MOTIVIC COHOMOLOGY OF TWISTED FLAG VARIETIES

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Abstract. In this paper, we study the $\text{mod}(p)$ motivic cohomology of twisted complete flag varieties X over some restricted fields k . Here we take fields k such that the Milnor K -theory $K_{n+2}^M(k)/p = 0$ for some $n \geq 2$. For these fields, we compute the $\text{mod}(p)$ motivic cohomologies of the Rost motives R_n and the flag variety X containing R_2 .

1. Introduction

Let G be a simply connected compact Lie group (and T its maximal torus). For a field $k \subset \mathbb{C}$, let G_k be the corresponding split reductive group over k and T_k its split torus. Let us write by $\mathbb{G}_k$ a non-trivial G_k -torsor. Then $\mathbb{G}_k/B_k$ is a (twisted form of the) complete flag variety for a Borel subgroup $B_k \supset T_k$. Let us simply write by X the flag variety $\mathbb{G}_k/B_k$. We will study the motivic cohomology $H^{*,*'}(X; \mathbb{Z}/p)$ for the twisted flag variety X .

For non-twisted cases, note that it is well known that

$$H^{*,*'}(G_k/B_k; \mathbb{Z}/p) \cong H^*(G/T; \mathbb{Z}/p) \otimes K_*^M(k) \otimes \mathbb{Z}/p[\tau],$$

where $0 \neq \tau \in H^{0,1}(\text{Spec}(\bar{k}); \mathbb{Z}/p)$. The cohomology $H^*(G/T; \mathbb{Z}/p)$ (for the topological flag manifold) was completely known by Toda, Watanabe and Nakagawa [To, To-Wa, Na]. In this paper, we give examples (which are the first for odd prime p) such that $H^{*,*'}(X; \mathbb{Z}/p)$ is computed for twisted complete flag varieties X .

Given a non-zero pure symbol $a \in K_{n+1}^M(k)/p$ the $\text{mod}(p)$ Milnor K -theory, Rost and Voevodsky found an (original Rost) motive R_n in the motive $M(V_a)$ for the norm variety V_a [Ro1, Ro2, Vo3, Su-Jo]. This motive R_n is used in the proof of the Bloch–Kato conjecture [Vo2, Vo4] by Voevodsky.

Petrov, Semenov and Zainoulline [Pe-Se-Za] show that the motive $M(X)$ of X is decomposed as a sum of (degree changing) generalized Rost motive $R(\mathbb{G}_k)$, namely,

$$M(X) \cong R(\mathbb{G}_k) \otimes \left(\bigoplus_s \mathbb{T}^{\otimes s} \right),$$

where $\mathbb{T}^{\otimes s}$ is the (degree changing) reduced Tate motive. Hence we can compute $H^{*,*'}(X; \mathbb{Z}/p)$ additively if $H^{*,*'}(R(\mathbb{G}_k); \mathbb{Z}/p)$ can be computed, while it is a difficult problem in general cases.

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We say that a Lie group G is of type (I) if it is simply connected and $H^*(G; \mathbb{Z}/p)$ contains only one even-dimensional truncated polynomial generator. In this case, $R(\mathbb{G}_k)$ is isomorphic to the original Rost motive R_2 [Pe-Se-Za] defined from a symbol in $K_3^M(k)/p$, and their Chow groups are known [Me-Su, Ya1, Ya3]. Moreover we can compute also the ring structure of $CH^*(X)/p$ (see [Ya4]).

The motivic cohomology $H^{*,*'}(R_n; \mathbb{Z}/p)$, of course, depends on the field k . For $k = \mathbb{C}$, the variety X is always non-twisted. Hence $H^{*,*'}(X; \mathbb{Z}/p) \cong H^*(G/T; \mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau]$. (In fact, $K_*^M(\mathbb{C})/p = 0$ for $*$ > 0 .) The cohomology $H^{*,*'}(R_n; \mathbb{Z}/2)$ is known for $k = \mathbb{R}$. Hence we can compute $H^{*,*'}(X; \mathbb{Z}/2)$ in this case. We also compute its ring structure (Theorem 8.5) by using the result from [Ya2].

There appears to be no literature in which $H^{*,*'}(R_n; \mathbb{Z}/p)$ is computed for an odd prime p . Hence we write it down (Theorem 3.2) when $K_{n+2}^M(k)/p = 0$. (For each $n \geq 1$, such fields exist [Ka].) Using these, we give the ring structure of $H^{*,*'}(X; \mathbb{Z}/p)$ (Theorem 5.2) when $K_4^M(k)/p = 0$ and $K_3^M(k)/p \neq 0$. We also compute the mod(p) algebraic cobordism $MGL^{*,*'}(X; \mathbb{Z}/p)$ for these fields k .

The plan of this paper is the following. In Section 2, we recall the motives of flag varieties. In Section 3, we give $H^{*,*'}(R_n; \mathbb{Z}/p)$ when $K_{n+2}^M(k)/p = 0$, and its proof is given in Section 4. In Section 5, we study $H^{*,*'}(X; \mathbb{Z}/p)$ when G is type (I) . In Section 6, we study the algebraic cobordism version. In Section 7 we study the $BP\langle n-1 \rangle^*$ theory version, which gives another short proof for the related result in Section 6. In Section 8, we compute $H^{*,*'}(X; \mathbb{Z}/2)$ when $k = \mathbb{R}$.

2. Lie groups and algebraic groups

Let p be a prime number and k a field with $k \subset \mathbb{C}$. Let Y be an algebraic variety over k such that $\bar{Y} = Y \otimes_k \bar{k}$ is cellular for the algebraic closure $\bar{k}$ of k . Let $H^{*,*'}(Y; \mathbb{Z}/p)$ (respectively $CH^*(Y) = CH^*(Y)_{(p)}$) be the mod(p) motivic cohomology (respectively Chow ring localized at p) over k . We recall that [Vo1, Vo2, Vo3]

$$H^{2*,*}(Y; \mathbb{Z}/p) \cong CH^*(Y)/p.$$

Note that $H^{*,*'}(\bar{Y}; \mathbb{Z}_{(p)})$ is torsion-free, since $\bar{Y}$ is cellular.

Let $a = (a_0, \dots, a_n)$, $a_i \in k^\times$, be a non-zero pure symbol in the mod(p) Milnor K -theory $K_{n+1}^M(k)/p$. By Rost [Ro1, Ro2, Su-Jo, Vo1, Vo2, Vo3, Ka-Me], we can construct the norm variety V_a of $\dim(V_a) = p^n - 1$ such that $\bar{V}_a \cong v_n$ is cellular, and $a = 0 \in k_*^M(k(V_a))/p$ for the function field $k(V_a)$ of V_a . Here v_n is the generator of the coefficient ring BP^* of the Brown–Peterson cohomology $BP^*(-)$ [Ra, Ha]. Rost and Voevodsky showed that there is $y \in CH^{b_n}(\bar{V}_a)$ for $b_n = (p^n - 1)/(p - 1)$ such that y^{p-1} is the fundamental class of $\bar{V}_a$.

For a graded ring R , let us write by $R\{a, \dots, b\}$ the free R -module generated by $a, \dots, b$. We have the following theorem [Ro1, Vo1, Vo4, Me-Su, Ya3].

THEOREM 2.1. *For a non-zero pure symbol $a \in K_{n+1}^M(k)/p$, there is an irreducible (Rost) motive R_a (written by R_n simply) in the motive $M(V_a)$ of the norm variety V_a such that*

$CH^*(\bar{R}_a) \cong \mathbb{Z}_{(p)}[y]/(y^p)$, and

$$H^{2*,*}(R_n; \mathbb{Z}_{(p)}) \cong CH^*(R_a) \cong \mathbb{Z}_{(p)}\{1\} \oplus \bigoplus_{i=1}^{p-1} M_n(y^i).$$

Here $M_n(y^i) = \mathbb{Z}_{(p)}\{c_0(y^i)\} \oplus \mathbb{Z}/p\{c_1(y^i), \dots, c_{n-1}(y^i)\}$ with $|c_j(y^i)| = |y^i| - |v_j| = 2(b_n i - (p^j - 1))$.

Let G be a simply connected compact Lie group (and T its maximal torus) and G_k be the corresponding split reductive group over k and T_k its split torus. Let $\mathbb{G}_k$ be a non-trivial G_k -torsor, and $\mathbb{G}_k/B_k$ the (twisted form of the) complete flag variety. Let us simply write by X the twisted flag variety $\mathbb{G}_k/B_k$. We are interested in the motivic cohomology $H^{*,*'}(X; \mathbb{Z}/p)$. (For a Borel subgroup $B \supset T$, note that G_k/B_k is cellular and $H^{*,*'}(G_k/B_k; \mathbb{Z}/p) \cong H^*(G/T; \mathbb{Z}/p) \otimes H^{*,*'}(pt.; \mathbb{Z}/p)$.)

We only consider here the cases

$$gr H^*(G; \mathbb{Z}/p) \cong \mathbb{Z}/p[y]/(y^p) \otimes \Lambda(x_1, \dots, x_\ell),$$

where the degree $|y|$ = even, $|x_i|$ = odd, and $\Lambda(x_1, \dots, x_\ell)$ is the exterior algebra generated by $x_1, \dots, x_\ell$. (When $p = 2$, let $y = x_1^2$ in $H^*(G; \mathbb{Z}/2)$.) The following simple Lie groups satisfy the above isomorphism **[Mi-To]**:

$$(G, p) = \begin{cases} G_2, F_4, E_6, \text{ Spin}(n) \text{ with } 7 \leq n \leq 10 & \text{for } p = 2, \\ F_4, E_6, E_7 & \text{for } p = 3, \\ E_8 & \text{for } p = 5. \end{cases}$$

We say that these (simply connected) groups G are of type (I) . It is known that $\ell \geq 2(p - 1)$.

There is a fibering $G \rightarrow G/T \rightarrow BT$ and the induced spectral sequence (for details, see **[To, To-Wa, Mi-Ni, Na]**)

$$\begin{aligned} E_2^{*,*'} &\cong H^*(G; \mathbb{Z}/p) \otimes H^*(BT) \\ &\cong \mathbb{Z}/p[y]/(y^p) \otimes \Lambda(x_1, \dots, x_\ell) \otimes S(t) \implies H^*(G/T; \mathbb{Z}/p), \end{aligned}$$

where $S(t) = H^*(BT) \cong \mathbb{Z}_{(p)}[t_1, \dots, t_\ell]$ with $|t_i| = 2$. Then it is known **[To]** that there is a regular sequence $(b_1, \dots, b_\ell)$ in $S(t)$ such that $d_{|x_i|+1}(x_i) = b_i$ for the differential in the spectral sequence. (These b_i are called transgressive elements.) Hence we can write

$$gr H^*(G/T; \mathbb{Z}/p) \cong \mathbb{Z}/p[y]/(y^p) \otimes S(t)/(b_1, \dots, b_\ell).$$

Let us write $S(t)/(p, b_1, \dots, b_\ell)$ simply by $S(t)/(b)$, and write $S(t)^+$ as the positive degree parts of $S(t)$.

THEOREM 2.2. **[To]** *We have the ring isomorphism*

$$H^*(G/T; \mathbb{Z}/p) \cong \mathbb{Z}/p[y] \otimes S(t)/(b, f_p), \quad f_p = y^p \bmod(S(t)^+).$$

Petrov, Semenov and Zainoulline **[Pe-Se-Za]** prove that when G is a group of type (I) , the motive $M(X)$ (which is localized at p) is decomposed as a sum of (degree changing) Rost

motive R_2 , namely,

$$M(X) \cong R_2 \otimes \left(\bigoplus_s \mathbb{T}^{\otimes s} \right), \quad (2.1)$$

where $\mathbb{T}^{\otimes s}$ is the (degree changing) reduced Tate motive. Note that $S(t) = H^*(BT) \cong CH^*(BB_k)$. Hence we easily see that $CH^*(\bigoplus_s \mathbb{T}^{\otimes s})/p \cong S(t)/(b)$, and

$$H^{*,*'} \left(\bigoplus_s \mathbb{T}^{\otimes s}; \mathbb{Z}/p \right) \cong H^{*,*'}(pt.; \mathbb{Z}/p) \otimes S(t)/(b),$$

where $pt. = \text{Spec}(k)$. Hence additively $H^{*,*'}(X; \mathbb{Z}/p) \cong H^{*,*'}(R_2; \mathbb{Z}/2) \otimes S(t)/(b)$. In particular, we have additively

$$CH^*(X)/p \cong \mathbb{Z}/p\{1, c_0(y), c_1(y), \dots, c_0(y^{p-1}), c_1(y^{p-1})\} \otimes S(t)/(b).$$

On the other hand, the following additive isomorphism is immediate:

$$S(t)/(b_i b_j, b_k) \cong \mathbb{Z}/p\{1, b_1, \dots, b_{2p-2}\} \otimes S(t)/(b),$$

where i, j, k range over $1 \leq i, j \leq 2p-2 < k \leq \ell$. We can take $c_1(y^i) = b_{2i-1}$ and $c_0(y^i) = b_{2i}$ [Ya4], and prove the following theorem.

THEOREM 2.3. [Ya4] *Let G be a simple Lie group of type (I), and $X = \mathbb{G}_k/B_k$ for a non-trivial torsor $\mathbb{G}_k$. Let $S(t) = H^*(BT) \cong CH^*(BB_k)$ and $b_1, \dots, b_\ell$ be transgressive elements in $S(t)$. Then we have the ring isomorphism*

$$H^{2*,*}(X; \mathbb{Z}/p) \cong CH^*(X)/p \cong S(t)/(p, b_i b_j, b_k),$$

where i, j, k range over $1 \leq i, j \leq 2(p-1) < k \leq \ell$.

3. Motivic cohomology of the Rost motive

Recall that V_a is a norm variety for a non-zero symbol $a \in K_{n+1}^*(k)/p$. Let $\chi_a = \check{C}(V_a)$ be the Čech simplex sheaf for V_a so that we have a map $\chi_a \rightarrow pt. = \text{Spec}(k)$. Let $\tilde{\chi}_a$ be the object defined by the cofiber $\tilde{\chi}_a \rightarrow \chi_a \rightarrow pt.$ in the stable $\mathbb{A}^1$ -homotopy category.

By the solution of the Bloch–Kato conjecture by Voevodsky, we see the exact sequence [Or-Vi-Vo, Ya3]

$$0 \rightarrow H^{*+1,*}(\chi_a; \mathbb{Z}/p) \xrightarrow{\times \tau} K_{*+1}^M(k)/p \rightarrow K_{*+1}^M(k(V_a))/p \quad (3.1)$$

identifying $H^{*+1,*+1}(\chi_a; \mathbb{Z}/p) \cong K_{*+1}^M(k)/p$. Since $a = 0 \in K_{n+1}^M(k(V_a))/p$, there is a unique element $a' \in H^{n+1,n}(\chi_a; \mathbb{Z}/p)$ such that $\tau a' = a$.

Let M_a be the object in the derived category $\mathbb{DM}$ of mixed motives [Vo2, Vo4] defined by the distinguished triangle

$$M(\chi_a(b_n))[2b_n] \rightarrow M_a \rightarrow M(\chi_a) \xrightarrow{\delta_a = Q_0 \cdots Q_{n-1}(a')} M(\chi_a(b_n)[2b_n + 1], \quad (3.2)$$

where $b_n = (p^n - 1)/(p - 1) = p^{n-1} + \dots + p + 1$ so that $\deg(\delta_a) = (2b_n + 1, b_n)$. For $i < p$, define the symmetric powers

$$M_a^i = S^i(M_a) = q_i(M_a^{\otimes i}) \subset M_a^{\otimes i},$$

where $q_i = (1/i!) \sum_{\sigma \in S_i} \sigma$ is the projector in $\mathbb{DM}$, and S_i is the symmetric group of i letters. One of the important results in [Vo2, Vo4] that Voevodsky proved is that M_a^{p-1} is a direct summand of a motive of V_a (for details see [Vo2, Vo4]). This M_a^{p-1} is written by R_a (and is called the Rost motive) in preceding sections.

Hence there are distinguished triangles ((5.5) and (5.6) in [Vo4]) in $\mathbb{DM}$:

$$M_a^{i-1}(b_n)[2b_n] \rightarrow M_a^i \rightarrow M(\chi_a) \xrightarrow{s_i} M_a^{i-1}(b_n)[2b_n + 1], \quad (3.3)$$

$$M(\chi_a)(b_n i)[2b_n i] \rightarrow M_a^i \rightarrow M_a^{i-1} \xrightarrow{r_i} M(\chi_a)(b_n i)[2b_n i + 1]. \quad (3.4)$$

For each prime p , we have the Milnor operation [Vo1, Vo2]

$$Q_i : H^{*,*'}(X; \mathbb{Z}/p) \rightarrow H^{*+2p^i-1, *'+p^i-1}(X; \mathbb{Z}/p).$$

Let us write by $Q(n)$ the exterior algebra $\Lambda(Q_0, \dots, Q_n)$ over $\mathbb{Z}/p$. (We see that $Q_i Q_j = -Q_j Q_i$ for all p and $Q_i^2 = 0$ also for $p = 2$.)

THEOREM 3.1. [Ya3, Theorem 8.5] *Let $0 \neq a = (a_0, \dots, a_n) \in K_{n+1}^M(k)/p$ and $\text{Ker}(a)$ be the kernel of the map $K_*^M(k)/p \xrightarrow{\times a} K_{*+a+1}^M(k)/p$. Then there is a $K_*^M(k) \otimes Q(n)$ -module isomorphism*

$$H^{*,*'}(\tilde{\chi}_a; \mathbb{Z}/p) \cong K_*^M(k)/(\text{Ker}(a)) \otimes Q(n) \otimes \mathbb{Z}/p[\xi_a]\{a'\},$$

where $\xi_a = Q_n Q_{n-1} \cdots Q_0(a')$ and $\deg(a') = (n+1, n)$.

Hence each element $x \in H^{*,*'}(\tilde{\chi}_a; \mathbb{Z}/p)$ is written as

$$x = \sum b \cdot \xi_a^r \cdot Q_{i_1} \cdots Q_{i_s}(a'), \quad b \in K_*^M(k), \quad 0 \leq i_1 < \cdots < i_s \leq n, \quad 0 \leq r.$$

Let $K_1^M(k)/p$ be finite, e.g. generated by $a_1, \dots, a_m$. When p is odd, $K_*^M(k)/p$ is isomorphic to a quotient of the exterior algebra $\Lambda(a_1, \dots, a_m)$. So $K_i^M(k)/p = 0$ for all $i > m$. Hereafter we consider some easy cases such that $K_*^M(k)/p = 0$ for $* > n+1$ and hence $K_*^M(k)/(\text{Ker}(a)) = \mathbb{Z}/p$ for a in Theorem 3.1.

Let us write $H^{*,*'} = H^{*,*'}(pt.; \mathbb{Z}/p) \cong K_*^M(k)/p[\tau]$ for $pt. = \text{Spec}(k)$, and

$$\tilde{H}^{*,*'} = H^{*,*'}(\mathbb{Z}/p\{1\}) = \bigoplus_{(*,*) \neq (0,0)} H^{*,*'}(pt., \mathbb{Z}/p).$$

We assume that a primitive p^2 th root of unity exists in k , that is, each primitive p th root ζ_p is zero in $K_1^M(k)/p$, so that $Q_0(\tau) = \zeta_p = 0$ in $H^{*,*'}(pt.; \mathbb{Z}/p)$.

We have the additive structure of $H^{*,*'}(M_a^{p-1}; \mathbb{Z}/p) \cong H^{*,*'}(R_n; \mathbb{Z}/p)$ by the following theorem. (Its proof will be given in the next section.)

THEOREM 3.2. *Let k contain a primitive p^2 th root of unity. Let $a \in K_{n+1}(k)/p$ be a non-zero pure symbol, and $K_*^M(k)/p = 0$ for $* > n+1$. Then there is an additive isomorphism*

$$H^{*,*'}(R_n; \mathbb{Z}/p) \cong H^{*,*'} \oplus (\tilde{Q}(n-1)\{a'\} \oplus \tilde{H}^{*,*'}\{t\})\{1, t, \dots, t^{p-2}\},$$

where $\tilde{Q}(n-1) = Q(n-1)/(\mathbb{Z}/p\{Q_0 \cdots Q_{n-1}\})$, $\deg(t) = (2b_n, b_n)$.

Hence we have a map $f : H^{*,*'}(R_n; \mathbb{Z}/p) \rightarrow H^{*,*'}[t]/(t^p)$ such that

$$\mathrm{Ker}(f) \cong H^{*,*'} \oplus \mathbb{Z}/p[t]/(t^{p-1}) \otimes \tilde{Q}(n-1)\{a'\}, \quad \mathrm{Coker}(f) \cong \mathbb{Z}/p[t]/(t^{p-1})\{t\}.$$

Remark. Note that t^i is a vertical element, i.e., t^i itself does not exist in $H^{*,*'}(M_a^{p-1}; \mathbb{Z}/p)$ but $a't^{i-1}$, τt^i and ct^i exist for $c \in K_1^M(k)/p$ and $1 \leq i \leq p-1$.

COROLLARY 3.3. [Ya3, Corollary 8.8] *Let $c_i = Q_0 \cdots \hat{Q}_i \cdots Q_{n-1}(a')$. Then $c_i \in H^{2*,*}(R_n; \mathbb{Z}/p)$, and there is the additive isomorphism*

$$H^{2*,*}(M_a^{p-1}; \mathbb{Z}/p) \cong \mathbb{Z}/p\{1\} \oplus \mathbb{Z}/p[t]/(t^{p-1})\{c_0, \dots, c_{n-1}\},$$

where $\deg(t) = (2b_n, b_n)$.

Identifying $c_j(y^i) = c_j \cdot t^{i-1}$, we have Theorem 2.1 in Section 2.

Recall the étale cohomology $H_{\text{ét}}^*(Y; \mathbb{Z}/p) \cong H^{*,*}(Y; \mathbb{Z}/p)$ for a (motive in a) smooth variety Y . Since $\tau t \in \tilde{H}^{*,*'}\{t\}$ and $\tau Q_i(a') = Q_i(a) = 0$, we can see the following.

COROLLARY 3.4. *We have $H_{\text{ét}}^*(R_n; \mathbb{Z}/p) \cong K_*^M(k)[t]/(p, t^p)$.*

This fact itself is easily seen from $H_{\text{ét}}^*(\chi_a; \mathbb{Z}/p) \cong H_{\text{ét}}^*(pt.; \mathbb{Z}/p)$.

Remark. As examples of the fields satisfying the assumption of Theorem 3.2, we can take the high-dimensional local fields defined by Kato and Parsin. Let k be a complete discrete valuation field with residue field F . Then it is well known that

$$K_r^M(k)/p \cong K_r^M(F)/p \oplus K_{r-1}^M(F)/p.$$

Let k_0 be a finite field, and let $k_1, \dots, k_n$ be the sequence of complete discrete valuation fields such that the residue field of k_i is k_{i-1} for each $1 \leq i \leq n$. Then the field $k = k_n$ obtained in this way is called an n -dimensional local field (see [Ka]). Then

$$K_n^M(k_n)/p \cong \mathbb{Z}/p \quad \text{and} \quad K_m^M(k_n)/p = 0 \quad \text{for } m > n.$$

4. The proof of Theorem 3.2

Let Y be an object of the category $\mathbb{DM}$ of mixed motives. For $x \in H^{*,*'}(Y; \mathbb{Z}/p)$, let us write by $f. \deg(x) = |x| = *$, and $s. \deg(x) = *'$, and define

$$w(x) = 2s. \deg(x) - f. \deg(x), \quad d(x) = f. \deg(x) - s. \deg(x).$$

Then it is well known [Vo1, Vo2] that if Y is a (motive of a) smooth variety and $0 \neq x \in H^{*,*'}(Y; \mathbb{Z}/p)$, then $0 \leq w(x)$ and $0 \leq d(x) \leq \dim(Y)$.

To prove Theorem 3.2, we need some lemmas and arguments. Let us assume that p is an odd prime (the $p = 2$ case is proved similarly). Let us write χ_a (respectively M_a , M_a^i , δ_a , ξ_a) by simply χ (respectively M , M^i , δ , ξ). Define

$$D_i(Y) = \{H^{*,*'}(Y; \mathbb{Z}/p) \mid 2b_n i + n < * \leq 2b_n(i+1) + n\}$$

so that $H^{*,*'}(Y; \mathbb{Z}/p) = \sum_{i=-1}^{\infty} D_i(Y)$ and $D_i \cap D_j = 0$ for $i \neq j$. Note that $|\delta| = 2b_n + 1$ and $|\xi| = 2b_{n+1} = 2pb_n + 2$.

To see arguments easily, here we use notation such that $\xrightarrow{D_i}_{a_i}$ or $\xrightarrow{a_i}_{D_i}$ simply means $a_i \in D_i$, e.g.

$$H^{*,*'}(\chi; \mathbb{Z}/p) : \xrightarrow[1]{D_{-1}} \xrightarrow[a', \delta]{D_0} \xrightarrow[0, \dots]{D_1, \dots} \xrightarrow[0]{D_{p-2}} \xrightarrow[Q_n(a'), \xi]{D_{p-1}} \xrightarrow[a' \xi, \delta \xi]{D_p}.$$

On the other hand, when $Y = M^{p-1}$, Theorem 3.2 implies that $D_i = 0$ for $i \geq p-1$, and

$$H^{*,*'}(M^{p-1}; \mathbb{Z}/p) : \xrightarrow[H^{*,*'}]{D_{-1}} \xrightarrow[a', \tilde{H}^{*,*'}_t]{D_0} \xrightarrow[a't, \tilde{H}^{*,*'}_{t^2}, \dots]{D_1, \dots} \xrightarrow[a't^{p-2}, \tilde{H}^{*,*'}_{t^{p-1}}]{D_{p-2}} \xrightarrow[0, 0, \dots]{D_{p-1}, D_p, \dots}.$$

We first compute $H^*(M; \mathbb{Z}/p)$. Consider the exact sequence induced from (3.2):

$$\dots \xleftarrow{\delta} H^{*,*'}(\chi(b_n)[2b_n]; \mathbb{Z}/p) \xleftarrow{j_1} H^{*,*'}(M; \mathbb{Z}/p) \xleftarrow{j_2} H^{*,*'}(\chi; \mathbb{Z}/p) \xleftarrow{\delta} \dots$$

Hence we have the isomorphism

$$H^{*,*'}(M; \mathbb{Z}/p) \cong \text{Coker}(\delta) \oplus \text{Ker}(\delta)[b_n](2b_n).$$

The map $\delta : \chi \rightarrow \chi(b_n)[2b_n + 1]$ in $\mathbb{DM}$ is given by the cup product $x \mapsto x \cup \delta$ for the element $\delta \in H^{2b_n+1, b_n}(\chi; \mathbb{Z}_{(p)})$. In fact, we note that [Vo3, Vo4]

$$\begin{aligned} \text{Hom}_{\mathbb{DM}}(M(\chi), M(\chi)(*)[*]) &\cong \text{Hom}_{\mathbb{DM}}(M(\chi) \otimes M(\chi), \mathbb{Z}_{(p)}(*)[*]) \\ &\cong \text{Hom}_{\mathbb{DM}}(M(\chi), \mathbb{Z}_{(p)}(*)[*]) \cong H^{*,*'}(\chi; \mathbb{Z}_{(p)}). \end{aligned}$$

Note that from $Q_i^2 = 0$, the map multiplying $\delta = Q_{n-1} \cdots Q_0(a')$ is a $Q(n-1)$ -module map, while it is not a $Q(n)$ -module map.

LEMMA 4.1. *In $H^{*,*'}(\chi; \mathbb{Z}/p)$, we have $\delta \cdot Q_n(a') = \xi \cdot a'$.*

Proof. We have

$$Q_n(\delta \cdot Q_n(a')) = Q_n(\delta) \cdot Q_n(a') = \xi \cdot Q_n(a') = Q_n(\xi \cdot a').$$

So $\delta \cdot Q_n(a') = \xi \cdot a' + Q_n(b)$ for some b since $H^{*,*'}(\tilde{\chi}; \mathbb{Z}/p)$ is $Q(n)$ -free. Write $b = \sum \lambda_J Q_J(a')$ for some $\lambda_J \in \mathbb{Z}/p$ and $Q_J \in Q(n-1)$. But $|b| = |\delta a'| > |\delta = Q_{n-1} \cdots Q_0(a')|$, and we see that $\lambda_J = 0 \in \mathbb{Z}/p$. $\square$

The map $\times \delta : H^{*,*'}(\chi; \mathbb{Z}/p) \rightarrow H^{*,*'}(\chi; \mathbb{Z}/p)$ is written as

$$\begin{array}{ccccccc} & \xrightarrow[1]{D_{-1}} & \xrightarrow[a', \delta]{D_0} & \xrightarrow[0, \dots, 0]{D_1, \dots, D_{p-3}} & \xrightarrow[0]{D_{p-2}} & \xrightarrow[Q_n(a'), \xi]{D_{p-1}} & \xrightarrow[a' \xi, \delta \xi]{D_p} \\ \delta \downarrow & & \delta \downarrow & \delta \downarrow & \delta \downarrow & \delta \downarrow & \delta \downarrow \\ 1 \xrightarrow{D_{-1}} & \xrightarrow[a', \delta]{D_0} & \xrightarrow[0, 0]{D_1} & \xrightarrow[0, \dots, 0]{D_2, \dots, D_{p-2}} & \xrightarrow[Q_n(a'), \xi]{D_{p-1}} & \xrightarrow[a' \xi, \delta \xi]{D_p} & \xrightarrow[0, 0]{D_{p+1}} \end{array},$$

which is not a commutative diagram, and it means only $\delta \cdot 1 = \delta$, $\delta \cdot a' = 0$, $\delta \cdot \delta = 0$, $\dots$, $\delta \cdot Q_n(a') = a' \xi$, $\dots$.

Recall that $Q(n-1)\{a'\}/\delta \cong \tilde{Q}(n-1)\{a'\}$. We also recall that $\delta \cdot Q(n-1)Q_n\{a'\} \cong Q(n-1)\{a' \xi\}$ from the above lemma. Using these, we can compute

$$\text{Coker}(\delta) \cong H^{*,*'} \oplus \tilde{Q}(n-1)(a') \oplus Q(n-1)Q_n(a') \otimes \mathbb{Z}/p[\xi],$$

$$\text{Ker}(\delta)[b_n](2b_n) \cong \text{Ker}(\delta)\{t\} \cong \tilde{H}^{*,*'}\{t\} \oplus (Q(n-1)(a't) \otimes \mathbb{Z}/p[\xi]),$$

where t is a vertical element with $\deg(t) = (2b_n, b_n)$ for changing degree. Hence we can write $H^{*,*'}(M; \mathbb{Z}/p)$ as

$$\begin{array}{ccccccc} \xrightarrow{H^{*,*'}} & \xrightarrow{a', \tilde{H}^{*,*'}t} & \xrightarrow{a't, \delta t} & \xrightarrow{0, \dots, 0} & \xrightarrow{Q_n(a'), \xi} & \xrightarrow{0, 0} & \xrightarrow{a'\xi t, \delta \xi t} \\ D_{-1} & D_0 & D_1 & D_2, \dots, D_{p-2} & D_{p-1} & D_p & D_{p+1} \end{array}.$$

Here note that $Q_{n-1} \cdots Q_0(Q_n(a')) = \xi$, e.g. $\xi \in Q(n-1)(Q_n(a'))$.

LEMMA 4.2. *We have the isomorphism*

$$H^{*,*'}(M; \mathbb{Z}/p) \cong H^{*,*'} \oplus \tilde{Q}(n-1)(a') \oplus \tilde{H}^{*,*'}\{t\} \oplus Q(n-1)\{a't, Q_n(a')\} \otimes \mathbb{Z}/p[\xi].$$

Next, we compute $H^{*,*'}(M^i; \mathbb{Z}/p)$ for $1 < i < p-1$. We consider the long exact sequence induced from (3.4):

$$\cdots \xleftarrow{r_i^*} H^{*-2b_n i, *'-b_n i}(\chi; \mathbb{Z}/p) \leftarrow H^{*,*'}(M^i; \mathbb{Z}/p) \leftarrow H^{*,*'}(M^{i-1}; \mathbb{Z}/p) \xleftarrow{r_i^*} \cdots$$

The map r_i^* is represented by an element in $H^{2b_n i+1, b_n i}(M^{i-1}; \mathbb{Z}/p)$ from

$$\begin{aligned} \text{Hom}_{\mathbb{DM}}(M^{i-1}, M(\chi)(*)[*]) &\cong \text{Hom}_{\mathbb{DM}}(M^{i-1} \otimes M(\chi), \mathbb{Z}/p(*)[*]) \\ &\cong \text{Hom}_{\mathbb{DM}}(M^{i-1}, \mathbb{Z}/p(*)[*]) \cong H^{*,*'}(M^{i-1}; \mathbb{Z}/p). \end{aligned}$$

Hence $H^{*,*'}(M^i; \mathbb{Z}/p) \cong \text{Coker}(r_i^*) \oplus \text{Ker}(r_i^*)\{t^i\}$. We first study the case $i = 2 < p-1$. Note that the map $\times \delta t = \times Q_{n-1} \cdots Q_0(a't)$ is a $Q(n-1)$ -module map.

LEMMA 4.3. *We have $r_2^*(1) = \delta t$ and $r_2^*(Q_n(a')) = a't\xi$ (i.e. $\delta t Q_n(a') = a't\xi$).*

Proof. For $* < 2b_n i$, it is immediate that $H^{*-2b_n i, *'-b_n i}(\chi; \mathbb{Z}/p) = 0$. By the above exact sequence induced from (3.4), we see that

$$H^{*,*'}(M^i; \mathbb{Z}/p) \cong H^{*,*'}(M^{i-1}; \mathbb{Z}/p) \quad \text{for } * < 2b_n i.$$

Therefore, if $0 \neq \delta t \in H^{*,*'}(M^2; \mathbb{Z}/p)$, then so it is in $H^{*,*'}(M^{p-1}; \mathbb{Z}/p)$, which is a direct summand of $H^{*,*'}(V_a; \mathbb{Z}/p)$. This is a contradiction, since $w(\delta) = (2s, \deg - f, \deg)(\delta) = -1 < 0$. Therefore $\delta t \in \text{Im}(r_i^*)$. By dimensional reasoning, we have $\delta t = r_2^*(1)$.

Let us write $x = r_2^*(Q_n(a'))$. Then

$$\begin{aligned} Q_{n-1} \cdots Q_0(x) &= r_2^*(Q_n Q_{n-1} \cdots Q_0(a')) = r_2^*(\xi) \\ &= \delta t \xi = Q_{n-1} \cdots Q_0(a't\xi). \end{aligned}$$

Hence it induces $x = a't\xi$. □

Remark. This lemma is an immediate consequence of Lemma 4.1, if t exists, however, which is not correct.

The map $r_2^* : H^{*,*'}(\chi; \mathbb{Z}/p) \rightarrow H^{*,*'}(M; \mathbb{Z}/p)$ is given as follows:

$$\begin{array}{ccccccc} \xrightarrow{D_{-1}} & \xrightarrow{D_0} & \xrightarrow{D_1, \dots, D_{p-4}} & \xrightarrow{D_{p-3}} & \xrightarrow{D_{p-2}} & \xrightarrow{D_{p-1}} & \\ 1 & a', \delta & 0, \dots, 0 & 0 & 0 & Q_n(a'), \xi & \\ r_2^* \downarrow & r_2^* \downarrow & r_2^* \downarrow & r_2^* \downarrow & r_2^* \downarrow & r_2^* \downarrow & \\ \xrightarrow{H^{*,*'}, a', \tilde{H}^{*,*'}t} & \xrightarrow{a't, \delta t} & \xrightarrow{0, 0} & \xrightarrow{0, \dots, 0} & \xrightarrow{Q_n(a'), \xi} & \xrightarrow{0} & \xrightarrow{a'\xi t, \delta \xi t} \\ D_{-1}, D_0 & D_1 & D_2 & D_3, \dots, D_{p-2} & D_{p-1} & D_p & D_{p+1} \end{array}.$$

where $1 \leq i \leq p-3$. The map r_{p-1}^* is injective only except for $H^{*,*'}\{1\}$ (namely injective for $* > n+1$). We see that $\text{Ker}(r_{p-1}^*) \cong \tilde{H}^{*,*'}\{t^{p-1}\}$. Then we have $H^{*,*'}(M^{p-1}) \cong \text{Coker}(r_{p-1}^*) \oplus \tilde{H}^{*,*'}\{t^{p-1}\}$. So we have proved Theorem 3.2.

5. The motivic cohomology of the twisted flag variety

We prepare a lemma.

LEMMA 5.1. *Let $b = Q_{n-1} \cdots Q_1(a't^{i-1})$ in $H^{*,*'}(M^{p-1}; \mathbb{Z}/p)$. Then b can be lifted to the element $H^{*,*'}(M^{p-1}; \mathbb{Z}/p^2)$ with $b = pt^i$.*

Proof. Recall the exact sequence

$$\cdots \leftarrow H^{*,*'}(M^{i-1}; \mathbb{Z}/p) \xleftarrow{r_i^*} H^{*,*'}(\chi; \mathbb{Z}/p) \xleftarrow{\delta_i} H^{*,*'}(M^i; \mathbb{Z}/p) \leftarrow \cdots$$

When $r_i^*(h) = \delta t^{i-1}h = 0$ and $\delta(x) = h$, this element $x \in H^{*,*'}(M^i; \mathbb{Z}/p)$ is written by $t^i h$ in the preceding section.

Since $\delta t^{i-1} = Q_0 \cdots Q_{n-1}(a't^{i-1})$ is the Q_0 -image, we know that

$$r_i^*(p) = p\delta t^{i-1} = 0 \quad \text{in } H^{*,*'}(M^i; \mathbb{Z}/p^2).$$

Hence there is an element $pt^i \in H^*(M^i; \mathbb{Z}/p^2)$, which is a $\mathbb{Z}/p^2$ -module generator (since t^i itself does not exist). So pt^i represents a non-zero element in $H^{*,*'}(M^i; \mathbb{Z}/p)$.

On the other hand, from Theorem 3.2, we see that $H^{2b_{ni}, b_{ni}}(M^i; \mathbb{Z}/p) \cong \mathbb{Z}/p\{b\}$. Hence we can take $pt^i = b$. $\square$

Let G be a Lie group of type (I) and $X = \mathbb{G}_k/B_k$. Using (2.1) and Theorem 3.2 for $n = 2$, we see that $H^{*,*'}(X; \mathbb{Z}/p)$ is additively isomorphic to

$$(H^{*,*'} \oplus (\mathbb{Z}/p\{a', Q_0(a'), Q_1(a')\} \oplus \tilde{H}^{*,*'}\{y\}) \otimes \mathbb{Z}/p[y]/(y^{p-2})) \otimes S(t)/(b).$$

Here we have $Q_0(a'y^{i-1}) = b_{2i-1}$ and $Q_1(a'y^{i-1}) = b_{2i}$ in Theorem 2.3. By the preceding lemma, we can take the i th product y^i in $H^{*,*'}(\tilde{X}; \mathbb{Z}/p)$ in the notation in Theorem 2.3 as the additive generator t^i in $H^{*,*'}(\tilde{M}^{p-1}; \mathbb{Z}/p) \cong \mathbb{Z}/p[t]/(t^p)$.

THEOREM 5.2. *Let k be a field in Theorem 3.2. Let G be a group of type (I), and $X = \mathbb{G}_k/B_k$ be a twisted flag variety. Then $H^{*,*'}(X; \mathbb{Z}/p)$ is isomorphic to the $H^{*,*}'$ -subalgebra generated by*

$$1, a'y^i, \tilde{H}^{*,*'}y^{i+1} \quad (0 \leq i \leq p-2) \quad \text{and} \quad S(t)$$

in $(H^{*,*'}[a', y] \otimes S(t))/(R'_1, R'_2, R'_3)$ where

$$R'_1 = (b_i b_j, b_k \mid 1 \leq i, j \leq 2p-2 < k \leq \ell),$$

$$R'_2 = (K_+^M(k)a', \tilde{H}^{*,*'}b_i, a'b_i \mid 1 \leq i \leq 2p-2), \quad R'_3 = (\tau a' - a, (a')^2, y^p),$$

where $K_+^M(k)$ is the ideal of positive degree parts in $K_*^M(k)$.

Proof. The relation R'_1 holds, since it does so in $CH^*(X)$. Since $K_+^M(k)a = 0$, we have $K_+^M(k)a' = 0$ by using (3.1). Let $b_i = Q_s(a')$ for $s = 0$ or 1 . Then we have

$$K_+^M(k)b_i = K_+^M(k)Q_s(a') = Q_s(K_+^M(k)a') = 0.$$

We also have $\tau b_i = \tau Q_s(a') = Q_s(\tau a') = Q_s(a) = 0$. Since $|a'| = 3$, we have $(a')^2 = 0$ for p odd, and so $b_i a' = Q_s(a')a' = 1/2 Q_s((a')^2) = 0$. Thus we have R'_2 for p odd.

For the last relation in R'_3 , note that y^p itself does not exist in $H^{*,*'}(X; \mathbb{Z}/p)$ but $(\tilde{H}^{*,*'})^2 y^p$ exists. At first, we consider

$$x = \tau y \cdot \tau y^{p-1} = \tau^2 y^p.$$

This element is zero by multiplying τ^N for sufficient large N (i.e. zero in $H_{\text{ét}}^*(X; \mathbb{Z}/p)$). From R'_2 , the kernel $\text{Ker}(\times \tau^N)$ is additively generated by

$$b_i y^j t \quad \text{where } 0 \leq j \leq p-2, t \in S(t).$$

Since $w(x) = 4$ from $w(\tau) = 2$ but $w(b_i y^j t) = 0$, we see that $x = 0$.

Similarly, we see that $(\tilde{H}^{*,*'})^2 y^p = 0$. The relation $(a')^2 = 0$, $b_i a' = 0$ for $p = 2$ is shown similarly from $a^2 = 0$ and the additive structure. $\square$

COROLLARY 5.3. *We have an isomorphism*

$$H_{\text{ét}}^*(X; \mathbb{Z}/p) \cong K_*^M(k)/p \otimes \mathbb{Z}[y]/(y^p) \otimes S(t)/(b).$$

6. The algebraic BP-theory $Ah^{*,*'}(Y) = ABP^{*,*'}(Y; \mathbb{Z}/p)$

In this section, we study the algebraic cobordism for the field k with $K_{n+2}^M(k)/p = 0$. Moreover, we assume that k contains a p^2 th root of unity. Let Y be a smooth algebraic variety over k . We study the BP version of the algebraic cobordism

$$\begin{aligned} ABP^{*,*'}(Y) &= MGL^{*,*'}(Y) \otimes_{MU^*} BP^*, \\ (MGL^{*,*'}(Y)_{(p)} &\cong ABP^{*,*'}(Y) \otimes_{BP^*} MU_{(p)}^*), \\ \Omega^*(Y) &= MGL^{2*,*}(Y) \otimes_{MU^*} BP^*, \end{aligned}$$

defined by Voevodsky and Levine–Morel [Vo1, Le-Mo1, Le-Mo2, Ya1, Ya3], with the coefficient ring $BP^* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$, where $|v_n| = -2(p^n - 1)$ [Ra, Ha].

The Chow ring $CH^*(Y)$ is written as a quotient of $\Omega^*(Y)$ [Vo1, Le-Mo1, Le-Mo2], that is

$$CH^*(Y) \cong \Omega^*(Y) \otimes_{BP^*} \mathbb{Z}_{(p)} \cong \Omega^*(Y)/(BP^{<0}),$$

where $BP^{<0}$ is the ideal $(v_1, v_2, \dots)$ of negative degree parts in BP^* .

Let us write by $ABP^{*,*'}(Y; \mathbb{Z}/p)$ the mod (p) ABP-theory defined by

$$\rightarrow ABP^{*,*'}(Y) \xrightarrow{\times p} ABP^{*,*'}(Y) \rightarrow ABP^{*,*'}(Y; \mathbb{Z}/p) \rightarrow \dots$$

and $ABP^{2*,*}(Y; \mathbb{Z}/p) \cong \Omega^{2*}(Y)/p$. We simply write

$$Ah^{*,*'}(Y) = ABP^{*,*'}(Y; \mathbb{Z}/p).$$

LEMMA 6.1. *If $n + 1 \leq 2p - 1$, then $Ah^{*,*'}(pt.) \cong H^{*,*'} \otimes BP^*$.*

Proof. Consider the Atiyah–Hirzebruch spectral sequence (AHss)

$$E_2^{*,*'},^{*''} \cong H^{*,*'}(pt.; \mathbb{Z}/p) \otimes BP^{*''} \Longrightarrow Ah^{*,*'}(pt.).$$

It is known that the first (possible) non-zero differential is

$$d_{2p-1}(x) = v_1 \otimes Q_1(x).$$

Here $H^{*,*'} = 0$ for $* > n + 1$ from the assumption for k . Recall that $H^{0,i}(pt.; \mathbb{Z}/p)$ is generated by τ^i . When p is odd, Q_i is a derivation, and we see $Q_1(\tau^i) = 0$ since $Q_1(\tau) = 0$. For $p = 2$, we have $Q_1(\tau^2) = \rho^3$ [Ya1, Lemma 6.3]. Since $\rho = -1 = 0 \in K_1^M(k)/2$, we see that $d_3 = 0$. Hence from the assumption of this lemma, the spectral sequence collapses. $\square$

Hereafter in this section we assume that k is a field $k_*^M(k)/p \cong 0$ for $* > n + 1$, $2p - 1 \geq n + 1$, and the primitive p^2 -root ζ_{p^2} is in k . Therefore $Ah^{*,*'}(X)$ is an $H^{*,*'} \otimes BP^*$ -algebra.

Remark. When $p = 2$ and $k = \mathbb{R}$, the algebra $Ah^{*,*'}(pt.) = Ah^{*,*'}(\text{Spec}(\mathbb{R}))$ is quite complicated [Ya1].

For smooth Y_1, Y_2 and $\theta \in Ah^{2d,d}(Y_1 \times Y_2)$ for $d = \dim(Y_2)$, we can define

$$f_\theta : Ah^{*,*'}(Y_1) \rightarrow Ah^{*,*'}(Y_2) \quad \text{by } f_\theta(x) = pr_{2*}(pr_1^*(x) \cup \theta),$$

where $pr_i : Y_1 \times Y_2 \rightarrow Y_i$ is the i th projection. For a projector pj (i.e. $pj = f_\theta$ with $pj \cdot pj = pj$), we can define the motive $M = (Y, pj)$ with $Ah^{*,*'}(M) = pj \cdot Ah^{*,*'}(Y)$. Thus we have the category of Ah -motives [Vi-Ya].

LEMMA 6.2. *For a smooth Y and a motive $M \subset M(Y)$, we have the AHss*

$$E_2^{*,*'},^{*''} \cong H^{*,*'}(M; \mathbb{Z}/p) \otimes BP^* \Longrightarrow Ah^{*,*'}(M).$$

Proof. (See [Ya3, Lemma 7.1].) We only need to see that $d_r f_\theta = f_\theta d_r$ for the differential d_r in the spectral sequence. For $\theta \in CH^*(Y \times Y)/p \subset Ah^{2*,*}(Y \times Y)$, we have

$$f_\theta(d_r(x)) = pr_{2*}(pr_1^*(d_r(x) \cdot \theta) = pr_{2*}(d_r(pr_1^*(x) \cdot \theta)).$$

The last equation follows from $d_r(\theta) = 0$, since $w(\theta) = 0$.

We know (e.g. [Ya3, Section 5] and [Vo2, Vo4])

$$pr_{2*}(x) = i^*(Th_Y(1) \cdot x),$$

where $Th_Y(1) \in H^{2m,m}(Th_Y(V); \mathbb{Z}/p)$ is the Thom class for some bundle V over Y and $i : \mathbb{T}^m \times Y \subset Th_Y(V) \times Y$. Since $w(Th_Y(1)) = 0$, we see that $d_r(Th_Y(1) \cdot x) = Th_Y(1) \cdot d_r(x)$. Therefore pr_{2*} commutes with d_r . Hence so does f_θ . $\square$

For the Rost motive R_n , we study the AHss

$$E_2^{*,*'},^{*''} \cong H^{*,*'}(R_n; \mathbb{Z}/p) \otimes BP^{*''} \Longrightarrow Ah^{*,*'}(R_n). \quad (6.1)$$

From Theorem 3.2, we still have

$$H^{*,*'}(R_n; \mathbb{Z}/p) \cong H^{*,*'} \oplus (\tilde{Q}(n-1)\{a'\} \oplus \tilde{H}^{*,*'}\{t\})\{1, t, \dots, t^{p-2}\}.$$

LEMMA 6.3. *The non-zero differential d_r in the AHss (6.1) has the form*

$$d_{2p^i-1}(x) = v_i \otimes Q_i(x) \pmod{I_i} \quad \text{with } I_i = (v_1, \dots, v_{i-1}).$$

The proof of this lemma will be given just before Corollary 6.8 below.

LEMMA 6.4. *The infinity term of the AHss (6.1) is written as*

$$E_{\infty}^{*,*,*'} \cong H^{*,*,*'} \otimes BP^* \oplus (A \oplus \tilde{H}^{*,*,*'} \otimes BP^*\{t\})\{1, t, \dots, t^{p-2}\},$$

where $A = BP^*/I_n\{c_0\} \oplus \bigoplus_{i=1}^{n-1} BP^*/I_i\{c_i\}$ for $c_i = Q_0 \cdots \hat{Q}_i \cdots Q_{n-1}(a')$.

Proof. Let us write $Q(i)' = \Lambda(Q_i, \dots, Q_1)$. For $x = Q_{i_1} \cdots Q_{i_s} \in \tilde{Q}(n-1)$, considering that Q_0 is contained in x or not, we have the decomposition

$$\tilde{Q}(n-1) \cong Q(n-1)' \oplus \tilde{Q}(n-1)'Q_0.$$

Considering Q_{n-1} is contained in x or not for $x \in \tilde{Q}(n-1)'$, we have

$$\tilde{Q}(n-1)'Q_0 \cong Q(n-2)'Q_0 \oplus \tilde{Q}(n-2)'Q_{n-1}Q_0.$$

Continuing this argument, we can see that

$$\tilde{Q}(n-1) \cong Q(n-1)' \oplus \bigoplus_{i=1}^{n-1} Q(i-1)'Q_{n-1} \cdots Q_{i+1}Q_0.$$

For a $BP^* \otimes Q(n-1)'$ -module B , let us write $E_0(B) = B$ and $E_{r+1}(B) = H(E_r(B); v_r \otimes Q_r)$. Then we easily see by induction on j that

$$E_j(BP^* \otimes Q(i)') \cong \begin{cases} BP^*/I_{j+1} \otimes \Lambda(Q_i, \dots, Q_{j+1})\{Q_j \cdots Q_1\} & j < i, \\ BP^*/I_{i+1}\{Q_i \cdots Q_1\} & \text{otherwise.} \end{cases}$$

Hence we see that

$$\begin{aligned} E_{n-1}(BP^* \otimes Q(i-1)'Q_{n-1} \cdots Q_{i+1}Q_0(a')) \\ \cong BP^*/I_i\{Q_{n-1} \cdots Q_{i+1}Q_{i-1} \cdots Q_1Q_0(a')\} \cong BP^*/I_i\{c_i\}. \end{aligned}$$

Therefore we get

$$E_{n-1}(BP^* \otimes \tilde{Q}(n-1)'(a')) \cong A = BP^*/I_n\{c_0\} \oplus \bigoplus_{i=1}^{n-1} BP^*/I_i\{c_i\}.$$

From Lemma 6.3, all differentials are of the form $d_{2p^i-1} = v_i \otimes Q_i$ (in particular, $d_r(BP^* \otimes \tilde{H}^{*,*,*'}y^j) = 0$), and thus we have this lemma. $\square$

We recall the following lemma to see the relations between c_i in $Ah^{*,*,*'}(R_n)$.

LEMMA 6.5. [Ya3, Corollary 3.4] *Let $x \in CH^*(Y) \cong E_{\infty}^{2*,*,0}$ and $v_s x = 0$ in $E_{\infty}^{2*,*,*}$ for the AHss converging to $ABP^{*,*,*'}(Y)$. Then there is $z \in H^{*,*,*'}(Y; \mathbb{Z}/p)$ with $Q_s(z) = x$ and in $ABP^{2*,*,*}(Y)$ there is a relation*

$$v_s x + v_{s+1}x_{s+1} + \cdots + v_k x_k + \cdots = 0 \pmod{I_{\infty}^2}$$

with $x_k = Q_k(z)$ in $H^{2*,*,*}(Y; \mathbb{Z}/p)$ for all $k \geq s$.

COROLLARY 6.6. *In $ABP^{2*,*}(R_n)$, we have $v_i c_j = v_j c_i \bmod(I_\infty^2)$. The restriction image is $\text{Res}(c_i) = v_i y \bmod(I_\infty^2)$ in $ABP^{2*,*}(\bar{R}_n)$.*

Proof. First note that c_i exists in (the integral) $ABP^{2*,*}(R_n)$, since c_i exists in $H^{2*,*}(R_n)$ (considering the integral AHss). Let $i > 0$. Since $pc_i = 0$ in the spectral sequence there is $x \in H^{*,*'}(R_2)$ such that $Q_0 x = c_i$ and there is a relation

$$pc_i + v_1 Q_1(x) + \cdots + v_i Q_i(x) = 0 \bmod(I_\infty^2).$$

By dimensional reasoning, we see that $x = Q_{n-1} \cdots \hat{Q}_i \cdots Q_1(a')$. Hence the above equation is written as

$$pc_i - v_i c_0 = 0 \bmod(I_\infty^2)$$

since $Q_i x = Q_1 \cdots Q_{n-1}(a') = c_0$.

From Lemma 5.1, we know that $\text{Res}(c_0) = py \bmod(p^2, BP^{<0})$. So $\text{Res}(c_i) = v_i y$ since $ABP^{*,*'}(\bar{R}_n)$ is BP^* -free. The first relation is proved similarly by using $v_j c_i = 0$ for $j < i$ in the spectral sequence. $\square$

THEOREM 6.7. *We have an isomorphism*

$$Ah^{*,*'}(R_n) \cong H^{*,*'} \otimes BP^* \oplus (A' \oplus BP^* \otimes \tilde{H}^{*,*'}\{t\})\{1, t, \dots, t^{p-2}\},$$

where $A' = BP^*/I_n\{c_0\} \oplus I_n\{t\}$.

Proof. We know the isomorphism [Ya3, Theorem 10.6]

$$gr(\text{Res}) : \bigoplus_{i=1}^{n-1} BP^*/I_i\{c_i\} \rightarrow \bigoplus_{i=1}^{n-1} BP^*/I_i\{v_i t\} \cong gr(I_n t).$$

Therefore, we get the result from Lemma 6.4 and Corollary 6.6. $\square$

Proof of Lemma 6.3. For an element $x = Q_i \cdots Q_1 Q_0(a')$ in E_{2p^i} , the next non-zero differential is d_{2p^i+1-1} . Otherwise

$$d_r(x) = v \otimes x' \neq 0 \quad |v_i| > |v| > |v_{i+1}|.$$

But by dimensional reasoning, there does not exist non-zero x such that $|x| < |x'| < |x| + |Q_{i+1}|$.

The elements $\tilde{H}^{*,*'}(t^j)$ $j \geq 1$ are generated as a $BP^* \otimes H^{*,*}'$ -algebra by τt^j and bt^j for $b \in K_1^M(k)/p$. Since $w(bt^j) = 1$, the differential image $w(d_r(bt^j)) = 0$. Hence $d_r(bt^j) \in \bigoplus_{i,k} BP^*/I_i\{c_i t^k\}$. But if $d_r(bt^j) \neq 0$, then it contradicts the existence of the restriction map $gr(\text{Res})$ in the proof of Theorem 6.7.

At last we consider τt^j , $j \geq 1$. Since $\tau c_1 t^{j-1} = 0$ in the spectral sequence, $\tau c_1 t^{j-1} = v c$ for $v \in BP^{<0}$ and some c in $Ah^{*,*'}(R_n)$. Restricting it to $Ah^{*,*'}(\bar{R}_n)$, we have $\tau v_1 t^j = v(c|_{\bar{k}})$. Since $Ah^{*,*'}(\bar{R}_n)$ is $BP^*/p[\tau]$ -free, we see that $v = v_1$ and $c|_{\bar{k}} = \tau t^j$. This means that τt^j exists in $Ah^{*,*'}(R_n)$, and it is a permanent cycle. $\square$

PROPOSITION 6.8. *Let G be a group of type (I). Let $X = \mathbb{G}_k/B_k$ be a twisted flag variety. Then there is a filtration such that $gr Ah^{*,*'}(X)$ is isomorphic to the $BP^* \otimes H^{*,*}'$ -subalgebra generated by*

$$1, \quad \tilde{H}^{*,*'} y^{i+1} \quad (0 \leq i \leq p-2) \quad \text{and} \quad S(t)$$

in $BP^* \otimes H^{*,*'}[y]/(y^p) \otimes S(t)/(R_1'', R_2'')$, where

$$R_1'' = (b_i b_j, b_k \mid 1 \leq i, j \leq 2p-2 < k \leq \ell),$$

$$R_2'' = (\tilde{H}^{*,*'} b_{2i-1}, \tilde{H}^{*,*'} b_{2i}, v_1 b_{2i} \mid 1 \leq i \leq p-1).$$

Note that $\tau b_{2i-1} = 0$ in the AHss, but $\tau b_{2i-1} = v_1 \tau y$ in $Ah^{*,*'}(X)$.

Define the étale cobordism theory by $Ah_{\text{ét}}^*(Y) = \lim_N (\tau^N Ah^{*,*'}(Y))$.

COROLLARY 6.9. *We have the isomorphism*

$$gr Ah_{\text{ét}}^*(X) \cong K_*^M(k)/p \otimes BP^*[y]/(y^p) \otimes S(t)/(b).$$

7. The truncated BP-theory $Ak^{*,*'}(Y) = ABP\langle n-1 \rangle^{*,*'}(Y; \mathbb{Z}/p)$

In algebraic topology [Ra], $BP\langle m \rangle^*(Y)$ is the cohomology theory with the coefficient ring $BP\langle m \rangle^* = \mathbb{Z}_{(p)}[v_1, \dots, v_m]$. The $BP\langle n-1 \rangle$ -version of Theorem 6.7 is a quite simple form (Theorem 7.1 below), and we try to give its direct proof in this section.

Throughout this section, we assume that

$$K_{n+2}^M(k)/p = 0, \quad Q_0(\tau) = \zeta_p = 0 \in K_1^M(k)/p, \quad n+1 \leq 2p-1,$$

as in the preceding section.

We consider its mod(p) algebraic geometric version [Ya1]. Let us write

$$Ak^{*,*'}(Y) = ABP\langle n-1 \rangle^{*,*'}(Y; \mathbb{Z}/p),$$

so that we have

$$Ak^{*,*'} = Ak^{*,*'}(pt.) \cong BP\langle n-1 \rangle \otimes H^{*,*'} \cong \mathbb{Z}/p[v_1, \dots, v_{n-1}] \otimes H^{*,*'}$$

and

$$\tilde{A}k^{*,*'} = Ak^{*,*'} / (\mathbb{Z}/p\{1\}) = \bigoplus_{(*,*) \neq (0,0)} Ak^{*,*'}.$$

We note that

$$I_n \cong \tilde{k}^* = BP\langle n-1 \rangle^{<0}/p, \quad \text{with } \tilde{k}^* = k^*/(\mathbb{Z}/p\{1\}).$$

Considering the $BP\langle n-1 \rangle$ -version of the AHss, we can show the following theorem.

THEOREM 7.1. *We have $Ak^{*,*'}(R_n) \cong Ah^{*,*'}(R_n) \otimes_{BP^*} k^*$, that is*

$$Ak^{*,*'}(R_n) \cong Ak^{*,*'} \oplus (\mathbb{Z}/p\{c_0\} \oplus \tilde{A}k^{*,*'}\{t\})\{1, \dots, t^{p-2}\}.$$

Hence we have a map $f : Ak^{*,*'}(R_n) \rightarrow Ak^{*,*'}[t]/(t^p)$ such that

$$\text{Ker}(f) \cong Ak^{*,*'} \oplus \mathbb{Z}/p[t]/(t^{p-1})\{c_0\}, \quad \text{Coker}(f) \cong \mathbb{Z}/p[t]/(t^{p-1})\{t\}.$$

In this section, we will give another proof for the above theorem, but assuming the existence of some properties for Ak -motives. For ease of notation, hereafter in this section, we simply write

$$c = c_0, \quad \delta = \delta_a, \quad \xi' = Q_n(c_0), \quad \xi = \xi_a, \quad \chi = \chi_a, \quad M = M_a, \quad M^i = M_a^i.$$

LEMMA 7.2. We have $Ak^{*,*'}(\chi) \cong Ak^{*,*'} \oplus Ak^{*,*'}(\tilde{\chi})$ and

$$Ak^{*,*'}(\tilde{\chi}) \cong \mathbb{Z}/p[\xi]\{c, \delta, \xi', \xi\},$$

where $c = Q_{n-1} \cdots Q_1(a')$, $\delta = Q_0c$ and $\xi' = Q_n(c)$, $Q_0(\xi') = \xi$.

Proof. We consider the AHss

$$E_2^{*,*',*''} \cong H^{*,*'}(\tilde{\chi}; \mathbb{Z}/p) \otimes BP\langle n-1 \rangle^{*''} \implies Ak^{*,*'}(\tilde{\chi}).$$

Here $H^{*,*'}(\tilde{\chi}; \mathbb{Z}/p) \cong Q(n)\{a'\} \otimes \mathbb{Z}/p[\xi]$ from Theorem 3.1.

We consider the decomposition

$$Q(n) \cong \Lambda(Q_0, Q_n) \otimes Q(n-1)' \quad \text{with } Q(n-1)' = \Lambda(Q_{n-1}, \dots, Q_1).$$

We easily see that all differentials are of the form $d_{2p^r-1}(x) = v_r \otimes Q_r(x)$. Hence we have

$$\begin{aligned} E_{2p^{n-1}-1}^{*,*',*''} &\cong \Lambda(Q_0, Q_n) \otimes BP\langle n-1 \rangle^*/I_n\{Q_{n-1} \cdots Q_1(a')\} \otimes \mathbb{Z}/p[\xi] \\ &\cong \Lambda(Q_0, Q_n)\{c\} \otimes \mathbb{Z}/p[\xi] \cong \mathbb{Z}/p\{c, Q_0c, Q_n(c), Q_0Q_n(c)\} \otimes \mathbb{Z}/p[\xi]. \end{aligned}$$

Here we used

$$BP\langle n-1 \rangle^*/I_n \cong \mathbb{Z}/p[v_1, \dots, v_{n-1}]/(v_1, \dots, v_{n-1}) \cong \mathbb{Z}/p.$$

This term is isomorphic to the infinity term, since $E_{2p^{n-1}-1}^{*,*',*''} = 0$ for $*'' \neq 0$. □

Elements $1, c, \delta, \xi', \xi$ in $Ak^{*,*'}(\chi)$ are contained in D_i as follows:

$$\begin{array}{ccccccc} \xrightarrow{D_{-1}} & \xrightarrow{D_0} & \xrightarrow{D_1, \dots} & \xrightarrow{D_{p-2}} & \xrightarrow{D_{p-1}} & \xrightarrow{D_p} & \\ 1 & c, \delta & 0, \dots & 0 & \xi', \xi & c\xi, \delta\xi & \end{array} .$$

Recall the category Ah -motives in the preceding section. More strongly, we assume the following.

ASSUMPTION 7.3. There is a triangular category $\mathbb{DM}(Ak)$ which contains the category of Ak -motives having the following properties. For $X \in \mathbb{DM}(Ak)$, we can define $Ak^{*,*'}(X)$, and for a cofiber $X \rightarrow Y \rightarrow Z$ in $\mathbb{DM}(Ak)$, we have the induced exact sequence

$$\cdots \rightarrow Ak^{*,*'}(Z) \rightarrow Ak^{*,*'}(Y) \rightarrow Ak^{*,*'}(X) \rightarrow Ak^{*+1,*'}(Z) \rightarrow \cdots .$$

Moreover, χ is an object of $\mathbb{DM}(Ak)$ and

$$\text{Hom}_{\mathbb{DM}(Ak)}(M^i, \chi(*')[*]) \cong Ak^{*,*'}(M^i).$$

From the assumption we can define the map $\delta : \chi \rightarrow \chi(*')[*]$ in the category $\mathbb{DM}(Ak)$. From Lemma 4.1, we see that $\delta Q_n(a') = a'\xi$, which induces $\delta\xi' = c\xi$.

The map $\times\delta : Ak^{*,*'}(\chi) \rightarrow Ak^{*,*'}(\chi)$ is written as

$$\begin{array}{ccccccc} \xrightarrow{D_{-1}} & \xrightarrow{D_0} & \xrightarrow{D_1, \dots, D_{p-3}} & \xrightarrow{D_{p-2}} & \xrightarrow{D_{p-1}} & \xrightarrow{D_p} & \\ 1 & c, \delta & 0, \dots, 0 & 0 & \xi', \xi & c\xi, \delta\xi & \\ \delta \downarrow & \delta \downarrow & \delta \downarrow & \delta \downarrow & \delta \downarrow & \delta \downarrow & \\ \xrightarrow{D_{-1}} & \xrightarrow{D_0} & \xrightarrow{D_1} & \xrightarrow{D_2, \dots, D_{p-2}} & \xrightarrow{D_{p-1}} & \xrightarrow{D_p} & \xrightarrow{D_{p+1}} \\ 1 & c, \delta & 0, 0 & 0, \dots, 0 & \xi', \xi & c\xi, \delta\xi & 0, 0 \end{array} .$$

We can easily see that

$$\text{Coker}(\delta) \cong Ak^{*,*'} \oplus \mathbb{Z}/p\{c\} \oplus \mathbb{Z}/p[\xi]\{\xi', \xi\},$$

and

$$\text{Ker}(\delta)\{t\} \cong \tilde{A}k^{*,*'}\{t\} \oplus \mathbb{Z}/p[\xi]\{ct, \delta t\}.$$

LEMMA 7.4. *We have the isomorphism*

$$Ak^{*,*'}(M) \cong Ak^{*,*'} \oplus \mathbb{Z}/p\{c\} \oplus \tilde{A}k^{*,*'}\{t\} \oplus \mathbb{Z}/p\{ct, \delta t, \xi', \xi\} \otimes \mathbb{Z}/p[\xi].$$

Next, we compute $Ak^{*,*}(M^i)$ for $1 < i < p-1$. As in Lemma 4.5, we can compute the next result.

LEMMA 7.5. *We have the isomorphism*

$$\begin{aligned} Ak^{*,*'}(M^{p-2}) &\cong Ak^{*,*'} \oplus (\mathbb{Z}/p\{c\} \oplus \tilde{A}k^{*,*'}\{t\})\{1, t, \dots, t^{p-3}\} \\ &\oplus (\mathbb{Z}/p\{ct^{p-2}, \delta t^{p-2}, \xi', \xi\}) \otimes \mathbb{Z}/p[\xi]. \end{aligned}$$

At last we compute $Ak^{*,*'}(M^{p-1})$. From Lemma 4.5, the map r_{p-1}^* is given in $H^{*,*'}(M^{p-2}; \mathbb{Z}/p)$ by

$$1 \mapsto \delta t^{p-2}, \quad a' \mapsto Q_n(a'), \quad Q_n(a') \mapsto a' \xi t^{p-2}.$$

Hence, in $Ak^{*,*'}(M^{p-2})$, we have $1 \mapsto \delta t^{p-2}$, $c \mapsto \xi'$, $\delta \mapsto \xi$, $\xi' \mapsto c \xi t^{p-2}$ and $\xi \mapsto \delta t^{p-2} \xi$. The map r_{p-1}^* is given as follows:

$$\begin{array}{ccccccccc} & & \xrightarrow{D_{-1}} & \xrightarrow{D_0} & \xrightarrow{D_{i+1}} & \xrightarrow{D_{p-1}} & \xrightarrow{D_p} & & \\ & & Ak^{*,*'}\{1\} & c, \delta & 0, 0 & \xi', \xi & c\xi, \delta\xi & & \\ & r_{p-1}^* \downarrow & r_{p-1}^* \downarrow & r_{p-1}^* \downarrow & r_{p-1}^* \downarrow & r_{p-1}^* \downarrow & & & \\ Ak^{*,*'}, ct^i, \tilde{A}k^{*,*'} t^{i+1} & \xrightarrow{ct^{p-2}, \delta t^{p-2}} & \xrightarrow{\xi', \xi} & \xrightarrow{0, 0} & \xrightarrow{c\xi t^{p-2}, \delta\xi t^{p-2}} & \xrightarrow{\xi' \xi, \xi^2} & & & \\ D_1, \dots, D_i, \dots & D_{p-2} & D_{p-1} & D_{p+i} & D_{2p-2} & D_{2p-1} & & & \end{array}$$

Note that $\text{Ker}(r_{p-1}^*)\{t\} \cong \tilde{A}k^{*,*'} t^{p-1}$. Using these, we see Theorem 7.1 with Assumption 7.2 (while Theorem 7.1 itself is true without Assumption 7.2).

Remark. When $Ak^{*,*'}(X) = ABP\langle n \rangle^{*,*'}(X)$, we see that δ_a does not exist in $Ak^{*,*'}(\chi)$. Hence we cannot make the above arguments.

8. Motivic cohomology over $\mathbb{R}$

When $p = 2$, important cases have the property with $K_*^M(k)/2 \neq 0$ for all $* \geq 0$. The field $\mathbb{R}$ of real numbers is such a case. Let $\rho = -1 \in K_1^M(\mathbb{R})/2 \cong \mathbb{R}^\times/(\mathbb{R}^\times)^2$. Then it is well known that

$$K_*^M(\mathbb{R})/2 \cong H_{\text{et}}^*(pt.; \mathbb{Z}/2) \cong \mathbb{Z}/2[\rho] \quad \text{where } pt. = \text{Spec}(\mathbb{R}).$$

Hence the motivic cohomology is given for $\tau \in H^{0,1}(pt., \mathbb{Z}/2)$:

$$H^{*,*'}(pt.; \mathbb{Z}/2) \cong K_*^M(\mathbb{R}) \otimes \mathbb{Z}/2[\tau] \cong \mathbb{Z}/2[\rho, \tau].$$

Let $a = \rho^{n+1} \in K_{n+1}^M(\mathbb{R})/2$. From Theorem 3.1, we have

$$H^*(\chi_a; \mathbb{Z}/2) \cong \mathbb{Z}/2[\rho] \otimes (\mathbb{Z}/2[\tau] \oplus Q(n)(a') \otimes \mathbb{Z}/2[\xi]), \quad (8.1)$$

where $\xi = \delta_a^2$. Note that $H_{\text{ét}}^*(\chi_a; \mathbb{Z}/2) \cong H_{\text{ét}}^*(pt.; \mathbb{Z}/2) \cong \mathbb{Z}/2[\rho]$.

LEMMA 8.1. *We have $\delta a' = Q_n(a')$ and $\delta Q_n(a') = a'\xi$.*

Proof. When $\deg(Q_n(a')) = (*, *)$, we see that $H^{*,*'}(R_n; \mathbb{Z}/2) \cong \mathbb{Z}/2$ from (8.1). The first equation follows from the fact that

$$Q_{n-1} \cdots Q_0(a'\delta) = \delta^2 = \xi = Q_{n-1} \cdots Q_0(Q_n(a')).$$

Then we have $\delta Q_n(a') = \delta^2 a' = \xi a'$. □

The map $\times \delta : H^{*,*'}(\chi_a; \mathbb{Z}/2) \rightarrow H^{*,*'}(\chi_a; \mathbb{Z}/2)$ is written as

$$\begin{array}{ccccccc} & & \xrightarrow{D_{-1}} & \xrightarrow{D_0} & \xrightarrow{D_1} & \xrightarrow{D_2} & \\ & & 1 & a', \delta & Q_n(a') \xi & a' \xi, \delta \xi & \\ \delta \downarrow & & \delta \downarrow & & \delta \downarrow & & \delta \downarrow \\ \xrightarrow{1} & \xrightarrow{a', \delta} & \xrightarrow{Q_n(a'), \xi} & \xrightarrow{a' \xi, \delta \xi} & \xrightarrow{Q_n(a') \xi, \xi^2} & & \\ D_{-1} & D_0 & D_1 & D_2 & D_3 & & \end{array}$$

From (3.2), we can compute $H^{*,*'}(M_a; \mathbb{Z}/2) \cong H^{*,*'}(R_n; \mathbb{Z}/2)$. In particular, it contains $Q(n-1)\{a'\}$. In fact, the following is known.

THEOREM 8.2. [Ya2, Theorem 1.3] *The motivic cohomology $H^{*,*'}(R_n; \mathbb{Z}/2)$ is isomorphic to the $\mathbb{Z}/2[\rho, \tau]$ -subalgebra of $\mathbb{Z}/2[\rho, \tau, \tau^{-1}]/(\rho^{2^{n+1}-1})$ generated by*

$$1 \quad \text{and} \quad Q^\epsilon(a') = \tau^{d(\epsilon)-1} \rho^{f(\epsilon)+n+1}.$$

Here $Q^\epsilon = Q_0^{\epsilon_0} \cdots Q_{n-1}^{\epsilon_{n-1}}$ for $\epsilon = (\epsilon_0, \dots, \epsilon_{n-1})$ with $\epsilon_i = 0$ or 1 , and $d(\epsilon) = \sum_i \epsilon_i 2^i$ and $f(\epsilon) = \sum_i \epsilon_i (2^{i+1} - 1)$.

COROLLARY 8.3. *We have the $\mathbb{Z}/2[\rho, \tau]$ -module isomorphism*

$$H^{*,*'}(R_2; \mathbb{Z}/2) \cong (\mathbb{Z}/2[\tau] \oplus \mathbb{Z}/2\{a', Q_0 a', Q_1 a'\}) \otimes \mathbb{Z}/2[\rho]/(\rho^7)$$

in $\mathbb{Z}/2[\rho, \tau, \tau^{-1}]/(\rho^7)$ identifying $a' = \tau^{-1} \rho^3$, $Q_0 a' = \tau^{-2} \rho^4$ and $Q_1 a' = \tau^{-3} \rho^6$.

COROLLARY 8.4. *We have a $\mathbb{Z}/2[\tau]$ -module isomorphism*

$$H^{*,*'}(R_2; \mathbb{Z}/2) \cong \mathbb{Z}/2[\tau]\{1, \rho, \rho^2, a', Q_0(a'), \rho Q_0(a'), Q_1(a')\}.$$

THEOREM 8.5. *Let $X = \mathbb{G}_k/B_k$ for G of type (I) for $p = 2$. Then for $* \geq *'$ we have the isomorphism*

$$H^{*,*'}(X; \mathbb{Z}/2) \cong \mathbb{Z}/2[\rho] \otimes S(t)\{1, a'\}/(R_1, R_2, R_3),$$

where $R_1 = (b_i b_j, b_k \mid 1 \leq i, j \leq 2 < k < \ell)$, $R_2 = (\rho^7, \rho^3 b_1, \rho b_2)$ and

$$R_3 = ((a')^2 - \rho^2 b_1, \rho^4 a', b_s a' \mid 1 \leq s \leq \ell).$$

Proof. Note that the inclusion $H^{*,*'}(R_2; \mathbb{Z}/2) \subset H^{*,*'}(X; \mathbb{Z}/2)$ is an $H^{*,*'}(pt.; \mathbb{Z}/2)$ -module map. Hence $\rho^7 = 0$ also in $H^{*,*'}(X; \mathbb{Z}/2)$.

In $H^{*,*'}(X; \mathbb{Z}/2)$, the relation R_1 holds since it does so in $CH^*(X)/2 \cong H^{2*,*}(X; \mathbb{Z}/2)$. By dimensional reasoning, we can take $b_1 = Q_0 a' + \lambda t$ for $0 \neq t \in S(t)/(b)$ and $\lambda \in \mathbb{Z}$. Since $b_1|_{\mathbb{C}} = Q_0 a'|_{\mathbb{C}} = 0$ but $t|_{\mathbb{C}} \neq 0$, we have $\lambda = 0 \in \mathbb{Z}/2$. Similarly we can take $Q_1 a' = b_2$. Hence R_2 also holds in $H^{*,*'}(X; \mathbb{Z}/2)$.

We can take a' satisfying R_3 from $H^{*,*'}(R_2; \mathbb{Z}/2)$. Hence we have the $\mathbb{Z}/2[\rho]$ -algebra map

$$A = \mathbb{Z}/2[\rho] \otimes S(t)/(R_1, R_2, R_3) \rightarrow H^{*,*'}(X; \mathbb{Z}/2).$$

When $* \geq *'$, this map is additively isomorphic from the Petrov–Semenov–Zainoulline theorem (equation (2.1) in Section 2). So it is isomorphic as a $\mathbb{Z}/2[\rho]$ -algebra map. $\square$

COROLLARY 8.6. *We have the $\mathbb{Z}/2[\rho]$ -algebra isomorphism*

$$H_{\text{ét}}^*(X; \mathbb{Z}/2) \cong \mathbb{Z}/2[\rho] \otimes S(t)/(R_1, R'_2),$$

where $R'_2 = (\rho^7, \rho^4 = b_1, \rho^6 = b_2)$.

For the étale cohomology of a smooth Y over k , we consider the (Borel) spectral sequence such that

$$E_2^{*,*'} \cong H^*(\text{Gal}(\bar{k}/k); H_{\text{ét}}^{*'}(\bar{Y}; \mathbb{Z}/p)) \implies H_{\text{ét}}^*(Y; \mathbb{Z}/p).$$

We study the case $Y = X$ and the Galois group $\text{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2$:

$$E_2^{*,*'} \cong H^*(B\mathbb{Z}/2; H_{\text{ét}}^{*'}(G_k/B_k; \mathbb{Z}/2)) \implies H_{\text{ét}}^*(X; \mathbb{Z}/2).$$

Here $H^*(B\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2[\rho]$ and

$$H_{\text{ét}}^*(G_k/B_k; \mathbb{Z}/2) \cong H^*(G/T; \mathbb{Z}/2) \cong \mathbb{Z}/2[y]/(y^2) \otimes S(t)/(b).$$

The element y is not a permanent cycle of this spectral sequence. We can see that $d_7(y) = \rho^7$, since $|y| = 6$, $|\rho| = 1$ and

$$E_8^{*,*'} \cong \mathbb{Z}/2[\rho]/(\rho^7) \otimes S(t)/(b).$$

This term becomes an E_{∞} -term and $gr H_{\text{ét}}^*(X; \mathbb{Z}/2) \cong E_8^{*,*'}.$ This means that $b_i = 0 \in E_{\infty}^{*,*}'$ but $b_1 = \rho^4 \neq 0$ and $b_2 = \rho^6 \neq 0$ in $H_{\text{ét}}^*(X; \mathbb{Z}/2)$. Hence Corollary 8.6 states stronger results than those using only the Borel spectral sequence.

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