f_{v|\pi_0}} \right| = |\pi_0^{f_{v|\pi_0}}| = q_v < |\pi|.$$

Hence the above congruence implies that $(a_{v,p^v})^{r_0} = (-1)^r \binom{r}{p^v}^{r_0} \pi_0^{f_{v|\pi_0}}$. Comparing the π_0 -adic valuations of both sides, we obtain $r_0 | f_{v|\pi_0}$, which is a contradiction. $\square$

Set

$$\begin{aligned}
C_6 &= C_6(K_s, n_K, q, r) := \max\{C_5(K_s, n_K, q, m, 1); m | n_K C_1(q, r)\} \\
C_7 &= C_7(K_s, n_K, q, r) := \max\{C_2(n_K, q, r), C_6(K_s, n_K, q, r)\}.
\end{aligned}$$

Then we have the following theorem.

THEOREM 4.9. *Suppose that $r = r_0 p^v$ as above and $\deg(\pi) > C_7$. Then the set $\mathcal{D}(K, r, \pi)$ is empty.*

Proof. Assume that $\mathcal{D}(K, r, \pi)$ is not empty and $[\phi] \in \mathcal{D}(K, r, \pi)$. By Proposition 4.6, there exist a monic irreducible element π_0 and a place v of K above π_0 such that $f_{v|\pi_0} = 1$, $\deg(\pi_0) < \deg(\pi)$ and $\chi(m_{\phi})(\text{Frob}_{\pi_0}) = 1$. However, since π_0 and v satisfy the assumptions of Proposition 4.8, we see that $\chi(m_{\phi})(\text{Frob}_v) = \chi(m_{\phi})(\text{Frob}_{\pi_0}) \neq 1$. $\square$

By the same argument, we can also prove a uniform version. For a fixed finite separable extension K_0 of F with degree $n_0 := [K_0 : F]$ and a positive integer n , set

$$C_8 = C_8(K_0, q, r, n) := \max\{C_2(nn_0, q, r), \max\{C_5(K_0, n_0, q, m, n); m | n_0 C_1(q, r)\}\}.$$

THEOREM 4.10. *Let $r = r_0 p^v$, K_0 , and n be as above. Suppose that r_0 does not divide n . If $\deg(\pi) > C_8$, then for any finite extension K of K_0 satisfying $[K : K_0] = n$, the set $\mathcal{D}(K, r, \pi)$ is empty, namely the union*

$$\bigcup_{[K:K_0]=n} \mathcal{D}(K, r, \pi)$$

is empty.

Proof. Let K be a finite extension of K_0 with $[K : K_0] = n$ and assume that $[\phi] \in \mathcal{D}(K, r, \pi)$. Applying Proposition 4.6 to K_0 , we can find a monic irreducible element π_0 and a finite place v_0 of K_0 above π_0 such that $f_{v_0|\pi_0} = 1$, $n \deg(\pi_0) < \deg(\pi)$ and $\chi(m_{\phi})(\text{Frob}_{\pi_0}) = 1$. Now we can take a place v of K above v_0 such that $f_{v|v_0} (= f_{v|\pi_0})$ is not divisible by r_0 . Indeed, if not, then r_0 must divide $n = \sum_{v|v_0} e_{v|v_0} f_{v|v_0}$. Since $f_{v|\pi_0} \deg(\pi_0) < n \deg(\pi_0) < \deg(\pi)$ by Proposition 4.8, we see that $\chi(m_{\phi})(\text{Frob}_v) = \chi(m_{\phi})(\text{Frob}_{\pi_0})^{f_{v|\pi_0}} \neq 1$. This is a contradiction. $\square$

5. Comparison with number field case

In this final section, we compare the Rasmussen–Tamagawa conjecture and its Drinfeld module analogue. After studying the similarity between them, we construct an example of a Drinfeld module satisfying Rasmussen–Tamagawa type conditions for any π and prove Theorem 5.6. We also prove the infiniteness of $\mathcal{D}(K, r, \pi)$ for $r \geq 2$ and $\pi = t$ in Proposition 5.15.

5.1. Defining conditions of $\mathcal{D}(K, r, \pi)$

In the number field case, $\mathcal{A}(k, g, \ell)$ is defined by the equivalent conditions (RT-1), (RT-2), and (RT-3) in Section 1. The equivalence of them follows from the criterion of Néron–Ogg–Shafarevich and the next group theoretic lemma.

LEMMA 5.1. (Rasmussen and Tamagawa [RT2, Lemma 3.4]) *Let $\mathbb{F}$ be a finite field of characteristic ℓ . Suppose G is a profinite group, $N \subset G$ is a pro- ℓ open normal subgroup, and $C = G/N$ is a finite cyclic subgroup with $\#C \mid \#\mathbb{F}^\times$. Let V be an $\mathbb{F}$ -vector space of dimension r on which G acts continuously. Fix a group homomorphism $\chi_0 : G \rightarrow \mathbb{F}^\times$ with $\text{Ker}(\chi_0) = N$. Then there exists a filtration*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_r = V$$

such that each V_s is G -stable and $\dim_{\mathbb{F}} V_s = s$ for any $0 \leq s \leq r$. Moreover, for each $1 \leq s \leq r$, the G -action on each quotient V_s/V_{s-1} is given by $\chi_0^{i_s}$ for some integer i_s satisfying $0 \leq i_s < \#C$.

Remark 5.2. In [RT2], this lemma is proved when $\mathbb{F} = \mathbb{F}_\ell$. The general case can be proved in the same way.

As an analogue, we give two conditions which are equivalent to (D1)+(D2). Let ϕ be a rank- r Drinfeld module over K and let $\pi \in A$ be a monic irreducible element. Consider the field $K(\phi[\pi^\infty]) := K(\bigcup_{n \geq 1} \phi[\pi^n])$ generated by all π -power torsion points of ϕ . Then it coincides with the fixed subfield of K^{sep} by the kernel of $\rho_{\phi, \pi} : G_K \rightarrow \text{GL}_r(A_\pi)$. Recall that the mod π Carlitz character $\chi_\pi : G_K \rightarrow \text{GL}_{\mathbb{F}_\pi}(\mathcal{C}[\pi]) \simeq \mathbb{F}_\pi^\times$ is an analogue of the mod ℓ cyclotomic character. For the field $L := K(\phi[\pi]) \cap K(\mathcal{C}[\pi])$, we can prove the next proposition in the same way as the abelian variety case.

PROPOSITION 5.3. *The following conditions are equivalent.*

- (DR-1) $K(\phi[\pi^\infty])/L$ is a pro- p extension which is unramified at any finite place of L not lying above π .
- (DR-2) ϕ has good reduction at any finite place of K not lying above π and $K(\phi[\pi])/L$ is a p -extension.
- (DR-3) ϕ satisfies (D1) and (D2).

Remark 5.4. Unlike the abelian variety case, the field $K(\phi[\pi])$ may not contain $K(\mathcal{C}[\pi])$. For example, for $x \in \mathbb{F}_q^\times \setminus \{1\}$, consider the rank-one Drinfeld module ϕ over F determined by $\phi_t = t + x\tau$ and suppose $q \neq 2$. Then the fields $F(\phi[t])$ and $F(\mathcal{C}[t])$ are generated by the roots of $t + xT^{q-1}$ and $t + T^{q-1}$, respectively. By Kummer theory, we see that $F(\phi[t]) \neq F(\mathcal{C}[t])$, so that $F(\phi[t]) \not\supset F(\mathcal{C}[t])$.

Proof of Proposition 5.3. Since the kernel of $\mathrm{GL}_r(A_\pi) \rightarrow \mathrm{GL}_r(\mathbb{F}_\pi)$ is a pro- p group, the extension $K(\phi[\pi^\infty])/K(\phi[\pi])$ is always pro- p . The extension $K(\mathcal{C}[\pi])/K$ is unramified at any finite place of K not lying above π (Example 2.7). Hence the conditions (DR-1) and (DR-2) are equivalent by Proposition 2.5. Suppose that (DR-2) holds. Since we have $[K(\phi[\pi]) : L] = [M : K(\mathcal{C}[\pi])]$ for the composite field $M := K(\phi[\pi])K(\mathcal{C}[\pi])$, $M/K(\mathcal{C}[\pi])$ is also a p -extension. We regard χ_π as a character of $\mathrm{Gal}(M/K)$. Then the condition (DR-3) follows from Lemma 5.1 for $G = \mathrm{Gal}(M/K)$, $\chi_0 = \chi_\pi : G \rightarrow \mathbb{F}_\pi^\times$, $N = \mathrm{Ker}(\chi_0) = \mathrm{Gal}(M/K(\mathcal{C}[\pi]))$, and $V = \phi[\pi]$. Conversely, if (DR-3) holds, then the image of $\mathrm{Gal}(K(\phi[\pi])/L)$ by $\bar{\rho}_{\phi,\pi}$ is contained in

$$\left\{ \begin{pmatrix} 1 & * & \cdots & * \\ & 1 & \ddots & \vdots \\ & & \ddots & * \\ & & & 1 \end{pmatrix} \in \mathrm{GL}_r(\mathbb{F}_\pi) \right\},$$

which is a Sylow p -subgroup of $\mathrm{GL}_r(\mathbb{F}_\pi)$. Since $\bar{\rho}_{\phi,\pi} : \mathrm{Gal}(K(\phi[\pi])/K) \rightarrow \mathrm{GL}_r(\mathbb{F}_\pi)$ is injective, we see that $K(\phi[\pi])/L$ is a p -extension. $\square$

Remark 5.5. The original conjecture of Rasmussen and Tamagawa is formulated for abelian varieties of arbitrary dimension, and so we would like to formulate its function field analogue for some higher-dimensional objects (recall that Drinfeld modules are analogues of elliptic curves). In [An], Anderson introduced objects called t -motives as analogues of abelian varieties of higher dimensions, which are also generalizations of Drinfeld modules. In fact the category of Drinfeld modules is anti-equivalent to that of t -motives of dimension one. It is known that t -motives have the notions of good reduction and Galois representations attached to their π -torsion points (see, for example, [Gal]), so that we can consider the conditions (D1) and (D2) for t -motives. Moreover, Proposition 5.3 is also generalized to t -motives since the Galois criterion of good reduction for t -motives holds. Therefore the set $\mathcal{M}(K, d, r, \pi)$ of isomorphism classes of d -dimensional t -motives over K of rank r satisfying the Rasmussen–Tamagawa type conditions can be defined and the following question makes sense: Is the set $\mathcal{M}(K, d, r, \pi)$ empty for any π with sufficiently large degree?

5.2. Non-emptiness of $\mathcal{D}(K, r, \pi)$

In this subsection, by constructing a concrete example, we prove the following theorem.

THEOREM 5.6. *If r divides $[K : F]_i$, then the set $\mathcal{D}(K, r, \pi)$ is never empty for any π .*

If $r = 1$, then Theorem 5.6 is trivial since the Carlitz module $\mathcal{C}$ satisfies both (D1) and (D2). Assume that $r \geq 2$ and $[K : F]_i$ is divisible by r . Then r is a p -power and the r -power map $A \rightarrow A; a \mapsto a^r$ is an injective ring homomorphism.

For any $a = \sum x_n t^n \in A$ with $x_n \in \mathbb{F}_q$, set $\hat{a} := \sum x_n^{1/r} t^n$. Then we see that $a \mapsto \hat{a}$ is a ring automorphism of A and the map $A \rightarrow A; a \mapsto \hat{a}^r$ is an injective $\mathbb{F}_q$ -algebra homomorphism.

LEMMA 5.7. *Set $[K : F]_i = p^\nu$. Then $K_s = K^{p^\nu}$.*

Proof. Since K is a purely inseparable extension of K_s of degree p^v , the field K^{p^v} is contained in K_s . Consider the sequence of fields $K \supset K^p \supset \cdots \supset K^{p^v}$. Proposition 7.4 of [Ro] implies that each extension $K^{p^n}/K^{p^{n+1}}$ is of degree p . Hence $[K : K^{p^v}] = p^v = [K : K_s]$, which means that $K_s = K^{p^v}$. $\square$

Since r divides $[K : F]_i$, Lemma 5.7 implies that K contains the field $F^{1/r}$. In particular the r th root $t^{1/r}$ of t is contained in K . Then we have a new injective A -field structure $\iota : A \rightarrow K$ defined by $\iota(t) = t^{1/r}$. Define the rank-one Drinfeld module

$$C' : A \rightarrow K\{\tau\}$$

over the A -field (K, ι) by $C'_t = t^{1/r} + \tau$.

Set $(^r)\mu := \sum c'_i \tau^i$ for any $\mu = \sum c_i \tau^i \in K\{\tau\}$. Then $K\{\tau\} \rightarrow K\{\tau\}; \mu \mapsto (^r)\mu$ is a ring endomorphism. We can relate C' with the Carlitz module C as follows.

LEMMA 5.8.

- (1) For any $a \in A$, $(^r)C'_a = C_a$.
- (2) For any element $\lambda \in C'[\hat{a}]$, there exists a unique $\delta \in C[a]$ such that $\lambda = \delta^{1/r}$.

Proof. Clearly $(^r)C'_x = x = C_x$ for any $x \in \mathbb{F}_q$ and $(^r)C'_t = (^r)C'_t = C_t$. Hence for any $a = \sum x_n t^n \in A$,

$$(^r)C'_a = (^r)\left(\sum x_n^{1/r} (C'_t)^n\right) = \sum x_n (C_t)^n = C_a.$$

For any $\lambda \in C'[\hat{a}]$, we see that

$$0 = (C'_a(\lambda))^r = (^r)C'_a(\lambda^r) = C_a(\lambda^r).$$

Hence $\lambda^r \in C[a]$ and we have the injective homomorphism $C'[\hat{a}] \rightarrow C[a]; \lambda \mapsto \lambda^r$ of finite groups. Since $\#C'[\hat{a}]$ is equal to $\#C[a]$ by $\deg(\hat{a}) = \deg(a)$, it is a bijection. $\square$

Define $\Phi_a := C'_{\hat{a}^r} = (C'_a)^r \in K\{\tau\}$ for any $a \in A$. Then by construction it gives an $\mathbb{F}_q$ -algebra homomorphism

$$\Phi : A \rightarrow K\{\tau\},$$

which is determined by $\Phi_t = (t^{1/r} + \tau)^r$. Since $\iota(\hat{a}^r) = a$ holds, Φ is a rank- r Drinfeld module over K . Moreover it has good reduction at every finite place v of K since $v(t^{1/r}) \geq 0$.

By the following proposition, we see that $[\Phi] \in \mathcal{D}(K, r, \pi)$, which implies Theorem 5.6.

PROPOSITION 5.9. *Let i be the positive integer satisfying $ir \equiv 1 \pmod{q_\pi - 1}$ and $i < q_\pi - 1$. Then the mod π representation attached to Φ is of the form*

$$\bar{\rho}_{\Phi, \pi} \simeq \begin{pmatrix} \chi_\pi^i & * & \cdots & * \\ & \chi_\pi^i & \ddots & \vdots \\ & & \ddots & * \\ & & & \chi_\pi^i \end{pmatrix}.$$

Proof. It suffices to prove that $\bar{\rho}_{\Phi, \pi}^{\text{ss}} = (\chi_\pi^i)^{\oplus r}$. For each $1 \leq s \leq r$, set

$$V_s := \{\lambda \in {}_\Phi K^{\text{sep}}; C'_{\hat{a}^s}(\lambda) = 0\}.$$

For any $a \in A$ and $\lambda \in V_s$, we see that $\Phi_a(\lambda) \in V_s$ since $\mathcal{C}'_{\hat{\pi}^s}(\Phi_a(\lambda)) = \mathcal{C}'_{\hat{\pi}^s}(\mathcal{C}'_{\hat{a}^r}(\lambda)) = \mathcal{C}'_{\hat{a}^r}(\mathcal{C}'_{\hat{\pi}^s}(\lambda)) = 0$. Hence V_s is an A -submodule of ${}_{\Phi}K^{\text{sep}}$ with the natural G_K -action. Moreover $\Phi_{\pi}(\lambda) = 0$ for any $\lambda \in V_s$, so that V_s is an $\mathbb{F}_{\pi}$ ($= A/\pi A$)-vector space. Here $\Phi[\pi] = V_r$ by the definition of Φ . Then we obtain the filtration

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_r = \Phi[\pi]$$

of G_K -stable $\mathbb{F}_{\pi}$ -subspaces of $\Phi[\pi]$. The map $V_s \rightarrow V_1; \lambda \mapsto \mathcal{C}'_{\hat{\pi}^{s-1}}(\lambda)$ induces a G_K -equivariant isomorphism $V_s/V_{s-1} \cong V_1$. Since $V_1 = \mathcal{C}'[\hat{\pi}]$ (as a set) and $\deg(\pi) = \deg(\hat{\pi})$, we have $\#V_1 = q_{\hat{\pi}} = q_{\pi}$ ($= \#\mathbb{F}_{\pi}$). Hence $\dim_{\mathbb{F}_{\pi}} V_1 = 1$ and the semisimplification of $\Phi[\pi]$ (as an $\mathbb{F}_{\pi}[G_K]$ -module) is $\Phi[\pi]^{\text{ss}} = \bigoplus_{s=1}^r V_s/V_{s-1} \cong V_1^{\oplus r}$. For any $\sigma \in G_K$ and $\lambda \in V_1$, we prove $\sigma(\lambda) = \chi_{\pi}(\sigma)^i \cdot \lambda$ as follows. Take an element $a_{\sigma} \in A$ satisfying $a_{\sigma} \equiv \chi_{\pi}(\sigma) \pmod{\pi}$. By Lemma 5.8(2), $\lambda = \delta^{1/r}$ for some $\delta \in \mathcal{C}[\pi]$. Then

$$\sigma(\lambda)^r = \sigma(\delta) = \chi_{\pi}(\sigma) \cdot \delta = \mathcal{C}_{a_{\sigma}}(\delta).$$

The $\mathbb{F}_{\pi}$ -vector space structure of V_1 is determined by Φ and so $\chi_{\pi}(\sigma)^i \cdot \lambda = \Phi_{a_{\sigma}^i}(\lambda) = \mathcal{C}'_{\hat{a}_{\sigma}^{ir}}(\lambda)$. Since $ir \equiv 1 \pmod{q_{\hat{\pi}} - 1}$ holds, we have $\hat{a}_{\sigma}^{ir} \equiv \hat{a}_{\sigma} \pmod{\hat{\pi}}$. This implies that $\mathcal{C}'_{\hat{a}_{\sigma}^{ir}}(\lambda) = \mathcal{C}'_{\hat{a}_{\sigma}}(\lambda)$. By Lemma 5.8 (1), we obtain

$$(\chi_{\pi}(\sigma)^i \cdot \lambda)^r = (\mathcal{C}'_{\hat{a}_{\sigma}}(\lambda))^r = {}^{(r)}\mathcal{C}'_{\hat{a}_{\sigma}}(\lambda^r) = \mathcal{C}_{a_{\sigma}}(\delta) = \sigma(\lambda)^r.$$

Since the r -power map is injective, we have $\sigma(\lambda) = \chi_{\pi}(\sigma)^i \cdot \lambda$. Hence the G_K -action on V_1 is given by χ_{π}^i . $\square$

Remark 5.10. Let u be a finite place of K above π . Now r divides $e_{u|\pi}$ by assumption. Set $e = e_{u|\pi}/r$. Since $ir \equiv 1 \pmod{q_{\pi} - 1}$, we see that

$$\chi_{\pi}^i|_{I_{K_u}} = (\omega_{1, K_u})^{i \cdot e_{u|\pi}} = (\omega_{1, K_u})^{i \cdot r \cdot e} = (\omega_{1, K_u})^e.$$

Hence the set of tame inertia weights of $\bar{\rho}_{\Phi, \pi}|_{I_{K_u}}$ is $\text{TI}_{K_u}(\bar{\rho}_{\Phi, \pi}) = \{e\}$.

5.3. Infiniteness of $\mathcal{D}(K, r, t)$

Finally, for $\pi = t$, we construct an infinite subset of $\mathcal{D}(K, r, t)$. In the number field case, the set $\mathcal{A}(k, g, \ell)$ is always finite because of the Shafarevich conjecture proved by Faltings [Fa], which states that there exist only finitely many isomorphism classes of abelian varieties over fixed k with fixed dimension g which have good reduction outside a fixed finite set of finite places of k . However, the Drinfeld module analogue of it does not hold.

Example 5.11. For any $a \in A$, consider the rank-two Drinfeld module $\phi^{(a)} : A \rightarrow F\{\tau\}$ given by $\phi_t^{(a)} = t + a\tau + \tau^2$. It is easily seen that $\phi^{(a)}$ has good reduction at any finite place of F . If $\phi^{(a)}$ is isomorphic to $\phi^{(a')}$ for some $a' \in A$ over F , then there exists an element $c \in F$ such that $c\phi_t^{(a')} = \phi_t^{(a)}c$. Hence

$$\phi_t^{(a')} = t + a'\tau + \tau^2 = t + c^{q-1}a\tau + c^{q^2-1}\tau^2.$$

This means that $c \in \mathbb{F}_q^{\times}$ and hence $a' = c^{q-1}a = a$. Therefore the set of isomorphism classes $\{[\phi^{(a)}]; a \in A\}$ is infinite.

Let W be a G_K -stable one-dimensional $\mathbb{F}_q$ -vector space contained in K^{sep} and write $\kappa_W : G_K \rightarrow \mathbb{F}_q^\times$ for the character attached to W . Set $P_W(T) := \prod_{\lambda \in W} (T - \lambda)$, which is an $\mathbb{F}_q$ -linear polynomial of the form

$$P_W(T) = T^q + c_W T, \quad c_W := \left(\prod_{\lambda \in W \setminus \{0\}} -\lambda \right) \in K^\times$$

by [Go, Corollary 1.2.2]. For any $c \in K^\times$, denote by $\bar{c} \in K^\times / (K^\times)^{q-1}$ the class of c and by $\kappa_{(c)} : G_K \rightarrow \mathbb{F}_q^\times$ the character corresponding to $\bar{c}$ by the map $K^\times / (K^\times)^{q-1} \xrightarrow{\sim} \text{Hom}(G_K, \mathbb{F}_q^\times)$ of Kummer theory.

LEMMA 5.12. *For the above element $c_W \in K^\times$, the character $\kappa_{(-c_W)}$ coincides with κ_W .*

Proof. Since $\lambda^{q-1} = -c_W$ for any $\lambda \in W \setminus \{0\}$, the character $\kappa_{(-c_W)}$ is given by $\kappa_{(-c_W)}(\sigma) = \sigma(\lambda)/\lambda = \kappa_W(\sigma)$ for any $\sigma \in G_K$. $\square$

Identify $\mathbb{F}_t = A/tA = \mathbb{F}_q$. Then $\mathcal{C}[t]$ is a one-dimensional $\mathbb{F}_q$ -subspace of K^{sep} and $P_{\mathcal{C}[t]}(T) = T^q + tT$ by the definition of $\mathcal{C}$. By Lemma 5.12, we see that $\chi_t = \kappa_{(-t)}$. Note that $\chi_t^i = \kappa_{((-t)^i)}$ for any integer i .

Take r elements $c_1, \dots, c_r \in K^\times$. For any $1 \leq s \leq r$, define $f_s(\tau) := (\tau + c_s)(\tau + c_{s-1}) \cdots (\tau + c_1) \in K\{\tau\}$. Set $W_s := \text{Ker}(f_s : K^{\text{sep}} \rightarrow K^{\text{sep}})$, which is a G_K -stable s -dimensional $\mathbb{F}_q$ -subspace of K^{sep} . Thus we obtain the filtration

$$0 = W_0 \subset W_1 \subset \cdots \subset W_r$$

of $\mathbb{F}_q[G_K]$ -modules.

LEMMA 5.13. *The $\mathbb{F}_q$ -linear representation $\bar{\rho} : G_K \rightarrow \text{GL}_{\mathbb{F}_q}(W_r) \simeq \text{GL}_r(\mathbb{F}_q)$ is of the form*

$$\bar{\rho} \simeq \begin{pmatrix} \kappa_{(-c_1)} & * & \cdots & * \\ & \kappa_{(-c_2)} & \ddots & \vdots \\ & & \ddots & * \\ & & & \kappa_{(-c_r)} \end{pmatrix}.$$

Proof. For any $1 \leq s \leq r$, the quotient W_s/W_{s-1} is isomorphic to $\text{Ker}(\tau + c_s : K^{\text{sep}} \rightarrow K^{\text{sep}})$ as an $\mathbb{F}_q[G_K]$ -module. Hence each W_s/W_{s-1} is embedded into K^{sep} . By Lemma 5.12, the action of G_K on W_s/W_{s-1} is given by $\kappa_{(-c_s)}$. $\square$

Fix r integers $i_1, \dots, i_r$ satisfying $\sum_{s=1}^r i_s = 1$. For any $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$ satisfying $\sum_{s=1}^r m_s = 0$, consider the $\mathbb{F}_q$ -algebra homomorphism $\phi^{\mathbf{m}} : A \rightarrow K\{\tau\}$ given by

$$\phi_t^{\mathbf{m}} = (-1)^{r-1} \prod_{s=1}^r (\tau - (-t)^{k_s}),$$

where $k_s = i_s + m_s(q-1)$ for any $1 \leq s \leq r$. Now $\sum_{s=1}^r k_s = 1$, so that the constant term of $\phi_t^{\mathbf{m}}$ is $(-1)^{r-1} \prod_{s=1}^r (-(-t)^{k_s}) = (-1)^{2r} t = t$. Hence $\phi^{\mathbf{m}}$ is a rank- r Drinfeld module over K .

PROPOSITION 5.14. *The isomorphism class $[\phi^{\mathbf{m}}]$ is contained in $\mathcal{D}(K, r, t)$. Moreover, the mod t representation attached to $\phi^{\mathbf{m}}$ is of the form*

$$\bar{\rho}_{\phi^{\mathbf{m}}, t} \simeq \begin{pmatrix} \chi_t^{i_1} & * & \cdots & * \\ & \chi_t^{i_2} & \ddots & \vdots \\ & & \ddots & * \\ & & & \chi_t^{i_r} \end{pmatrix},$$

where $i_1, \dots, i_r$ are the integers fixed as above.

Proof. For any finite place v of K not lying above t , since $-t \in \mathcal{O}_{K_v}$ and the leading coefficient of $\phi_t^{\mathbf{m}}$ is $(-1)^{r-1}$, we see that $\phi^{\mathbf{m}}$ has good reduction at v . Now $\phi^{\mathbf{m}}[t]$ coincides with the kernel of $\prod_{s=1}^r (\tau - (-t)^{k_s})$. Applying Lemma 5.13 to $f_s = (\tau - (-t)^{k_s}) \cdots (\tau - (-t)^{k_1})$, we see that $\bar{\rho}_{\phi^{\mathbf{m}}, t}$ is given as above since $\kappa_{((-t)^{k_s})} = \chi_t^{k_s} = \chi_t^{i_s}$ for any $1 \leq s \leq r$. $\square$

PROPOSITION 5.15. *If $r \geq 2$, then the set $\mathcal{D}(K, r, t)$ is infinite.*

Proof. We construct an infinite subset of $\mathcal{D}(K, r, t)$ as follows. Fix r integers $i_1, \dots, i_r$ satisfying $\sum_{s=1}^r i_s = 1$. For any positive integer m , consider $(-m, 0, \dots, 0, m) \in \mathbb{Z}^r$ and define $\phi^m := \phi^{(-m, 0, \dots, 0, m)}$, which is a Drinfeld module satisfying $[\phi^m] \in \mathcal{D}(K, r, t)$ by Proposition 5.14. Write $\phi_t^m = t + c_1 \tau + \cdots + c_{r-1} \tau^{r-1} + (-1)^{r-1} \tau^r \in K\{\tau\}$. Then by construction the coefficient c_{r-1} is given by

$$c_{r-1} = (-t)^{i_1 - m(q-1)} + (-t)^{i_r + m(q-1)} + \sum_{s=2}^{r-1} (-t)^{i_s}.$$

For any finite place u of K above t , if m is sufficiently large, then

$$u(c_{r-1}) = (i_1 - m(q-1))u(-t) < 0.$$

Hence we see that $u(c_{r-1}) \rightarrow -\infty$ as $m \rightarrow \infty$. On the other hand, for two positive integers m and m' , if $\phi^{m'}$ is isomorphic to ϕ^m , then $\phi_t^{m'} = x^{-1} \phi_t^m x$ for some $x \in \mathbb{F}_K^\times$ by the same argument as in Example 5.11. These facts imply that, if m' is sufficiently large, then ϕ^m and $\phi^{m'}$ are not isomorphic. Therefore the subset $\{[\phi^m]; m \in \mathbb{Z}_{>0}\}$ of $\mathcal{D}(K, r, t)$ is infinite. $\square$

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