REMARKS ON DIMENSION OF HOMOLOGY SPHERES WITH ODD NUMBERS OF FIXED POINTS OF FINITE GROUP ACTIONS

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Abstract. For each positive integer m , an arbitrary finite non-solvable group acts smoothly on infinitely many standard spheres with exactly m fixed points. However, for a given finite non-solvable group G and a given positive integer m , all standard spheres do not admit smooth actions of G with exactly m fixed points. In this paper, for each of the alternating group A_6 on six letters, the symmetric group S_6 on six letters, the projective general linear group $\mathrm{PGL}(2, 9)$ of order 720, the Mathieu group M_{10} of order 720, the automorphism group $\mathrm{Aut}(A_6)$ of A_6 and the special linear group $\mathrm{SL}(2, 9)$ of order 720, we will give the dimensions of homology spheres whose fixed point sets of smooth actions of the group do not consist of odd numbers of points.

1. Introduction

Let G be a finite group and M a smooth manifold. A *smooth action* of G on M is a group homomorphism Φ from G to the diffeomorphism group of M . Throughout this paper, all manifolds and all actions of finite groups on manifolds will be assumed smooth. For a manifold M with G -action and a subgroup H of G , let M^H denote the H -fixed point set of M : the set of all points $x \in M$ such that $\Phi(h)(x) = x$ for all $h \in H$. The H -fixed point set M^H is a submanifold of M with $N_G(H)/H$ -action, where $N_G(H)$ is the normalizer of H in G . An action of G on M is called a *one-fixed-point action* if M^G consists of exactly one point. More generally, if M^G consists of an odd number of points, we call such an action G on M an *odd-fixed-point action*.

For a commutative ring R with unity, an *R -homology sphere* is a closed manifold having the same homology groups with coefficients in R as a sphere. In this paper, a $\mathbb{Z}$ -homology sphere will be simply called a homology sphere. A finite group G is called a *mod- $\mathcal{P}$ hyper-elementary group* if there exists a normal sequence $P \triangleleft H \triangleleft G$ such that P and G/H are of prime power order, and H/P is cyclic. It is easy to see that spheres do not admit one-fixed-point actions of mod- $\mathcal{P}$ hyper-elementary groups. The converse was proven by Laitinen and Morimoto [10]: Every finite Oliver group G , i.e. G is not a mod- $\mathcal{P}$ hyper-elementary group, acts on some spheres with exactly one fixed point. Thus, the answer to the determination of finite groups acting on spheres with exactly one fixed point is already known. Furthermore,

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the problem of the dimension of spheres admitting one-fixed-point actions of finite groups has also been completed. Furuta [9], Demichelis [8] and Buchdahl, Kwasik and Schultz [6] showed that, if there exists a one-fixed-point action on a finite group on a sphere S , then the dimension of S is greater than or equal to six. Conversely, the n -dimensional sphere S^n , $n \geq 6$, has one-fixed-point actions of the alternating group A_5 , which was proven by Morimoto [11–13] for $n = 6$ and $n \geq 9$, and by Bak and Morimoto [1, 2] for $n = 7, 8$.

In addition to the above results, we will describe some results on one- or odd-fixed-point actions of finite groups on spheres. Stein [19] constructed an effective one-fixed-point action of the special linear group $SL(2, 5)$ on S^7 . On the other hand, Borowiecka [3] proved that $SL(2, 5)$ does not act effectively on any eight-dimensional homology sphere with exactly one fixed point. Recently, Borowiecka and Mizerka [4] gave many pairs (G, S) , where G is a finite Oliver group with $|G| \leq 216$ and S is a sphere with $\dim S \leq 10$, such that G does not act on S with exactly one fixed point, and Morimoto and Tamura [15] showed that, if there exists an effective odd-fixed-point action of the symmetric group S_5 (respectively $SL(2, 5)$) on a $\mathbb{Z}$ (respectively $\mathbb{Z}_2$)-homology sphere Σ , then $\dim \Sigma$ does not lie in $\{0, 1, 2, 3, 4, 5, 7, 8, 9, 13\}$ (respectively $\{0, 1, 2, 3, 4, 5, 6, 8, 9\}$). In this paper, we denote by $[a..b]$ the set of integers m such that $a \leq m \leq b$, by $\mathbb{N}$ the set of positive integers and by $\mathbb{Z}_{\geq 0}$ the set of non-negative integers, i.e. $\mathbb{Z}_{\geq 0} = \mathbb{N} \cup \{0\}$. Our main results are the following.

THEOREM 1.1. *Let Σ be a $\mathbb{Z}_2$ -homology sphere. There are no effective odd-fixed-point actions of A_6 and $SL(2, 9)$ on Σ if the dimension of Σ lies in*

$$T_{A_6} = [0..7] \cup [9..12] \cup \{14, 15\} \cup \{19, 20\}$$

and

$$T_{SL(2,9)} = [0..15] \cup [17..20] \cup \{22, 23\} \cup \{27\},$$

respectively.

THEOREM 1.2. *Let Σ be a homology sphere. There are no odd-fixed-point actions of S_6 , $PGL(2, 9)$, M_{10} and $\text{Aut}(A_6)$ on Σ if the dimension of Σ lies in*

$$T_{S_6} = [0..15] \cup [17..20] \cup [22..24] \cup [27..29] \cup \{33\} \cup \{38\},$$

$$T_{PGL(2,9)} = [0..7] \cup [9..15] \cup [19..23] \cup [29..31] \cup \{39\},$$

$$T_{M_{10}} = [0..15] \cup [17..24] \cup [27..31] \cup \{33\} \cup [37..40] \cup \{47\} \cup \{49\}$$

and

$$T_{\text{Aut}(A_6)} = [0..15] \cup [17..24] \cup [27..31] \cup \{33\} \cup [37..40] \cup \{47\} \cup \{49\},$$

respectively.

Here, A_6 and S_6 denote the alternating and the symmetric groups on six letters, respectively, and $SL(2, 9)$ and $PGL(2, 9)$ denote the special linear and the projective general linear groups of degree two over the finite field $\mathbb{F}_9$ with nine elements, respectively. And M_{10} denotes the Mathieu group of degree 10: the definition in this paper is the subgroup of the symmetric group S_{10} defined by

$$\langle g, h \in S_{10} \mid g^2 = h^8 = (gh^4)^3 = ghghghgh^{-2}gh^3gh^{-2} = e \rangle,$$

where $g = (2, 3)(4, 6)(5, 7)(8, 9)$, $h = (1, 2)(3, 4, 7, 9, 10, 8, 6, 5)$, and e is the identity element in S_{10} . Finally, $\text{Aut}(A_6)$ denotes the automorphism group of A_6 .

Theorem 1.1 is obtained from Theorem 3.4 for A_6 , and from Theorem 3.6 for $\text{SL}(2, 9)$, and Theorem 1.2 follows from Theorem 4.3.

2. Preliminaries for proofs of Theorem 1.1 and Theorem 1.2

The following three facts are useful for our study.

PROPOSITION 2.1. (cf. [5, Ch. III, Theorem 4.3]) *Let p be a prime number, G a finite group of p -power order and X a compact manifold with G -action. Then the Euler characteristic $\chi(X^G)$ is congruent modulo p to the Euler characteristic $\chi(X)$.*

LEMMA 2.2. (Smith's theorem [18], cf. [5, Ch. III, Theorem 5.1]) *Let p be a prime number, G a finite group of p -power order and X a $\mathbb{Z}_p$ -homology sphere with G -action. Then X^G is also a $\mathbb{Z}_p$ -homology sphere.*

LEMMA 2.3. (Conner and Floyd [7, Theorem 25.1], cf. [14, Proposition 3.2]) *Let G be a finite group of order two and X a connected closed manifold of positive dimension with G -action. If X^G is a non-empty finite set, then X^G consists of an even number of points.*

Let G be a finite group and M a manifold with G -action. For each $x \in M^G$, the tangent space $T_x(M)$ of M at x linearly inherits the G -action on M , i.e. $T_x(M)$ is a real G -module, and we call the real G -module $T_x(M)$ the *tangential G -module* at x .

The next proposition is a generalization of [15, Proposition 2.9].

PROPOSITION 2.4. *Let G be a finite group and Σ a homology sphere with G -action. Suppose G satisfies the following conditions.*

- (1) *All elements in G have prime power order.*
- (2) *For any g in G such that the order of g is 2-power and greater than four, there is at most one irreducible real G -module V with $V^g = \{0\}$.*

Then the tangential G -modules at all points in Σ^G are mutually isomorphic as real G -modules.

Proof. We may assume that $|\Sigma^G| \geq 2$. It suffices to prove that, for any $g \in G$ and for any $x, y \in \Sigma^G$, the restrictions to $\langle g \rangle$ of $T_x(\Sigma)$ and $T_y(\Sigma)$ are isomorphic as real $\langle g \rangle$ -modules, where $\langle g \rangle$ denotes the cyclic subgroup of G generated by $g \in G$.

Fix two points a and b in Σ^G . By the hypothesis on the order of elements in G , the g -fixed point set Σ^g either is a connected manifold or consists of the two points $\{a, b\}$ for each $g \in G$ (by Smith's theorem). If Σ^g is connected, then $\text{res}_{\langle g \rangle}^G T_a(\Sigma)$ is isomorphic to $\text{res}_{\langle g \rangle}^G T_b(\Sigma)$, and therefore we will verify the isomorphism in the case of $\Sigma^g = \{a, b\}$. If the order of $g \in G$ is two, four or an odd-prime power and $\Sigma^g = \{a, b\}$, we can obtain $\text{res}_{\langle g \rangle}^G T_a(\Sigma) \cong \text{res}_{\langle g \rangle}^G T_b(\Sigma)$ from character theory for g with $\text{ord}(g) = 2$ or 4, and from Sanchez' theorem [17, Corollary 1.11] for g with an odd-prime power order. By the condition (2) in Proposition 2.4, for any $g \in G$ with $\text{ord}(g) = 2^r$ ($r \geq 3$), both of the tangential $\langle g \rangle$ -modules at $\{a, b\} = \Sigma^g (= \Sigma^G)$ are generated only by the restriction to $\langle g \rangle$ of one irreducible real G -module, and they are thus isomorphic as real $\langle g \rangle$ -modules. $\square$

TABLE 3.1. The dimensions of U^H .

	C_2	D_4	D_8	A_4	$\mathfrak{A}_4$	$C_3^2 \rtimes C_2$	S_4	$\mathfrak{S}_4$	G
$\mathbb{R}_G$	1	1	1	1	1	1	1	1	1
$U_{5,1}$	3	2	1	2	0	1	1	0	0
$U_{5,2}$	3	2	1	0	2	1	0	1	0
$U_{8,i(i=1,2)}$	4	2	1	0	0	0	0	0	0
U_9	5	3	2	1	1	1	1	1	0
U_{10}	4	1	0	1	1	0	0	0	0

Note that the Mathieu group M_{10} contains an element of order eight, and A_6 and M_{10} fulfill the conditions (1) and (2) in Proposition 2.4.

Remark 1. As a corollary of Proposition 2.4, for an arbitrary finite group G fulfilling the conditions (1) and (2) in Proposition 2.4, the Smith set $Sm(G)$ is trivial. Hence the Smith sets $Sm(A_6)$ and $Sm(M_{10})$ are trivial, which has already been known by [16, Theorem C3] for A_6 , and by [20, Corollary 4.4] for M_{10} .

PROPOSITION 2.5. *Let X and Y be closed submanifolds of positive dimensions m and n in a $\mathbb{Z}_2$ -homology sphere Σ of dimension $m+n$ that intersect in all points, transversely. Then the intersection $X \cap Y$ consists of an even number of points.*

Proof. Since Σ is a $\mathbb{Z}_2$ -homology sphere of dimension $m+n$, the $\mathbb{Z}_2$ -intersection form on Σ ,

$$\varphi : H_m(\Sigma; \mathbb{Z}_2) \times H_n(\Sigma; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2,$$

is trivial. This implies that $|X \cap Y| \equiv 0 \pmod{2}$. $\square$

3. Actions of A_6 and $SL(2, 9)$ on $\mathbb{Z}_2$ -homology spheres

In this section, we let $G = A_6$, $\tilde{G} = SL(2, 9)$ and Z be the center of $\tilde{G}$. Then G and $\tilde{G}$ have precisely seven and 13 irreducible real G - and $\tilde{G}$ -modules (up to isomorphisms), respectively. We denote the irreducible real G -modules by

$$\mathbb{R}_G, U_{5,1}, U_{5,2}, U_{8,1}, U_{8,2}, U_9 \text{ and } U_{10},$$

where $\dim \mathbb{R}_G = 1$ and $\dim U_l = \dim U_{l,m} = l$, and the irreducible real $\tilde{G}$ -modules by

$$\mathbb{R}_{\tilde{G}}, \tilde{U}_{5,1}, \tilde{U}_{5,2}, \tilde{U}_{8,1}, \tilde{U}_{8,2}, \tilde{U}_9, \tilde{U}_{10}, W_{8,1}, W_{8,2}, W_{16,1}, W_{16,2}, W_{20,1} \text{ and } W_{20,2},$$

where $\dim \mathbb{R}_{\tilde{G}} = 1$ and $\dim \tilde{U}_l = \dim \tilde{U}_{l,m} = \dim W_{l,m} = l$. Note that $|Z| = 2$, $\tilde{G}/Z \cong G$,

$$\tilde{U}_l^Z = \tilde{U}_l \quad \text{and} \quad \tilde{U}_{l,m}^Z = \tilde{U}_{l,m},$$

$$\tilde{U}_l^Z \cong U_l \quad \text{and} \quad \tilde{U}_{l,m}^Z \cong U_{l,m} \quad (\text{as real } G\text{-modules}),$$

and that $W_{l,m}^Z = 0$. Table 3.1 shows $\dim U^H$ for some subgroups H of G and for the irreducible real G -modules U .

Here, C_n and D_n denote cyclic and dihedral subgroups of G of order n , respectively; A_4 and S_4 denote subgroups of G being isomorphic to the alternating and the symmetric groups on four letters, respectively; $\mathfrak{A}_4$ and $\mathfrak{S}_4$ are subgroups of G that are isomorphic but not conjugate to A_4 and S_4 as subgroups of G , respectively; and $C_3^2 \rtimes C_2$ denotes a subgroup of G being isomorphic to the semidirect product of the elementary abelian group of order nine with a cyclic group of order two acting via the inverse map. Note that C_2 , D_8 and A_4 (respectively C_2 , D_8 and $\mathfrak{A}_4$) are subgroups of S_4 (respectively $\mathfrak{S}_4$), and that C_2 is a subgroup of both $C_3^2 \rtimes C_2$ and D_4 .

LEMMA 3.1. *Let Σ be a $\mathbb{Z}_2$ -homology sphere with G -action. If the G -fixed point set Σ^G is a non-empty finite set, then the set of all points x in Σ^G such that the tangential G -module $T_x(\Sigma)$ is isomorphic to $U_{5,1}^{a_1} \oplus U_{5,2}^{a_2} \oplus U_9^b \oplus U_{10}^c$ for non-negative integers a_1, a_2, b and c consists of an even number of points.*

Proof. Let F denote the set of all points $x \in \Sigma^G$ such that the tangential G -module $T_x(\Sigma)$ satisfies $\dim T_x(\Sigma)^{D_8} = \dim T_x(\Sigma)^{S_4} + \dim T_x(\Sigma)^{\mathfrak{S}_4}$. For a point x in F and a subgroup H of G , we let Σ_x^H denote the connected component of Σ^H including x , and Σ_F^H denote the union of all connected components C of Σ^H such that $C \cap F \neq \emptyset$. We note that, since $0 < |\Sigma^G| < \infty$ (i.e. $\mathbb{R}_G \not\subset T_x(\Sigma)$ for each $x \in \Sigma^G$), $x \in F$ if and only if the tangential G -module $T_x(\Sigma)$ is isomorphic to $U_{5,1}^{a_1} \oplus U_{5,2}^{a_2} \oplus U_9^b \oplus U_{10}^c$ for non-negative integers a_1, a_2, b and c (by Table 3.1). Therefore, we will prove that F consists of an even number of points in the following four cases.

Case 1. $\dim T_x(\Sigma)^{D_8} = 0$ for some $x \in F$.

Case 2. $\dim T_x(\Sigma)^{D_8} = \dim T_x(\Sigma)^{S_4} (> 0)$ for some $x \in F$.

Case 3. $\dim T_x(\Sigma)^{D_8} = \dim T_x(\Sigma)^{\mathfrak{S}_4} (> 0)$ for some $x \in F$.

Case 4. $\dim T_x(\Sigma)^{D_8} = \dim T_x(\Sigma)^{S_4} (> 0) + \dim T_x(\Sigma)^{\mathfrak{S}_4} (> 0)$ for all $x \in F$.

Proof in Case 1. By Smith's theorem, Σ^{D_8} consists of exactly two points, say $\{x, y\}$. Then $\Sigma_x^{A_4}$ (respectively $\Sigma_x^{\mathfrak{A}_4}$) is a c -dimensional closed manifold with S_4/A_4 (respectively $\mathfrak{S}_4/\mathfrak{A}_4$)-action, and hence the S_4/A_4 (respectively $\mathfrak{S}_4/\mathfrak{A}_4$)-fixed point set consists of the two points $\{x, y\}$ by Lemma 2.3. Thus, we get

$$\{x, y\} = \Sigma^{D_8} = \Sigma^{S_4} \cap \Sigma^{\mathfrak{S}_4} = \Sigma^G = F,$$

and the proof in Case 1 is complete.

Proof in Case 2. For $x \in F$ with $\dim T_x(\Sigma)^{D_8} = \dim T_x(\Sigma)^{S_4}$, $\Sigma_x^{S_4}$ is a connected, closed submanifold of a $\mathbb{Z}_2$ -homology sphere Σ^{D_8} with the same dimension as Σ^{D_8} . This means that $\Sigma_x^{S_4} = \Sigma^{D_8} = \Sigma^{S_4}$. By Table 3.1, $\Sigma^{S_4} = \Sigma^{D_8}$ and $|\Sigma^G| < \infty$ imply that, for some $a_1 > 0$ and $c \geq 0$, $T_y(\Sigma) \cong U_{5,1}^{a_1} \oplus U_{10}^c$ for all $y \in \Sigma^G$ (hence $F = \Sigma^G$). Since $\Sigma^{S_4} = \Sigma^{D_8} \supset \Sigma^{\mathfrak{S}_4}$ and $\Sigma^{S_4} \cap \Sigma^{\mathfrak{S}_4} = \Sigma^G$, Σ^G coincides with $\Sigma^{\mathfrak{S}_4}$. Therefore, to complete the proof in Case 2, we need to show that Σ^G or $\Sigma^{\mathfrak{S}_4}$ consists of an even number of points.

We first consider the case $c = 0$. According to Table 3.1, Σ^{D_4} and $\Sigma_F^{C_3^2 \rtimes C_2}$ are closed submanifolds of positive dimensions in a $\mathbb{Z}_2$ -homology sphere Σ^{C_2} , and intersect in all points, transversely. By Proposition 2.5, the intersection $\Sigma^{D_4} \cap \Sigma_F^{C_3^2 \rtimes C_2} = \Sigma^G$ consists of an even number of points.

Next consider the case $c > 0$. Then $\Sigma_F^{\mathfrak{A}_4}$ is a c -dimensional closed manifold with $\mathfrak{S}_4/\mathfrak{A}_4$ -action. Therefore, $|\Sigma^{\mathfrak{S}_4}| \equiv 0 \pmod{2}$ follows from Lemma 2.3.

Proof in Case 3. We can prove the proof in Case 3 just as the proof in Case 2, by replacing $a_1, U_{5,1}, S_4, \mathfrak{A}_4$ and $\mathfrak{S}_4$ in the proof in Case 2 with $a_2, U_{5,2}, \mathfrak{S}_4, A_4$ and S_4 , respectively.

Proof in Case 4. First, we show that $\Sigma_F^{S_4} \cap \Sigma_F^{\mathfrak{S}_4} = F$. It suffices to prove that $\Sigma_x^{S_4} \cap \Sigma_x^{\mathfrak{S}_4} \subset F$ for any $x \in F$. Fix a point x in F . Then, the tangential G -module $T_x(\Sigma)$ is isomorphic to $U_{5,1}^{a_1} \oplus U_{5,2}^{a_2} \oplus U_9^b \oplus U_{10}^c$ for some a_1, a_2, b and c . Let n_1, n_2, n_3, n_4 and n_5 be non-negative integers such that

$$\begin{aligned} \dim \Sigma &= 5a_1 + 5a_2 + 9b + 10c = 5n_1 + 5n_2 + 8n_3 + 9n_4 + 10n_5, \\ \dim \Sigma^{C_2} &= 3a_1 + 3a_2 + 5b + 4c = 3n_1 + 3n_2 + 4n_3 + 5n_4 + 4n_5, \\ \dim \Sigma^{D_8} &= a_1 + a_2 + 2b = n_1 + n_2 + n_3 + 2n_4, \\ \dim \Sigma_x^{S_4} &= a_1 + b = n_1 + n_4, \\ \dim \Sigma_x^{\mathfrak{S}_4} &= a_2 + b = n_2 + n_4. \end{aligned}$$

Since $\Sigma, \Sigma^{C_2}, \Sigma^{D_8}, \Sigma_x^{S_4}$ and $\Sigma_x^{\mathfrak{S}_4}$ are connected, and the determinant of the matrix

$$\begin{bmatrix} 5 & 5 & 8 & 9 & 10 \\ 3 & 3 & 4 & 5 & 4 \\ 1 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

is non-zero (i.e. n_1, n_2, n_3, n_4 and n_5 are uniquely determined), the tangential G -module $T_y(\Sigma)$ is isomorphic to $U_{5,1}^{a_1} \oplus U_{5,2}^{a_2} \oplus U_9^b \oplus U_{10}^c$ for any $y \in \Sigma_x^{S_4} \cap \Sigma_x^{\mathfrak{S}_4} (\subset \Sigma^G)$. This implies that $\Sigma_x^{S_4} \cap \Sigma_x^{\mathfrak{S}_4} \subset F$.

Now, by the condition in Case 4, $\Sigma_F^{S_4}$ and $\Sigma_F^{\mathfrak{S}_4}$ are closed submanifolds of a $\mathbb{Z}_2$ -homology sphere Σ^{D_8} intersecting in all points, transversely. Therefore, F consists of an even number of points (by Proposition 2.5). $\square$

COROLLARY 3.2. *Let Σ be a $\mathbb{Z}_2$ -homology sphere with G -action. If the G -fixed point set Σ^G consists of an odd number of points, then the set of all points x in Σ^G such that the tangential G -module $T_x(\Sigma)$ contains some irreducible real G -submodule of dimension eight consists of an odd number of points.*

Proof. Lemma 3.1 also says that if Σ^G is a finite set, the set of all points $x \in \Sigma^G$ with the tangential G -module $T_x(\Sigma)$ not containing any irreducible real G -submodule of dimension eight consists of an even number of points. This proves Corollary 3.2. $\square$

The next corollary follows directly from Proposition 2.4 and Corollary 3.2.

COROLLARY 3.3. *Let Σ be a homology sphere with G -action. If the G -fixed point set Σ^G consists of an odd number of points, then the tangential G -module $T_x(\Sigma)$ contains some irreducible real G -submodule of dimension eight for each point x in Σ^G .*

THEOREM 3.4. *Let Σ be a $\mathbb{Z}_2$ -homology sphere. If the dimension of Σ lies in $T_G = [0..7] \cup [9..12] \cup \{14, 15\} \cup \{19, 20\}$, then Σ does not admit an odd-fixed-point action of G .*

Proof. Let n be the dimension of Σ , and a, b, c and d non-negative integers such that $n = 5a + 8b + 9c + 10d$. Note that T_G coincides with the set of non-negative integers defined by

$$\mathbb{Z}_{\geq 0} \setminus \{5k + 8l + 9m + 10n \mid (k, l, m, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{N} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}\},$$

and if $n \in T_G$ then $b = 0$. Thus, by Corollary 3.2, Σ^G does not consist of an odd number of points if $n \in T_G$. $\square$

Next we will discuss odd-fixed-point actions of $\tilde{G}$ on $\mathbb{Z}_2$ -homology spheres.

We can obtain the following proposition from Corollary 3.2.

PROPOSITION 3.5. *Let Σ be a $\mathbb{Z}_2$ -homology sphere with $\tilde{G}$ -action. If the $\tilde{G}$ -fixed point set $\Sigma^{\tilde{G}}$ consists of an odd number of points, then the set of all points x in $\Sigma^{\tilde{G}}$ such that the tangential $\tilde{G}$ -module $T_x(\Sigma)$ contains a real $\tilde{G}$ -submodule being isomorphic to $\tilde{U}_{8,1}$ or $\tilde{U}_{8,2}$ consists of an odd number of points.*

Proof. First, we recall that $|Z| = 2$, $\tilde{G}/Z \cong G$, $\tilde{U}_{8,i}^Z = \tilde{U}_{8,i}$ and $\tilde{U}_{8,i}^Z \cong U_{8,i}$ for $i = 1, 2$. By Smith's theorem, the Z -fixed point set Σ^Z is a $\mathbb{Z}_2$ -homology sphere with G -action.

Suppose that $|(\Sigma^Z)^G| \equiv 1 \pmod{2}$. Then, by Corollary 3.2, there are an odd number of points $x \in (\Sigma^Z)^G$ such that the tangential G -module $T_x(\Sigma^Z)$ contains a real G -submodule being isomorphic to $U_{8,1}$ or $U_{8,2}$. Since $(\Sigma^Z)^G = \Sigma^{\tilde{G}}$ and $T_y(\Sigma^Z)$ is isomorphic to $T_y(\Sigma)^Z$ as real G -modules for all $y \in \Sigma^{\tilde{G}}$, we see that for the odd number of points $x \in \Sigma^{\tilde{G}}$, the tangential $\tilde{G}$ -module $T_x(\Sigma)$ contains a real $\tilde{G}$ -submodule being isomorphic to $\tilde{U}_{8,1}$ or $\tilde{U}_{8,2}$. $\square$

Let Σ be a $\mathbb{Z}_2$ -homology sphere with $\tilde{G}$ -action. An action $\Phi : \tilde{G} \rightarrow \text{Diff}(\Sigma)$, where $\text{Diff}(\Sigma)$ is the diffeomorphism group of Σ , is called *effective* if the kernel

$$\ker \Phi = \{g \in \tilde{G} \mid \Phi(g)(x) = x \text{ for all } x \in \Sigma\}$$

of this group homomorphism Φ is trivial. Hence if an action of $\tilde{G}$ on Σ is effective, the tangential $\tilde{G}$ -module $T_x(\Sigma)$ at each point $x \in \Sigma^{\tilde{G}}$ must contain a real $\tilde{G}$ -submodule being isomorphic to $W_{8,1}$, $W_{8,2}$, $W_{16,1}$, $W_{16,2}$, $W_{20,1}$ or $W_{20,2}$.

THEOREM 3.6. *Let Σ be a $\mathbb{Z}_2$ -homology sphere with effective $\tilde{G}$ -action. If the dimension of Σ lies in $T_{\tilde{G}} = [0..15] \cup [17..20] \cup \{22, 23\} \cup \{27\}$, then Σ does not admit an odd-fixed-point action of $\tilde{G}$.*

Proof. By Proposition 3.5, if the $\tilde{G}$ -fixed point set of the effective action of $\tilde{G}$ on Σ consists of an odd number of points, then there are an odd number of points $x \in \Sigma^{\tilde{G}}$ such that the tangential $\tilde{G}$ -module $T_x(\Sigma)$ contains a real $\tilde{G}$ -submodule being isomorphic to $\tilde{U}_{8,1} \oplus W$ or $\tilde{U}_{8,2} \oplus W$, where W is one of $W_{8,1}$, $W_{8,2}$, $W_{16,1}$, $W_{16,2}$, $W_{20,1}$ and $W_{20,2}$.

Let n be the dimension of Σ , and a, b, c, d, x, y and z non-negative integers such that $n = 5a + 8b + 9c + 10d + 8x + 16y + 20z$. Note that $T_{\tilde{G}}$ coincides with the set of non-negative integers defined by

$$\mathbb{Z}_{\geq 0} \setminus \bigcup_{\alpha=8,16,20} \{5i + 8j + 9k + 10l + \alpha m \mid (i, j, k, l, m) \in \mathbb{Z}_{\geq 0} \times \mathbb{N} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{N}\},$$

and if $n \in T_{\tilde{G}}$ then $b = 0$ or $x = y = z = 0$. This implies that there is no effective odd-fixed-point action of $\tilde{G}$ on Σ if $n \in T_{\tilde{G}}$. $\square$

4. Actions of S_6 , $\mathrm{PGL}(2,9)$, M_{10} and $\mathrm{Aut}(A_6)$ on homology spheres

In this section, we will prove Theorem 1.2, and let $L = A_6$, $G_1 = S_6$, $G_2 = \mathrm{PGL}(2, 9)$, $G_3 = M_{10}$ and $G_4 = \mathrm{Aut}(A_6)$. Each of G_1 , G_2 and G_3 has a subgroup of index two being isomorphic to L , and G_4 has subgroups of index two being isomorphic to G_1 , G_2 and G_3 (hence G_4 has a subgroup of index four being isomorphic to L). Thus we may assume that G_1 , G_2 , G_3 and G_4 have L as a subgroup.

PROPOSITION 4.1. *Let G be a finite group, H a subgroup of G with index two and Σ a homology sphere with G -action. Suppose that either Σ^H is a finite set or each connected component of Σ^H has positive dimension. Then $|\Sigma^G| \equiv 1 \pmod{2}$ if and only if $|\Sigma^H| \equiv 1 \pmod{2}$.*

Proof. By Proposition 2.1, if $|\Sigma^H| \equiv 1 \pmod{2}$ then $|\Sigma^G| \equiv 1 \pmod{2}$.

Next assume that $|\Sigma^H| \not\equiv 1 \pmod{2}$. By the hypothesis on Σ^H , Σ^H either consists of an even number of points or is a closed manifold with G/H -action such that each connected component has positive dimension. Therefore, in either case of Σ^H , $|\Sigma^G| \not\equiv 1 \pmod{2}$ follows from Proposition 2.1 and Lemma 2.3. $\square$

We recall that L and G_3 fulfill the conditions (1) and (2) in Proposition 2.4. In other words, for $H = L$ or G_3 and for a homology sphere Σ with H -action, the tangential H -modules at all points in Σ^H are mutually isomorphic. So the dimensions of all connected components of Σ^H are equal (i.e. $0 < |\Sigma^H| < \infty$ or $\dim \Sigma^H > 0$).

COROLLARY 4.2. *Let G be one of G_1 , G_2 , G_3 and G_4 , and Σ a homology sphere with G -action. Then $|\Sigma^G| \equiv 1 \pmod{2}$ if and only if $|\Sigma^L| \equiv 1 \pmod{2}$.*

Proof. Corollary 4.2 is obtained by applying Proposition 4.1 once or twice. $\square$

Let G be one of G_1 , G_2 , G_3 and G_4 , and $\mathbb{R}_L$, $U_{5,1}$, $U_{5,2}$, $U_{8,1}$, $U_{8,2}$, U_9 and U_{10} the irreducible real L -modules (up to isomorphisms). The irreducible real G -modules V_* (up to isomorphisms) and the restrictions to L of V_* are as Tables 4.1 to 4.4.

Here, the real irreducible G - and L -modules in Tables 4.1–4.4 have the following dimensions

$$\dim \mathbb{R}_L = \dim \mathbb{R}_{1,m} = 1, \quad \dim U_l = \dim U_{l,m} = \dim V_l = \dim V_{l,m} = l.$$

We note that T_{G_1} , T_{G_2} , T_{G_3} and T_{G_4} in Theorem 1.2 are alternatively given by

$$\begin{aligned} T_{G_1} &= [0..15] \cup [17..20] \cup [22..24] \cup [27..29] \cup \{33\} \cup \{38\} \\ &= \mathbb{Z}_{\geq 0} \setminus \{5a + 9b + 10c + 16d \mid (a, b, c, d) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{N}\}, \\ T_{G_2} &= [0..7] \cup [9..15] \cup [19..23] \cup [29..31] \cup \{39\} \\ &= \mathbb{Z}_{\geq 0} \setminus \{8a + 9b + 10c \mid (a, b, c) \in \mathbb{N} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}\}, \\ T_{G_3} &= [0..15] \cup [17..24] \cup [27..31] \cup \{33\} \cup [37..40] \cup \{47\} \cup \{49\} \\ &= \mathbb{Z}_{\geq 0} \setminus \{9a + 10b + 16c + 20d \mid (a, b, c, d) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{N} \times \mathbb{Z}_{\geq 0}\}, \\ T_{G_4} &= [0..15] \cup [17..24] \cup [27..31] \cup \{33\} \cup [37..40] \cup \{47\} \cup \{49\} \\ &= \mathbb{Z}_{\geq 0} \setminus \{9a + 10b + 16c + 20d \mid (a, b, c, d) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{N} \times \mathbb{Z}_{\geq 0}\}. \end{aligned}$$

TABLE 4.1. Case of $G = G_1$.

V_*	$\mathbb{R}_{1,i(i=1,2)}$	$V_{5,i(i=1,2)}$	$V_{5,j(j=3,4)}$	$V_{9,i(i=1,2)}$	$V_{10,i(i=1,2)}$	V_{16}
$\text{res}_L^G V_*$	$\mathbb{R}_L$	$U_{5,1}$	$U_{5,2}$	U_9	U_{10}	$U_{8,1} \oplus U_{8,2}$

TABLE 4.2. Case of $G = G_2$.

V_*	$\mathbb{R}_{1,i(i=1,2)}$	$V_{8,i(i=1,2)}$	$V_{8,j(j=3,4)}$	$V_{9,i(i=1,2)}$	$V_{10,i(i=1,2)}$	$V_{10,3}$
$\text{res}_L^G V_*$	$\mathbb{R}_L$	$U_{8,1}$	$U_{8,2}$	U_9	U_{10}	$U_{5,1} \oplus U_{5,2}$

TABLE 4.3. Case of $G = G_3$.

V_*	$\mathbb{R}_{1,i(i=1,2)}$	$V_{9,i(i=1,2)}$	V_{10}	V_{16}	$V_{20,i(i=1,2)}$
$\text{res}_L^G V_*$	$\mathbb{R}_L$	U_9	$U_{5,1} \oplus U_{5,2}$	$U_{8,1} \oplus U_{8,2}$	$U_{10}^{\oplus 2}$

TABLE 4.4. Case of $G = G_4$.

V_*	$\mathbb{R}_{1,k(k=1,2,3,4)}$	$V_{9,k(k=1,2,3,4)}$	$V_{10,i(i=1,2)}$	$V_{16,i(i=1,2)}$	V_{20}
$\text{res}_L^G V_*$	$\mathbb{R}_L$	U_9	$U_{5,1} \oplus U_{5,2}$	$U_{8,1} \oplus U_{8,2}$	$U_{10}^{\oplus 2}$

THEOREM 4.3. *Let G be one of G_1 , G_2 , G_3 and G_4 , and Σ a homology sphere with G -action. If the dimension of Σ lies in T_G , then the G -fixed point set Σ^G does not consist of an odd number of points.*

Proof. By Corollary 4.2, it suffices to prove that if $\dim \Sigma \in T_G$ then $|\Sigma^L| \not\equiv 1 \pmod{2}$.

Suppose that $\dim \Sigma \in T_G$ and $0 < |\Sigma^L| < \infty$. Then, by the definitions of T_{G_1} , T_{G_2} , T_{G_3} and T_{G_4} and by Tables 4.1–4.4, the restriction to L of the tangential G -module $T_x(\Sigma)$ does not contain any irreducible real L -submodule of dimension eight for each x in Σ^G . Thus $|\Sigma^L|$ does not consist of an odd number of points by Corollary 3.3. $\square$

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