ON THE ASYMPTOTIC EXPANSION OF THE QUANTUM $SU(2)$ INVARIANT AT $\zeta = e^{4\pi i/r}$ FOR LENS SPACE $L(p, q)$

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Abstract. We give a formula for the quantum $SU(2)$ invariant at $\zeta = e^{4\pi i/r}$ for Lens space $L(p, q)$, and we prove that the asymptotic expansion is represented by a sum of contributions from $SL_2\mathbb{C}$ flat connections whose coefficients are square roots of the Reidemeister torsions.

1. Introduction

When $\zeta = e^{2\pi i/r}$, the asymptotic expansions of the quantum $SU(2)$ invariants of 3-manifolds M have been studied by some researchers [1–4, 6–9, 14, 15], and they are represented by a sum of contributions from $SL_2\mathbb{C}$ flat connections on M . Jeffrey [9] gave a formula for the quantum $SU(2)$ invariant of Lens space $L(p, q)$ and proved that the asymptotic expansion is represented by a sum of contributions from $SU(2)$ flat connections whose coefficients are square roots of the Reidemeister torsions.

When $\zeta = e^{4\pi i/r}$ with odd r , Chen and Yang [5] recently observed that the quantum $SU(2)$ invariant is of exponential order as $r \rightarrow \infty$ for some hyperbolic 3-manifolds and conjectured that the quantum $SU(2)$ invariant for a closed hyperbolic 3-manifold is of exponential order as $r \rightarrow \infty$ whose growth is given by the complex volume of M . Ohtsuki and the first author [11] showed that when $\zeta = e^{4\pi i/r}$, the asymptotic expansion of the quantum $SU(2)$ invariant of some Seifert 3-manifolds and Lens space $L(p, 1)$ with odd p is represented by a sum of contributions from some $SL_2\mathbb{C}$ flat connections on them, and square roots of the Reidemeister torsions appear as coefficients of such contributions.

We note that the quantum $SU(2)$ and $SO(3)$ invariants for a rational homology 3-sphere are equal when $\zeta = e^{4\pi i/r}$ with odd r (see [11, Appendix]), and we denote by $\tau_r(M)$ the quantum $SO(3)$ invariant of a 3-manifold of a M in this paper.

In this paper, we give a formula for the quantum $SO(3)$ invariant of Lens space $L(p, q)$ at $\zeta = e^{4\pi i/r}$ and prove that the asymptotic expansion is represented by a sum of contributions from $SL_2\mathbb{C}$ flat connections whose coefficients are square roots of the Reidemeister torsions. Since the fundamental group of Lens space $L(p, q)$ is the abelian group $\mathbb{Z}/p\mathbb{Z}$, the $SL_2\mathbb{C}$ flat connections correspond one-to-one to the $SU(2)$ flat connections, and so the asymptotic expansion is represented by a sum of contributions from $SU(2)$ flat connections.

Let ρ_k be a representation of $\pi_1(L(p, q)) = \mathbb{Z}/p\mathbb{Z}$ to $SL_2\mathbb{C}$ such that the eigenvalues of $\rho_k(g)$ for a generator g of $\pi_1(L(p, q)) = \mathbb{Z}/p\mathbb{Z}$ are $e^{\pm 2\pi i k/p}$. The main theorem is the following.

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THEOREM 1.1. *Let p and q be coprime integers with $p \geq 3$ and $p > q > 0$. Then, the quantum invariant $\tau_r(L(p, q))$ of $L(p, q)$ for odd r is expanded as $r \rightarrow \infty$ in the form*

$$\tau_r(L(p, q)) \sim e^{\pi i/2} e^{(\pi i/4)(n_0(p, q) - 3t)r} e^{(\pi i/2)(-1)^{(r-1)/2} n_1(p, q)} c(p)^r \\ \times \begin{cases} \sum_{\substack{1 \leq k < p \\ k \text{ is even}}} e^{\pi i \text{CS}(L(p, q); \text{ad} \circ \rho_k) r} \omega(L(p, q); \rho_k) & \text{if } p \equiv 0(8), \\ \sum_{\substack{1 \leq k < p \\ k \text{ is odd}}} e^{\pi i \text{CS}(L(p, q); \text{ad} \circ \rho_k) r} \omega(L(p, q); \rho_k) & \text{otherwise,} \end{cases}$$

where $n_0(p, q)$ and $n_1(p, q)$ are defined in Proposition 3.6 and $p/q = [m_1, \dots, m_t]$ with $m_i \geq 2$,

$$c(p) := \begin{cases} 1 & \text{if } p \text{ is odd,} \\ \frac{1}{2} & \text{if } p \equiv 0(4), \\ \frac{1}{2}(1 + (-1)^{p/4}) & \text{if } p \equiv 2(4), \end{cases}$$

$\text{CS}(L(p, q); \text{ad} \circ \rho_k)$ is the Chern–Simons invariant, and we put

$$\omega(L(p, q); \rho_k) = \frac{1}{\pi \sqrt{p}} \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p}.$$

Further, we have that

$$\omega(L(p, q); \rho_k)^2 = \pm \frac{1}{32\pi^2} \text{Tor}(L(p, q); \text{ad} \circ \rho_k),$$

where $\text{Tor}(L(p, q); \text{ad} \circ \rho_k)$ is the Reidemeister torsion.

This paper is organized as follows. In Section 2, we give a formula for the quantum $\text{SO}(3)$ invariant at $\zeta = e^{4\pi i/r}$ for Lens space $L(p, q)$. In Section 3, using our formula, we calculate the asymptotic expansion of it and prove the main theorem.

2. A Formula for $\tau_r(L(p, q))$

In this section, we give a formula for the quantum $\text{SO}(3)$ invariant at $\zeta = e^{4\pi i/r}$ for Lens spaces $L(p, q)$.

We briefly recall the definition of the quantum $\text{SU}(2)$ and $\text{SO}(3)$ invariants of a closed oriented 3-manifold M , following the construction of [10].

For a surface F and a complex number A , the linear skein $\mathcal{S}(F)$ of F is the vector space over $\mathbb{C}$ spanned by link diagrams on F subject to the isotopy and the relations

$$(i) \quad \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = A \quad \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} + A^{-1} \begin{array}{c} \diagdown \quad \diagup \\ \diagdown \quad \diagup \end{array} \right),$$

$$(ii) \quad D \bigcirc = (-A^2 - A^{-2})D \text{ for any diagram } D.$$

We have that the linear skein of the plane, $\mathcal{S}(\mathbb{R}^2)$, is isomorphic to a one-dimensional vector space with the empty diagram $\emptyset$ as a base. That is, for any diagram D in $\mathbb{R}^2$, we can write

$$D = \langle D \rangle \emptyset \quad \text{in } \mathcal{S}(\mathbb{R}^2),$$

where $\langle D \rangle$ denotes the Kauffman bracket of D , with $\langle \emptyset \rangle = 1$.

We introduce the box over n strands in $\mathcal{S}(D^2, 2n)$ recursively by

$$\frac{1}{\text{box}} \frac{1}{\text{box}} = \text{box} \quad , \quad (2)$$

We put

$$\mu_0 = \frac{A^2 - A^{-2}}{\sqrt{-r}}.$$

$$\omega = \mu_0 \sum_{n=0}^{r-2} \Delta_n \quad \text{,} \quad \omega_0 = \mu_0 \sum_{n=0}^{(r-3)/2} \Delta_{2n} \quad .$$

Let M be a closed oriented 3-manifold obtained from S^3 by surgery along a framed link L and D a diagram of L . Then, the quantum $SU(2)$ invariant $\tau_r^{SU(2)}(M)$ and the quantum $SO(3)$ invariant $\tau_r(M)$ at $\zeta = A^4 = e^{4\pi i/r}$ with $A = e^{\pi i/r}$ are defined by

$$\begin{aligned}\tau_r^{\text{SU}(2)}(M) &= \langle \omega \rangle_{U_+}^{-b_+} \langle \omega \rangle_{U_-}^{-b_-} \langle \omega, \dots, \omega \rangle_D, \\ \tau_r(M) &= \langle \omega_0 \rangle_{U_+}^{-b_+} \langle \omega_0 \rangle_{U_-}^{-b_-} \langle \omega_0, \dots, \omega_0 \rangle_D,\end{aligned}$$

Now we calculate the quantum $\mathrm{SO}(3)$ invariant $\tau_r(L(p, q))$ of Lens space $L(p, q)$. The Lens space $L(p, q)$ is specified by a pair p, q of coprime integers with $0 < |q| < p$, and $L(p, q)$ is obtained from S^3 by surgery along a framed link with the diagram $D(p, q)$ shown

FIGURE 1. $D(p, q)$.

in Figure 1, where $m_1, m_2, \dots, m_t$ are integers determined by

$$\frac{p}{q} = m_t - \frac{1}{m_{t-1} - \frac{1}{\ddots - \frac{1}{m_2 - \frac{1}{m_1}}}} =: [m_1, \dots, m_t]$$

and m_i denotes framing.

In what follows, we assume $0 < q < p$ and all $m_i \geq 2$; then the continued fraction expansion $[m_1, \dots, m_t]$ of p/q is determined uniquely.

The linking matrix W of the framed link presented by the diagram $D(p, q)$ is given by

$$W = \begin{pmatrix} m_1 & 1 & & & \\ 1 & m_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & m_t \end{pmatrix}.$$

Since $m_i \geq 2$, we see that W has t positive eigenvalues. Therefore, the quantum $\mathrm{SO}(3)$ invariant of Lens space $L(p, q)$ is given by

$$\tau_r(L(p, q)) = \langle \omega_0 \rangle_{U_+}^{-t} \langle \omega_0, \dots, \omega_0 \rangle_{D(p, q)}. \quad (3)$$

First we calculate the term $\langle \omega_0, \dots, \omega_0 \rangle_{D(p, q)}$. To calculate it, we use the following lemma.

LEMMA 2.1. [10]

- (1) $\Delta_n = \left\langle \begin{array}{c} \text{circle with framing } n \end{array} \right\rangle = (-1)^n \frac{A^{2(n+1)} - A^{-2(n+1)}}{A^2 - A^{-2}}.$
- (2) $\begin{array}{c} n \\ \text{square} \end{array} \text{ with a loop} = (-1)^n A^{n(n+2)} \begin{array}{c} n \\ \text{square} \end{array}.$
- (3) $\begin{array}{c} n \\ \text{square} \end{array} \text{ with a loop of framing } a = (-1)^n \frac{A^{2(n+1)(a+1)} - A^{-2(n+1)(a+1)}}{A^{2(n+1)} - A^{-2(n+1)}} \begin{array}{c} n \\ \text{square} \end{array}.$

From Lemma 2.1, we have that

$$\mu_0 \left\langle \begin{array}{c} \text{Two circles with a vertical bar at the bottom of each, labeled } 2j \text{ and } 2l \\ \hline \end{array} \right\rangle = \frac{1}{\sqrt{-r}} \{A^{2(2j+1)(2l+1)} - A^{-2(2j+1)(2l+1)}\},$$

$$\left\langle \begin{array}{c} \text{A circle with a vertical bar at the bottom, labeled } 2j \\ \hline \end{array} \right\rangle = A^{(2j+1)^2-1} \left\langle \begin{array}{c} \text{A circle with a vertical bar at the bottom, labeled } 2j \\ \hline \end{array} \right\rangle.$$

For $0 \leq j, l \leq (r-3)/2$, we put

$$S_{j,l} := \frac{1}{\sqrt{-r}} \{A^{2(2j+1)(2l+1)} - A^{-2(2j+1)(2l+1)}\},$$

$$T_j := A^{(2j+1)^2-1}.$$

We have the following lemma.

LEMMA 2.2. *These have the following symmetries:*

$$S_{j,l} = S_{j,r+l} = -S_{j,r-1-l},$$

$$T_j = T_{r+j} = T_{r-1-j},$$

$$S_{(r-1)/2,l} = S_{j,(r-1)/2} = 0.$$

From Lemma 2.1 and the definitions of S_{jl} and T_j , it is easy to see the following.

LEMMA 2.3.

$$\langle \omega_0, \dots, \omega_0 \rangle_{D(p,q)} = \mu_0^{-1} \sum_{k_1, \dots, k_t=0}^{(r-3)/2} S_{0,k_t} T_{k_t}^{m_t} S_{k_t, k_{t-1}} T_{k_{t-1}}^{m_{t-1}} \dots S_{k_2, k_1} T_{k_1}^{m_1} S_{k_1, 0}. \quad (4)$$

We introduce new notation and give some lemmas concerning the Gauss sum and $\text{PSL}(2, \mathbb{Z})$ for calculating the sum given in Lemma 2.3.

We put

$$e(\alpha) := \exp(2\pi i \alpha),$$

$$e_n(\alpha) := \exp\left(\frac{2\pi i \alpha}{n}\right).$$

LEMMA 2.4. (Gauss sum reciprocity formula in one dimension, [9])

$$\sum_{k \pmod{n}} e_{2n}(mk^2) e(\psi k) = \sqrt{\frac{in}{m}} \sum_{k \pmod{m}} e_{2m}(-n(k + \psi)^2),$$

if $k \in \mathbb{Z}$, $n, m \in \mathbb{Z}$, nm is even and $n\psi \in \mathbb{Z}$.

Definition 2.1. Suppose $U \in \text{PSL}(2, \mathbb{Z})$. A continued fraction expansion $[m_1, \dots, m_t]$ for U is a sequence of integers $m_1, \dots, m_t$ such that

$$U = T^{m_t} S \dots T^{m_1} S,$$

where S and T are generators of $\mathrm{PSL}(2, \mathbb{Z})$ with

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

satisfying the relations

$$S^2 = (ST)^3 = 1.$$

Such an expansion always exists, and is unique if each $m_i \geq 2$.

LEMMA 2.5. [9] *Suppose*

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Z}),$$

and suppose $[m_1, \dots, m_t]$ is a continued fraction expansion for U . Then

- (i) $\frac{a}{c} = [m_1, \dots, m_t]$,
- (ii) $\frac{b}{a} = -\left(\frac{1}{a_1} + \frac{1}{a_2 a_1} + \dots + \frac{1}{a_t a_{t-1}}\right)$.

Here we define a_i, b_i, c_i and d_i by

$$\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} = T^{m_i} S \cdots T^{m_1} S \quad \text{for } i \geq 1,$$

with the convention that

$$a_0 = d_0 = 1, \quad b_0 = c_0 = 0.$$

Then these satisfy the recurrence relation for $i \geq 2$:

- (iii) $a_i = m_i a_{i-1} - c_{i-1}, \quad c_i = a_{i-1},$
- (iv) $b_i = m_i b_{i-1} - d_{i-1}, \quad d_i = b_{i-1},$

and if $m_j \geq 2$ for $1 \leq j \leq i$, then

- (v) $a_i > 0.$

The right-hand side of the formula (4) is a special case of the following proposition.

PROPOSITION 2.6. *For any $U \in \mathrm{PSL}(2, \mathbb{Z})$ and its continued fraction expansion $[m_1, \dots, m_t]$ with $m_i \geq 2$, let $\varphi(U)_t$ denote the sum,*

$$\varphi(U)_t := \sum_{k_1, k_2, \dots, k_t=0}^{(r-3)/2} S_{k_{t+1}, k_t} T_{k_t}^{m_t} S_{k_t, k_{t-1}} T_{k_{t-1}}^{m_{t-1}} \cdots T_{k_1}^{m_1} S_{k_1, 0}.$$

Then we have

$$\begin{aligned} \varphi(U)_t &= K_t \sum_{\gamma=0}^{4a_t-1} \{x_t(-1)^\gamma + y_t e_8(m_t r) e_4((-1)^{(r-1)/2} m_t) (e_4(\gamma) + e_4(-\gamma)) + z_t(-1)^{m_t}\} \\ &\quad \times \sum_{\pm} \pm e_{8a_t r} \left(-c_t \left\{ r\gamma + 2 \left(2k_{t+1} + 1 \pm \frac{1}{c_t} \right) \right\}^2 \right), \end{aligned} \quad (5)$$

where K_t , x_t and y_t and z_t are given by

$$K_t = \left(-\frac{1}{2\sqrt{ra_t}} \right) \left(\frac{i}{2} \right)^{t-1} (e^{\pi i/4})^t A^{-(m_1+\dots+m_t)} A^{b_t/a_t+1/a_t c_t},$$

$$\begin{aligned} x_t &= x_{t-1} + 2y_{t-1}e_8(m_{t-1}r)e_4((-1)^{(r-1)/2}m_{t-1}) + z_{t-1}(-1)^{m_{t-1}}, & x_1 &= 1, \\ y_t &= x_{t-1} - (-1)^{m_{t-1}}z_{t-1}, & y_1 &= 0, \\ z_t &= x_{t-1} - 2y_{t-1}e_8(m_{t-1}r)e_4((-1)^{(r-1)/2}m_{t-1}) + z_{t-1}(-1)^{m_{t-1}}, & z_1 &= 0. \end{aligned}$$

Proof. We will show the proposition by induction. In the case that $t = 1$, we calculate $\varphi(U)_1$:

$$\begin{aligned} \varphi(U)_1 &= \sum_{k_1=0}^{(r-3)/2} S_{k_2, k_1} T_{k_1}^{m_1} S_{k_1, 0} \\ &= \frac{1}{2} \sum_{k_1=0}^{r-1} S_{k_2, k_1} T_{k_1}^{m_1} S_{k_1, 0} \\ &= -\frac{1}{2r} A^{-m_1} \sum_{k_1=0}^{r-1} e_{2r}(m_1(2k_1+1)^2) \\ &\quad \times \left\{ \sum_{\pm} \pm e_{2r}(\pm 2(2k_2+1)(2k_1+1)) \sum_{\pm} \pm e_{2r}(\pm 2(2k_1+1)) \right\} \\ &= -\frac{1}{2r} A^{-m_1} \sum_{k_1=0}^{r-1} e_{2r}(m_1(2k_1+1)^2) \\ &\quad \times \left\{ \sum_{\pm} \pm e_{2r}(2(2k_1+1)\{(2k_2+1) \pm 1\}) \right. \\ &\quad \left. + \sum_{\pm} \pm e_{2r}(-2(2k_1+1)\{(2k_2+1) \pm 1\}) \right\}, \end{aligned}$$

where the second equality follows from Lemma 2.2. Noting that

$$\begin{aligned} e_{2r}(m_1(2(r-1-k)+1)^2) &= e_{2r}(m_1(2k+1)^2), \\ e_{2r}(-2(2(r-1-k)+1)\{(2k_2+1) \pm 1\}) &= e_{2r}(2(2k+1)\{(2k_2+1) \pm 1\}), \end{aligned}$$

we calculate

$$\begin{aligned} \varphi(U)_1 &= \frac{-1}{r} A^{-m_1} \sum_{k_1=0}^{r-1} e_{2r}(m_1(2k_1+1)^2) \left\{ \sum_{\pm} \pm e_{2r}(2(2k_1+1)\{(2k_2+1) \pm 1\}) \right\} \\ &= \frac{-1}{r} \sum_{k_1=0}^{r-1} e_{2r}(4m_1k_1^2) \left\{ \sum_{\pm} \pm e_{2r}(\{4m_1+4(2k_2+1 \pm 1)\}k_1) e_{2r}(2(2k_2+1 \pm 1)) \right\}. \end{aligned}$$

Applying the reciprocity formula in Lemma 2.4, we have that

$$\begin{aligned} \varphi(U)_1 &= \frac{-1}{r} \sqrt{\frac{ir}{4m_1}} \sum_{k_1=0}^{4m_1-1} \sum_{\pm} \pm e_{8m_1} \left(-r \left\{ k_1 + \frac{2(m_1 + 2k_2 + 1 \pm 1)}{r} \right\}^2 \right) e_{2r}(2(2k_2 + 1 \pm 1)) \\ &= -\frac{1}{2\sqrt{ra_1}} (e^{\pi\sqrt{-1}/4}) A^{-m_1} \sum_{k_1=0}^{4a_1-1} \sum_{\pm} \pm (-1)^{k_1} e_{8a_1r} (-\{rk_1 + 2(2k_2 + 1 \pm 1)\}^2), \end{aligned}$$

where we use that $m_1 = a_1$. This confirms the first step in the induction.

We assume the result of the lemma inductively, and replace by the sum over k_t from 0 to $r-1$ using the symmetries in Lemma 2.2, and then

$$\begin{aligned} \varphi(U)_t &= \sum_{k_t=0}^{(r-3)/2} S_{k_{t+1}, k_t} T_{k_t}^{m_t} \varphi(U)_{t-1} \\ &= \frac{1}{2} \sum_{k_t=0}^{r-1} \left\{ \frac{1}{\sqrt{-r}} \sum_{\pm} \pm e_{2r}(\pm 2(2k_{t+1} + 1)(2k_t + 1)) \right\} \{e_{2r}(m_t(2k_t + 1)^2 - m_t)\} \\ &\quad \times \left\{ K_{t-1} \sum_{k_{t-1}=0}^{4a_{t-1}-1} (x_{t-1}(-1)^{k_{t-1}} \right. \\ &\quad \left. + y_{t-1}e_8(m_{t-1}r)e_4((-1)^{(r-1)/2}m_{t-1})(e_4(k_{t-1}) + e_4(-k_{t-1})) + z_{t-1}(-1)^{m_{t-1}}) \right. \\ &\quad \left. \times \sum_{\pm} \pm e_{8a_{t-1}r} \left(-c_{t-1} \left\{ rk_{t-1} + 2 \left(2k_t + 1 \pm \frac{1}{c_{t-1}} \right) \right\}^2 \right) \right\}. \end{aligned}$$

We decompose the sum of k_{t-1} to that of modulo 4 in the following:

$$\varphi(U)_t = \frac{1}{2\sqrt{-r}} e_{2r}(-m_t) K_{t-1} \{x_t \varphi(U)_{t,0} - y_t \varphi(U)_{t,1} + z_t \varphi(U)_{t,2} - y_t \varphi(U)_{t,3}\},$$

where

$$\begin{aligned} \varphi(U)_{t,j} &= \sum_{k_t=0}^{r-1} \left\{ e_{2r}(m_t(2k_t + 1)^2) \left\{ \sum_{\pm} \pm e_{2r}(\pm 2(2k_{t+1} + 1)(2k_t + 1)) \right\} \right. \\ &\quad \left. \times \sum_{k=0}^{a_{t-1}-1} \sum_{\pm} \pm e_{8a_{t-1}r} \left(-c_{t-1} \left\{ 4kr + jr + 2 \left(2k_t + 1 \pm \frac{1}{c_{t-1}} \right) \right\}^2 \right) \right\}. \quad (6) \end{aligned}$$

Using Lemma A.1, Lemma A.2 and Lemma A.3, we get

$$\begin{aligned} \varphi(U)_t &= \frac{-1}{2\sqrt{-r}} e_{2r}(-m_t) K_{t-1} e_{2r} \left(-\frac{1}{a_{t-1}c_{t-1}} \right) \sqrt{\frac{a_{t-1}ri}{a_t}} \\ &\quad \times \sum_{\gamma=0}^{4a_t-1} ((-1)^\gamma x_t + e_8(m_t r) e_4((-1)^{(r-1)/2} m_t) (e_4(\gamma) + e_4(-\gamma)) y_t + (-1)^{m_t} z_t) \\ &\quad \times \sum_{\pm} \pm e_{8a_t r} \left(-a_{t-1} \left\{ r\gamma + 2 \left(2k_{t+1} + 1 \pm \frac{1}{a_{t-1}} \right) \right\}^2 \right). \end{aligned}$$

Moreover, we have

$$\begin{aligned}
 & \frac{i}{2} \sqrt{\frac{a_{t-1}i}{a_t}} A^{-m_t} A^{-1/(a_{t-1}c_{t-1})} K_{t-1} \\
 &= \left(\frac{i}{2}\right)^{t-1} \sqrt{i} \frac{-1}{2\sqrt{ra_t}} (e^{\pi i/4})^{t-1} A^{-(m_1+\dots+m_t)} A^{b_{t-1}/a_{t-1}} \\
 &= \left(\frac{i}{2}\right)^{t-1} (e^{\pi i/4})^t \left(-\frac{1}{2\sqrt{ra_t}}\right) A^{-(m_1+\dots+m_t)} A^{b_t/a_t+1/a_t c_t} \\
 &= K_t,
 \end{aligned}$$

where the second equality is obtained by Lemma 2.5(ii).

Hence, the claim holds. $\square$

To simplify the formula for $\tau_r(L(p, q))$, we consider the Rademacher phi function and the Dedekind sum.

Definition 2.2. (Rademacher phi function and Dedekind sum [12]) The Rademacher phi function Φ is an integer-valued function on $\text{PSL}(2, \mathbb{Z})$ given by

$$\Phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} (a+d)/c - 12(\text{sign}(c))s(d, |c|), & \text{if } c \neq 0, \\ b/d, & \text{if } c = 0. \end{cases}$$

Here, for $c > 0$ the Dedekind sum $s(d, c)$ is defined by

$$\begin{aligned}
 s(0, 1) &= 0, \\
 s(d, c) &= \sum_{k=1}^{c-1} \left(\left(\frac{k}{c}\right)\right) \left(\left(\frac{dk}{c}\right)\right),
 \end{aligned}$$

where for a real number x ,

$$\llbracket x \rrbracket = \begin{cases} 0, & x \in \mathbb{Z}, \\ x - [x] - \frac{1}{2}, & \text{otherwise.} \end{cases}$$

The Dedekind sum $s(d, c)$ is determined by d modulo c , and has the properties

$$\begin{aligned}
 s(-d, c) &= -s(d, c), \\
 s(d^*, c) &= s(d, c),
 \end{aligned}$$

where d^* is the inverse of $d \pmod{c}$. The function Φ is almost a homomorphism from $\text{PSL}(2, \mathbb{Z})$ to $\mathbb{Z}$; more precisely, Rademacher proves that if $M'' = M'M$, then

$$\Phi(M'') = \Phi(M) + \Phi(M') - 3 \text{sign}(cc'c''). \quad (7)$$

We define

$$\Phi_t := \Phi(T^{m_t} S \cdots T^{m_1} S).$$

We have $\Phi_1 = \Phi(T^{m_1} S) = m_1$, and we find from (7) and Lemma 2.5 that

$$\Phi_t - \Phi_{t-1} = m_t - 3 \text{sign}(a_{t-1}c_{t-1}) \quad \text{for } t \geq 2, \quad \Phi_1 = m_1.$$

Then we have

$$\Phi_t = (m_1 + \cdots + m_t) - 3\{\text{sign}(a_1 a_0) + \cdots + \text{sign}(a_{t-1} a_{t-2})\}.$$

Since $a_i > 0$,

$$\Phi_t = (m_1 + \cdots + m_t) - 3(t-1).$$

Also from (7), for $U = T^{m_t} S \cdots T^{m_1} S$,

$$\Phi(\text{SU}) = \Phi(S) + \Phi(U) - 3 = (m_1 + \cdots + m_t) - 3t \quad (\text{by } \Phi(S) = 0). \quad (8)$$

For $U = \begin{pmatrix} p & b \\ q & d \end{pmatrix}$,

$$\text{SU} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p & b \\ q & d \end{pmatrix} = \begin{pmatrix} -q & -d \\ p & b \end{pmatrix},$$

and by the definition of Φ ,

$$\Phi(\text{SU}) = \frac{b-q}{p} - 12s(b, p). \quad (9)$$

Hence, from (8) and (9) with $U = T^{m_t} S \cdots T^{m_1} S = \begin{pmatrix} p & b \\ q & d \end{pmatrix}$,

$$m_1 + \cdots + m_t = \frac{b-q}{p} - 12s(b, p) + 3t.$$

As $-bq \equiv 1 \pmod{p}$,

$$s(b, p) = -s(q, p).$$

So we have the following result.

LEMMA 2.7. *Let $T^{m_t} S \cdots T^{m_1} S = \begin{pmatrix} p & b \\ q & d \end{pmatrix}$. We have*

$$(m_1 + \cdots + m_t) - \left(\frac{b}{p} + \frac{1}{pq} \right) = 12s(q, p) - \frac{q^2 + 1}{pq} + 3t.$$

We easily obtain the next lemma.

LEMMA 2.8. *It follows that $\langle \omega_0 \rangle_{U_+} = -A^{-3} e_8(3-r)$.*

Summing up, we obtain a formula for the quantum $\text{SO}(3)$ invariant of $L(p, q)$.

PROPOSITION 2.9.

$$\begin{aligned} & \tau_r(L(p, q)) \\ &= -\frac{1}{\sqrt{p}(A^2 - A^{-2})} A^{-12s(q, p) + (q^2 + 1)/pq} \left(\frac{e_8(-3r)}{2} \right)^t \\ & \quad \times \sum_{\gamma=0}^{4p-1} \left\{ x_t (-1)^\gamma + y_t e_8(m_t r) e_4((-1)^{(r-1)/2} m_t) (e_4(\gamma) + e_4(-\gamma)) + z_t (-1)^{m_t} \right\} \\ & \quad \times \sum_{\pm} \pm e_{8rp} \left(-q \left(r\gamma + 2 \pm \frac{2}{q} \right)^2 \right) \Bigg\}. \end{aligned}$$

Proof. We recall that

$$\tau_r(L(p, q)) = \langle \omega_0 \rangle_{U_t}^{-t} \langle \omega_0, \dots, \omega_0 \rangle_{D(p, q)}.$$

From Proposition 2.6, it follows that

$$\begin{aligned} & \langle \omega_0, \dots, \omega_0 \rangle_{D(p, q)} \\ &= \mu_0^{-1} \varphi(U)_t|_{k_{t+1}=0} \\ &= \mu_0^{-1} K_t \sum_{\gamma=0}^{4p-1} \left\{ \{x_t(-1)^\gamma + y_t e_8(m_t r) e_4((-1)^{(r-1)/2} m_t) (e_4(\gamma) + e_4(-\gamma)) + z_t(-1)^{m_t}\} \right. \\ & \quad \left. \times \sum_{\pm} \pm e_{8rp} \left(-q \left(r\gamma + 2 \pm \frac{2}{q} \right)^2 \right) \right\}. \end{aligned}$$

From Lemmas 2.7 and 2.8 with $U = \begin{pmatrix} p & b \\ q & d \end{pmatrix} = \begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix}$, we can obtain that

$$\begin{aligned} \langle \omega_0 \rangle_{U_+}^{-t} K_t &= (-1)^t A^{3t} e_8(tr - 3t) \frac{\sqrt{-1}}{\sqrt{rp}} \left(\frac{e_4(1)}{2} \right)^t e_8(t) A^{-(m_1 + \dots + m_t) + (b/p + 1/pq)} \\ &= \frac{\sqrt{-1}}{\sqrt{rp}} A^{-12s(q, p) + (q^2 + 1)/pq} \left(\frac{e_8(-3r)}{2} \right)^t. \end{aligned}$$

Since $\mu_0 = (A^2 - A^{-2})/\sqrt{-r}$, we obtain the proposition. $\square$

3. The semi-classical limit of $\tau_r(L(p, q))$

In this section, we prove Theorem 1.1.

First we show some propositions.

PROPOSITION 3.1. *Let p and q be odd.*

$$\begin{aligned} (1) \quad & \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (-1)^\gamma \sum_{\pm} \pm e_{8p}(-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma) \\ &= - \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} \sum_{\pm} \pm e_{8p}(-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma) \\ &= -8e_8(-qrp) \sum_{\substack{n \in \mathbb{Z}/p\mathbb{Z} \\ n \text{ is odd}}} e_{2p}(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p}. \\ (2) \quad & \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (e_4(\gamma) + e_4(-\gamma)) \sum_{\pm} \pm e_{8p}(-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma) \\ &= 16 \sum_{\substack{n \in \mathbb{Z}/p\mathbb{Z} \\ n \text{ is odd}}} e_{2p}(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p}. \end{aligned}$$

Proof. We show (1). We note that

$$e_{8p}(-qr(\gamma + 2p)^2) e_{2p}(-(q \pm 1)(\gamma + 2p)) = e_{8p}(-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma) (-1)^{q(p+\gamma)}.$$

Since p and q are odd, by replacing γ with $\gamma + 2p$, we see that

$$\begin{aligned} & \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (-1)^\gamma \sum_{\pm} \pm e_{8p}(-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma) \\ &= - \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} \sum_{\pm} \pm e_{8p}(-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma), \end{aligned}$$

and so we can restrict this sum for odd γ . Therefore, by putting $\gamma = 2n + p$, we have

$$\begin{aligned} & \sum_{\substack{\gamma \in \mathbb{Z}/4p\mathbb{Z} \\ \gamma \text{ is odd}}} (-1)^\gamma \sum_{\pm} \pm e_{8p}(-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma) \\ &= -e_8(-qrp) \sum_{n \in \mathbb{Z}/2p\mathbb{Z}} (-1)^n \sum_{\pm} \pm e_{2p}(-qrn^2) e_p(-(q \pm 1)n). \end{aligned}$$

We note that

$$\begin{aligned} & (-1)^{n+p} e_{2p}(-qr(n+p)^2) e_p(-(q \pm 1)(n+p)) \\ &= (-1)^{p(q+1)} (-1)^n e_{2p}(-qrn^2) e_p(-(q \pm 1)n). \end{aligned}$$

Then, since $p(q+1)$ is even, it follows that

$$\begin{aligned} & \sum_{n \in \mathbb{Z}/2p\mathbb{Z}} (-1)^n \sum_{\pm} \pm e_{2p}(-qrn^2) e_p(-(q \pm 1)n) \\ &= 2 \sum_{k \in \mathbb{Z}/p\mathbb{Z}} (-1)^k \sum_{\pm} \pm e_{2p}(-qrk^2) e_p(-(q \pm 1)k) \\ &= \sum_{k \in \mathbb{Z}/p\mathbb{Z}} (-1)^k e_{2p}(-qrk^2) \\ & \quad \times \{e_p(-(q+1)k) - e_p(-(q-1)k) + e_p((q+1)k) - e_p((q-1)k)\} \\ &= \sum_{k \in \mathbb{Z}/p\mathbb{Z}} (-1)^k e_{2p}(-qrk^2) (e_p(qk) - e_p(-qk)) (e_p(k) - e_p(-k)) \\ &= -4 \sum_{k \in \mathbb{Z}/p\mathbb{Z}} (-1)^k e_{2p}(-qrk^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p}, \end{aligned} \tag{10}$$

where we obtain the second equality by making the average of the second line for $k = \pm n$.

Since we have that

$$e_{2p}(-qr(p-k)^2) \sin \frac{2\pi q(p-k)}{p} \sin \frac{2\pi(p-k)}{p} = -e_{2p}(-qrk^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p},$$

by replacing even k with $p-k$,

$$\begin{aligned} & \sum_{n \in \mathbb{Z}/2p\mathbb{Z}} (-1)^n \sum_{\pm} \pm e_{2p}(-qrn^2) e_p(-(q \pm 1)n) \\ &= 8 \sum_{\substack{k \in \mathbb{Z}/p\mathbb{Z} \\ k \text{ is odd}}} e_{2p}(-qrk^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p}. \end{aligned} \tag{11}$$

Hence, (1) holds.

We show (2). As in (1), we see that

$$\begin{aligned}
 & \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (e_4(\gamma) + e_4(-\gamma)) \sum_{\pm} \pm e_{8p}(-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma) \\
 &= \sum_{\substack{\gamma \in \mathbb{Z}/4p\mathbb{Z} \\ \gamma \text{ is even}}} (e_4(\gamma) + e_4(-\gamma)) \sum_{\pm} \pm e_{8p}(-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma) \\
 &= 2 \sum_{n \in \mathbb{Z}/2p\mathbb{Z}} (-1)^n \sum_{\pm} \pm e_{2p}(-qrn^2) e_p(-(q \pm 1)n).
 \end{aligned}$$

Hence, from (11), we obtain (2). $\square$

PROPOSITION 3.2. *Let p be even and q be odd.*

$$\begin{aligned}
 (1) \quad & \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (-1)^\gamma \sum_{\pm} \pm e_{8p}(-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma) \\
 &= \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} \sum_{\pm} \pm e_{8p}(-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma) \\
 &= \begin{cases} -4 \sum_{\substack{n \in \mathbb{Z}/p\mathbb{Z} \\ n \text{ is even}}} e_{2p}(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } p \equiv 0(8), \\ -4 \sum_{\substack{n \in \mathbb{Z}/p\mathbb{Z} \\ n \text{ is odd}}} e_{2p}(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } p \equiv 4(8), \\ -4(1 - (-1)^{p/4}) \sum_{\substack{n \in \mathbb{Z}/p\mathbb{Z} \\ n \text{ is odd}}} e_{2p}(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } p \equiv \pm 2(8). \end{cases} \\
 (2) \quad & \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (e_4(\gamma) + e_4(-\gamma)) \sum_{\pm} \pm e_{8p}(-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma) \\
 &= \begin{cases} -8 \sum_{\substack{n \in \mathbb{Z}/p\mathbb{Z} \\ n \text{ is even}}} e_{2p}(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } p \equiv 0(8), \\ 8 \sum_{\substack{n \in \mathbb{Z}/p\mathbb{Z} \\ n \text{ is odd}}} e_{2p}(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } p \equiv 4(8), \\ -8(-1 - (-1)^{p/4}) \sum_{\substack{n \in \mathbb{Z}/p\mathbb{Z} \\ n \text{ is odd}}} e_{2p}(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } p \equiv \pm 2(8). \end{cases}
 \end{aligned}$$

Proof. We show (1). We see that

$$\begin{aligned}
 & \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (-1)^\gamma \sum_{\pm} \pm e_{8p}(-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma) \\
 &= \sum_{\substack{\gamma \in \mathbb{Z}/4p\mathbb{Z} \\ \gamma \text{ is even}}} (-1)^\gamma \sum_{\pm} \pm e_{8p}(-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma) \\
 &= \sum_{n \in \mathbb{Z}/2p\mathbb{Z}} \sum_{\pm} \pm e_{2p}(-qrn^2) e_p(-(q \pm 1)n).
 \end{aligned}$$

Since $p(q+1)$ is even, from the equality (10), it follows that

$$\begin{aligned} & \sum_{n \in \mathbb{Z}/2p\mathbb{Z}} (-1)^n \sum_{\pm} \pm e_{2p}(-qrn^2) e_p(-(q \pm 1)n) \\ &= -4 \sum_{k \in \mathbb{Z}/p\mathbb{Z}} (-1)^k e_{2p}(-qrk^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p}. \end{aligned}$$

We note that

$$\begin{aligned} & (-1)^{p/2+k} e_{2p} \left(-qr \left(\frac{p}{2} + k \right)^2 \right) \sin \frac{2\pi q}{p} \left(\frac{p}{2} + k \right) \sin \frac{2\pi}{p} \left(\frac{p}{2} + k \right) \\ &= (-1)^{p/2} e_8(-qrp) e_{2p}(-qrk^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p}. \end{aligned}$$

Therefore, if $p \equiv 0, 4(8)$, then we get that

$$\begin{aligned} & \sum_{k \in \mathbb{Z}/p\mathbb{Z}} (-1)^k e_{2p}(-qrk^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p} \\ &= \begin{cases} \sum_{\substack{n \in \mathbb{Z}/p\mathbb{Z} \\ n \text{ is even}}} e_{2p}(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } p \equiv 0(8), \\ - \sum_{\substack{n \in \mathbb{Z}/p\mathbb{Z} \\ n \text{ is odd}}} e_{2p}(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } p \equiv 4(8), \end{cases} \end{aligned}$$

and so the equalities in these cases of (1) hold. If $p \equiv \pm 2(4)$, then, by replacing even k with $p/2 + k$, we see that the equality in this case of (1) holds.

Hence, (1) holds.

In the same way as Proposition 3.1 (2), we get (2). $\square$

PROPOSITION 3.3. *Let p be odd and q be even.*

$$\begin{aligned} (1) \quad & \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (-1)^\gamma \sum_{\pm} \pm e_{8p}(-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma) \\ &= \begin{cases} -16 \sum_{\substack{n \in \mathbb{Z}/p\mathbb{Z} \\ n \text{ is odd}}} e_{2p}(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } q \equiv 0(8), \\ 0 & \text{if } q \equiv 4(8), \\ -8(1 + e_8(-qrp)) \sum_{\substack{n \in \mathbb{Z}/p\mathbb{Z} \\ n \text{ is odd}}} e_{2p}(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } q \equiv \pm 2(8). \end{cases} \end{aligned}$$

$$(2) \quad \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (e_4(\gamma) + e_4(-\gamma)) \sum_{\pm} \pm e_{8p}(-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma) = 0.$$

$$\begin{aligned}
(3) \quad & \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} \sum_{\pm} \pm e_{8p}(-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma) \\
&= \begin{cases} 0 & \text{if } q \equiv 0(8), \\ -16 \sum_{\substack{n \in \mathbb{Z}/p\mathbb{Z} \\ n \text{ is odd}}} e_{2p}(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } q \equiv 4(8), \\ -8(1 - e_8(-qrp)) \sum_{\substack{n \in \mathbb{Z}/p\mathbb{Z} \\ n \text{ is odd}}} e_{2p}(-qrn^2) \sin \frac{2\pi qn}{p} \sin \frac{2\pi n}{p} & \text{if } q \equiv \pm 2(8). \end{cases}
\end{aligned}$$

Proof. We show (1). We see that

$$\begin{aligned}
& \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (-1)^\gamma \sum_{\pm} \pm e_{8p}(-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma) \\
&= 2 \sum_{\gamma \in \mathbb{Z}/2p\mathbb{Z}} (-1)^\gamma \sum_{\pm} \pm e_{8p}(-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma).
\end{aligned}$$

We note that

$$\begin{aligned}
& (-1)^{n+p} e_{8p}(-qr(n+p)^2) e_{2p}(-(q \pm 1)(n+p)) \\
&= (-1)^{p+q+1} e_8(-qrp) (-1)^{n+(q/2)n} e_{8p}(-qrn^2) e_{2p}(-(q \pm 1)n).
\end{aligned}$$

By replacing n by $n + p$, noting that $p + q + 1$ is even, we have that

$$\begin{aligned}
& \sum_{n \in \mathbb{Z}/2p\mathbb{Z}} (-1)^n \sum_{\pm} \pm e_{8p}(-qrn^2) e_{2p}(-(q \pm 1)n) \\
&= e_8(-qrp) \sum_{n \in \mathbb{Z}/2p\mathbb{Z}} (-1)^{n+(q/2)n} \sum_{\pm} \pm e_{8p}(-qrn^2) e_{2p}(-(q \pm 1)n).
\end{aligned}$$

We consider the right-hand side in the three cases.

If $q \equiv 0(8)$, then we have that $(-1)^{n+(q/2)n} = (-1)^n$ and $e_8(-qrp) = 1$. Therefore, from the equality (10), we obtain that

$$\begin{aligned}
& \sum_{n \in \mathbb{Z}/2p\mathbb{Z}} (-1)^n \sum_{\pm} \pm e_{8p}(-qrn^2) e_{2p}(-(q \pm 1)n) \\
&= -4 \sum_{k \in \mathbb{Z}/p\mathbb{Z}} (-1)^k e_{8p}(-qrk^2) \sin \frac{\pi qk}{p} \sin \frac{\pi k}{p} \\
&= 8 \sum_{\substack{k \in \mathbb{Z}/p\mathbb{Z} \\ k \text{ is odd}}} e_{8p}(-qrk^2) \sin \frac{\pi qk}{p} \sin \frac{\pi k}{p} \\
&= -8 \sum_{\substack{k \in \mathbb{Z}/p\mathbb{Z} \\ k \text{ is odd}}} e_{2p}(-qrk^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p},
\end{aligned}$$

where we obtain the second equality by replacing even k with $p - k$ and the third equality by replacing odd k with $p - 2k$ in the previous line.

If $q \equiv 4(8)$, then we have that $(-1)^{n+(q/2)n} = (-1)^n$ and $e_8(-qrp) = -1$. Therefore, we have that

$$\sum_{n \in \mathbb{Z}/2p\mathbb{Z}} (-1)^n \sum_{\pm} \pm e_{8p}(-qrn^2) e_{2p}(-(q \pm 1)n) = 0.$$

If $q \equiv \pm 2(8)$, then we have that $(-1)^{n+(q/2)n} = 1$. By replacing odd n with $p + n$,

$$\begin{aligned}
 & \sum_{n \in \mathbb{Z}/2p\mathbb{Z}} (-1)^n \sum_{\pm} \pm e_{8p}(-qrn^2) e_{2p}(-(q \pm 1)n) \\
 &= (1 + e_8(-qrp)) \sum_{\substack{n \in \mathbb{Z}/2p\mathbb{Z} \\ n \text{ is even}}} \sum_{\pm} \pm e_{8p}(-qrn^2) e_{2p}(-(q \pm 1)n) \\
 &= (1 + e_8(-qrp)) \sum_{k \in \mathbb{Z}/p\mathbb{Z}} \sum_{\pm} \pm e_{2p}(-qrk^2) e_p(-(q \pm 1)k) \\
 &= -4(1 + e_8(-qrp)) \sum_{\substack{1 \leq k < p \\ k \text{ is odd}}} e_{2p}(-qrk^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p},
 \end{aligned}$$

where we obtain the second equality by replacing n with $2k$ and the last equality in the same way as the equality (10). Hence, we obtain (1). Similarly, (3) holds.

In the same way as Proposition 3.1 (2), we get (2). $\square$

We put

$$X(m) := e_8(mr) e_4((-1)^{(r-1)/2} m),$$

for an odd $r \geq 3$ and an integer m . From Proposition 2.6, x_t, z_t and y_t are determined by

$$\begin{aligned}
 x_t &= x_{t-1} + (-1)^{m_{t-1}} z_{t-1} + 2X(m_{t-1})(x_{t-2} - (-1)^{m_{t-2}} z_{t-2}), & x_1 &= 1, \\
 z_t &= x_{t-1} + (-1)^{m_{t-1}} z_{t-1} - 2X(m_{t-1})(x_{t-2} - (-1)^{m_{t-2}} z_{t-2}), & z_1 &= 0, \\
 y_t &= x_{t-1} - (-1)^{m_{t-1}} z_{t-1}, & y_1 &= 0,
 \end{aligned}$$

and it follows that $x_2 = 1$ and $z_2 = 1$. Then, we have

$$x_t + z_t = 2(x_{t-1} + (-1)^{m_{t-1}} z_{t-1}), \quad x_t - z_t = 4X(m_{t-1})(x_{t-2} - (-1)^{m_{t-2}} z_{t-2}). \quad (12)$$

Let t be an integer with $t \geq 3$ and $m_1, \dots, m_t$ be integers with $m_i \geq 2$. Let $I_1^\pm$ and $I_2^\pm$ be empty sets. For $t \geq 3$, we define the subsets I_t^- and I_t^+ of $\{2, \dots, t-1\}$ inductively in the following way: if m_t is even, then $I_t^- := \{t-1\} \cup I_{t-2}^-$; and $I_t^+ := I_{t-1}^+$ and if m_t is odd, then $I_t^- := I_{t-1}^+$ and $I_t^+ := \{t-1\} \cup I_{t-2}^-$.

We put $M(I) := \sum_{j=1}^s m_{i_j}$ for $I = \{i_1, \dots, i_s\}$ with integers m_{i_j} .

PROPOSITION 3.4. For $t \geq 2$,

$$x_t \pm (-1)^{m_t} z_t = 2^{t-2} X(M(I_t^\pm)) d_t^\pm, \quad (13)$$

where for $t \geq 3$,

$$d_t^\pm := \begin{cases} 2 & \text{if } 2 \in I_t^\pm, \\ 1 + (-1)^{m_2} & \text{if } 2, 3 \notin I_t^\pm, \\ 1 - (-1)^{m_2} & \text{if } 2 \notin I_t^\pm, 3 \in I_t^\pm, \end{cases}$$

and $d_2^\pm := 1 \pm (-1)^{m_2}$.

Proof. The claim for $t = 2$ is obvious. We will prove the claim for $t \geq 3$ by induction on t .

When $t = 3$, from the formula (12), we have

$$x_3 - (-1)^{m_3} z_3 = \begin{cases} x_3 - z_3 = 4X(m_2)(x_1 - (-1)^{m_1} z_1) = 4X(m_2) & \text{if } m_3 \text{ is even,} \\ x_3 + z_3 = 2(x_2 + (-1)^{m_2} z_2) = 2(1 + (-1)^{m_2}) & \text{if } m_3 \text{ is odd.} \end{cases}$$

On the other hand, if m_3 is even, then $I_3^- = \{2\} \cup I_1^- = \{2\}$ and so

$$2^{3-2}X(M(I_3^-))d_3^- = 2X(m_2)2 = 4X(m_2).$$

If m_3 is odd, then $I_3^- = I_2^+ = \emptyset$ and so

$$2^{3-2}X(M(I_3^-))d_3^- = 2X(0)(1 + (-1)^{m_2}) = 2(1 + (-1)^{m_2}).$$

Since $x_3 + (-1)^{m_3}z_3 = x_3 - (-1)^{m_3+1}z_3$, the claim for $t = 3$ holds.

When $t = 4$, from the formula (12), we have

$$\begin{aligned} x_4 - (-1)^{m_4}z_4 \\ = \begin{cases} x_4 - z_4 = 4X(m_3)(x_2 - (-1)^{m_2}z_2) = 4X(m_3)(1 - (-1)^{m_2}) & \text{if } m_4 \text{ is even,} \\ x_4 + z_4 = 2(x_3 + (-1)^{m_3}z_3) = 2 \cdot 2X(M(I_3^+))d_3^+ & \text{if } m_4 \text{ is odd.} \end{cases} \end{aligned}$$

On the other hand, if m_4 is even, then $I_4^- = \{3\} \cup I_2^- = \{3\}$ and so

$$2^{4-2}X(M(I_4^-))d_4^- = 4X(m_3)(1 - (-1)^{m_2}).$$

If m_4 is odd, then $I_4^- = I_3^+$, $d_4^- = d_3^+$ and so

$$2^{4-2}X(M(I_4^-))d_4^- = 2^2X(M(I_3^+))d_3^+.$$

Since $x_4 + (-1)^{m_4}z_4 = x_4 - (-1)^{m_4+1}z_4$, the claim for $t = 4$ holds.

We assume the claim up to $t \geq 4$ inductively. If m_{t+1} is even, then from the induction hypothesis, we have

$$\begin{aligned} x_{t+1} - (-1)^{m_{t+1}}z_{t+1} &= x_{t+1} - z_{t+1} = 4X(m_t)(x_{t-1} - (-1)^{m_{t-1}}z_{t-1}) \\ &= 4X(m_t)2^{t-3}X(M(I_{t-1}^-))d_{t-1}^- = 2^{t-1}X(m_t)X(M(I_{t-1}^-))d_{t-1}^-. \end{aligned}$$

On the other hand, we have that $I_{t+1}^- = \{t\} \cup I_{t-1}^-$ and $d_{t+1}^- = d_{t-1}^-$ since $t \geq 4$. Thus, we get

$$2^{t+1-2}X(M(I_{t+1}^-))d_{t+1}^- = 2^{t-1}X(m_t)X(M(I_{t-1}^-))d_{t-1}^-.$$

If m_{t+1} is odd, then from the induction hypothesis, we have

$$\begin{aligned} x_{t+1} - (-1)^{m_{t+1}}z_{t+1} &= x_{t+1} + z_{t+1} = 2(x_t + (-1)^{m_t}z_t) \\ &= 2 \cdot 2^{t-2}X(M(I_t^+))d_t^+ = 2^{t-1}X(M(I_t^+))d_t^+. \end{aligned}$$

On the other hand, we have that $I_{t+1}^- = I_t^+$ and $d_{t+1}^- = d_t^+$. Thus we get

$$2^{t+1-2}X(M(I_{t+1}^-))d_{t+1}^- = 2^{t-1}X(M(I_t^+))d_t^+.$$

Hence, the claim for $x_{t+1} - (-1)^{m_{t+1}}z_{t+1}$ holds. Noting that $x_{t+1} + (-1)^{m_{t+1}}z_{t+1} = x_{t+1} - (-1)^{m_{t+1}+1}z_{t+1}$, the claim for $x_{t+1} + (-1)^{m_{t+1}}z_{t+1}$ holds. $\square$

LEMMA 3.5. Let $t \geq 3$ and $m_i \geq 2$ be integers. We put $p_\ell/q_\ell := [m_1, \dots, m_\ell]$. We define the condition (C_I) for I_δ^ε by

$$(C_I) \quad \begin{cases} 2 \in I_\delta^\varepsilon & \text{if } m_1 \text{ is odd,} \\ 2, 3 \notin I_\delta^\varepsilon \ (t \geq 4), \ I_\delta^\varepsilon = I_3^- \ (t = 3) & \text{if } m_1 \text{ is even, } m_2 \text{ is even,} \\ 2 \notin I_\delta^\varepsilon, 3 \in I_\delta^\varepsilon \ (t \geq 4), \ I_\delta^\varepsilon = I_2^- \ (t = 3) & \text{if } m_1 \text{ is even, } m_2 \text{ is odd.} \end{cases}$$

- (a) If p_t and q_t are odd, then for $I_\delta^\varepsilon = I_t^-, I_{t-1}^-$, the term in (13) for I_δ^ε satisfying (C_I) is non-zero and the term in (13) for another I_δ^ε is zero.
- (b) If p_t is even and q_t is odd, then for $I_\delta^\varepsilon = I_t^+, I_{t-1}^-$, the term in (13) for I_δ^ε satisfying (C_I) is non-zero and the term in (13) for another I_δ^ε is zero.
- (c) If p_t is odd and q_t is even, then for $I_\delta^\varepsilon = I_t^+, I_t^-$, the term in (13) for I_δ^ε satisfying (C_I) is non-zero and the term in (13) for another I_δ^ε is zero.

Proof. We will show the lemma by induction on t .

We consider (a) for $t = 3$. Since p_3 and q_3 are odd, we have p_2 is odd. When m_3 is even, q_2 is odd, and so m_1 is odd and m_2 is even. Since $I_3^- = \{2\} \cup I_1^- = \{2\}$ and $I_2^- = \emptyset$, I_3^- satisfies (C_I) and from Proposition 3.4,

$$\begin{aligned} x_3 - (-1)^{m_3} z_3 &= 2^{3-2} X(M(I_3^-)) d_3^- = 4X(m_2) \neq 0, \\ x_2 - (-1)^{m_2} z_2 &= 2^0 X(M(I_2^-)) d_2^- = 1 - (-1)^{m_2} = 0. \end{aligned}$$

When m_3 is odd, q_2 is even and so m_1 is even. Since $I_3^- = I_2^+ = \emptyset$ and $I_2^- = \emptyset$, if m_2 is even, then I_3^- satisfies (C_I) and from Proposition 3.4,

$$\begin{aligned} x_3 - (-1)^{m_3} z_3 &= 2X(M(I_3^-)) d_3^- = 2X(M(I_3^-))(1 + (-1)^{m_2}) = 4, \\ x_2 - (-1)^{m_2} z_2 &= X(M(I_2^-)) d_2^- = 1 - (-1)^{m_2} = 0, \end{aligned}$$

and if m_2 is odd, then I_2^- satisfies (C_I) and

$$\begin{aligned} x_3 - (-1)^{m_3} z_3 &= 2X(M(I_3^-))(1 + (-1)^{m_2}) = 0, \\ x_2 - (-1)^{m_2} z_2 &= 1 - (-1)^{m_2} = 2. \end{aligned}$$

Hence, the claim (a) for $t = 3$ holds.

Similarly, the claims (b) and (c) for $t = 3$ hold.

Assume that the claims (a), (b) and (c) hold for $3, \dots, t$. We consider (a) for $t + 1$. Since p_{t+1} and q_{t+1} are odd, we have that p_t is odd.

When m_{t+1} is even, q_t is odd and from (12),

$$x_{t+1} - (-1)^{m_{t+1}} z_{t+1} = 4X(m_t)(x_{t-1} - (-1)^{t-1} z_{t-1}).$$

Since p_t and q_t are odd, by the induction hypothesis (a), if I_t^- satisfies (C_I) , then $x_t - (-1)^{m_t} z_t \neq 0$ and $x_{t-1} - (-1)^{m_{t-1}} z_{t-1} = 0$. Then, I_t^- satisfies (C_I) , $x_t - (-1)^{m_t} z_t \neq 0$, and $x_{t+1} - (-1)^{m_{t+1}} z_{t+1} = 0$. If I_{t-1}^- satisfies (C_I) , then $x_{t-1} - (-1)^{m_{t-1}} z_{t-1} \neq 0$ and $x_t - (-1)^{m_t} z_t = 0$. Since $I_{t+1}^- = \{t\} \cup I_{t-1}^-$, I_{t+1}^- satisfies (C_I) , $x_{t+1} - (-1)^{m_{t+1}} z_{t+1} \neq 0$, and $x_t - (-1)^{m_t} z_t = 0$. We note that when $t = 3$, if I_2^- satisfies (C_I) , then m_1 is even and m_2 is odd and $I_4^- = \{3\} \cup I_2^- = \{3\}$ satisfies (C_I) .

When m_{t+1} is odd, q_t is even and from (12),

$$x_{t+1} - (-1)^{m_{t+1}} z_{t+1} = 2(x_t + (-1)^{m_t} z_t).$$

Since p_t is odd and q_t is even, by the induction hypothesis (c), if I_t^- satisfies (C_I) , then $x_t - (-1)^{m_t} z_t \neq 0$ and $x_t + (-1)^{m_t} z_t = 0$. Then, I_t^- satisfies (C_I) , $x_t - (-1)^{m_t} z_t \neq 0$, and $x_{t+1} - (-1)^{m_{t+1}} z_{t+1} = 0$. If I_t^+ satisfies (C_I) , then $x_t + (-1)^{m_t} z_t \neq 0$ and $x_t - (-1)^{m_t} z_t = 0$. Since $I_{t+1}^- = I_t^+$, I_{t+1}^- satisfies (C_I) , $x_{t+1} - (-1)^{m_{t+1}} z_{t+1} \neq 0$, and $x_t - (-1)^{m_t} z_t = 0$.

Hence the claim (a) for $t + 1$ holds. Similarly, the claims (b) and (c) for $t + 1$ hold. $\square$

Let us compute the asymptotic expansion of $\tau_r(L(p, q))$. Since, as $r \rightarrow \infty$,

$$\sum_{\pm} \pm e_{8rp} \left(-q \left(r\gamma + 2 \pm \frac{2}{q} \right)^2 \right) \sim \sum_{\pm} \pm e_{8p} (-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma),$$

we have that

$$\begin{aligned} & \sum_{\gamma=0}^{4p-1} \left\{ x_t (-1)^\gamma + e_8(m_t r) e_4((-1)^{(r-1)/2} m_t) (e_4(\gamma) + e_4(-\gamma)) y_t + (-1)^{m_t} z_t \right\} \\ & \quad \times \sum_{\pm} \pm e_{8rp} \left(-q \left(r\gamma + 2 \pm \frac{2}{q} \right)^2 \right) \Big\} \\ & \sim F(p, q, r), \end{aligned}$$

where

$$\begin{aligned} & F(p, q, r) \\ &= x_t \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (-1)^\gamma \sum_{\pm} \pm e_{8p} (-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma) + e_8(m_t r) e_4((-1)^{(r-1)/2} m_t) y_t \\ & \quad \times \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} (e_4(\gamma) + e_4(-\gamma)) \sum_{\pm} \pm e_{8p} (-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma) \\ & \quad + (-1)^{m_t} z_t \sum_{\gamma \in \mathbb{Z}/4p\mathbb{Z}} \sum_{\pm} \pm e_{8p} (-qr\gamma^2) e_{2p}(-(q \pm 1)\gamma). \end{aligned}$$

We give an explicit formula of $F(p, q, r)$.

PROPOSITION 3.6. *Let $p/q = p_t/q_t = [m_1, \dots, m_t]$ and put*

$$\begin{aligned} G_e(p, q, r) &:= \sum_{\substack{1 \leq k < p \\ k \text{ is even}}} e_{2p}(-qrk^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p}, \\ G(p, q, r) &:= \sum_{\substack{1 \leq k < p \\ k \text{ is odd}}} e_{2p}(-qrk^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p}. \end{aligned}$$

(1) *If p and q are odd, then*

$$F(p, q, r) = 2^{t+2} e_8(n_0(p, q)r) e_4((-1)^{(r-1)/2} n_1(p, q)) G(p, q, r),$$

where for $t \geq 3$,

$$(n_0(p, q), n_1(p, q)) = \begin{cases} (4 - pq + M(I_t^-), M(I_t^-)) & \text{if } (C_t) \text{ with } I_\delta^\varepsilon = I_t^-, \\ (m_t + M(I_{t-1}^-), m_t + M(I_{t-1}^-)) & \text{if } (C_t) \text{ with } I_\delta^\varepsilon = I_{t-1}^-, \end{cases}$$

for $t = 2$, $(n_0(p, q), n_1(p, q)) = (m_2, m_2)$, and for $t = 1$, $(n_0(p, q), n_1(p, q)) = (4 - pq, 0)$.

(2) *If p is even and q is odd, then*

$$\begin{aligned} & F(p, q, r) \\ &= \begin{cases} 2^{t+1} e_8(n_0(p, q)r) e_4((-1)^{(r-1)/2} n_1(p, q)) G_e(p, q, r) & \text{if } p \equiv 0(8), \\ 2^{t+1} e_8(n_0(p, q)r) e_4((-1)^{(r-1)/2} n_1(p, q)) G(p, q, r) & \text{if } p \equiv 4(8), \\ 2^{t+1} e_8(n_0(p, q)r) e_4((-1)^{(r-1)/2} n_1(p, q)) (1 - (-1)^{p/4}) G(p, q, r) & \text{if } p \equiv \pm 2(8), \end{cases} \end{aligned}$$

where for $t \geq 3$,

$$(n_0(p, q), n_1(p, q)) = \begin{cases} (4 + M(I_t^+), M(I_t^+)) & \text{if } p \equiv 0(8), (C_I) \text{ with } I_\delta^\varepsilon = I_t^+, \\ (4 + m_t + M(I_{t-1}^-), m_t + M(I_{t-1}^-)) & \text{if } p \equiv 0(8), (C_I) \text{ with } I_\delta^\varepsilon = I_{t-1}^-, \\ (4 + M(I_t^+), M(I_t^+)) & \text{if } p \equiv 4(8), (C_I) \text{ with } I_\delta^\varepsilon = I_t^+, \\ (m_t + M(I_{t-1}^-), m_t + M(I_{t-1}^-)) & \text{if } p \equiv 4(8), (C_I) \text{ with } I_\delta^\varepsilon = I_{t-1}^-, \\ (4 + M(I_t^+), M(I_t^+)) & \text{if } p \equiv \pm 2(8), (C_I) \text{ with } I_\delta^\varepsilon = I_t^+, \\ (-pq + m_t + M(I_{t-1}^-), m_t + M(I_{t-1}^-)) & \text{if } p \equiv \pm 2(8), (C_I) \text{ with } I_\delta^\varepsilon = I_{t-1}^-, \end{cases}$$

for $t = 2$,

$$(n_0(p, q), n_1(p, q)) = \begin{cases} (4 + m_2, m_2) & \text{if } p \equiv 0(8), \\ (m_2, m_2) & \text{if } p \equiv 4(8), \\ (-pq + m_2, m_2) & \text{if } p \equiv \pm 2(8). \end{cases}$$

and for $t = 1$, $(n_0(p, q), n_1(p, q)) = (4, 0)$.

(3) If p is odd and q is even, then

$$F(p, q, r) = 2^{t+2} e_8(n_0(p, q)r) e_4((-1)^{(r-1)/2} n_1(p, q)) G(p, q, r),$$

where for $t \geq 3$,

$$(n_0(p, q), n_1(p, q)) = \begin{cases} (4 + M(I_t^\pm), M(I_t^\pm)) & \text{if } q \equiv 0(8), (C_I) \text{ with } I_\delta^\varepsilon = I_t^\pm, \\ (4 + M(I_t^+), M(I_t^+)) & \text{if } q \equiv 4(8), (C_I) \text{ with } I_\delta^\varepsilon = I_t^+, \\ (M(I_t^-), M(I_t^-)) & \text{if } q \equiv 4(8), (C_I) \text{ with } I_\delta^\varepsilon = I_t^-, \\ (4 + M(I_t^+), M(I_t^+)) & \text{if } q \equiv \pm 2(8), (C_I) \text{ with } I_\delta^\varepsilon = I_t^+, \\ (4 - pq + M(I_t^-), M(I_t^-)) & \text{if } q \equiv \pm 2(8), (C_I) \text{ with } I_\delta^\varepsilon = I_t^-, \end{cases}$$

and for $t = 2$, when m_2 is odd,

$$(n_0(p, q), n_1(p, q)) = \begin{cases} (4, 0) & \text{if } q \equiv 0(8), \\ (0, 0) & \text{if } q \equiv 4(8), \\ (4 - pq, 0) & \text{if } q \equiv \pm 2(8), \end{cases}$$

and when m_2 is even, $(n_0(p, q), n_1(p, q)) = (4, 0)$.

Proof. From Propositions 3.1, 3.2 and 3.3, we obtain the following formulas.

If p and q are odd, then

$$F(p, q, r) = \{-8e_8(-qrp)(x_t - (-1)^{m_t} z_t) + 16e_8(m_t r) e_4((-1)^{(r-1)/2} m_t) y_t\} G(p, q, r),$$

if p is even and q is odd, then

$$F(p, q, r) = \begin{cases} \{-4(x_t + (-1)^{m_t} z_t) - 8e_8(m_t r) e_4((-1)^{(r-1)/2} m_t) y_t\} G_e(p, q, r) & \text{if } p \equiv 0(8), \\ \{-4(x_t + (-1)^{m_t} z_t) + 8e_8(m_t r) e_4((-1)^{(r-1)/2} m_t) y_t\} G(p, q, r) & \text{if } p \equiv 4(8), \\ \{-4(x_t + (-1)^{m_t} z_t) + 8e_8(-qpr) e_8(m_t r) e_4((-1)^{(r-1)/2} m_t) y_t\} \\ \quad \times (1 - (-1)^{p/4}) G(p, q, r) & \text{if } p \equiv \pm 2(8), \end{cases}$$

and if p is odd and q is even, then

$$F(p, q, r) = \begin{cases} -16x_t G(p, q, r) & \text{if } q \equiv 0(8), \\ -16(-1)^{m_t} z_t G(p, q, r) & \text{if } q \equiv 4(8), \\ (-8(x_t + (-1)^{m_t} z_t) - 8e_8(-qrp)(x_t - (-1)^{m_t} z_t))G(p, q, r) & \text{if } q \equiv \pm 2(8). \end{cases}$$

We note that

$$x_t = \frac{1}{2}(x_t + (-1)^{m_t} z_t + x_t - (-1)^{m_t} z_t), \\ (-1)^{m_t} z_t = \frac{1}{2}(x_t + (-1)^{m_t} z_t - (x_t - (-1)^{m_t} z_t)).$$

We recall that $y_t = x_{t-1} - (-1)^{m_{t-1}} z_{t-1}$. From Lemma 3.5 for $t \geq 3$ and a direct calculation for $t = 1, 2$, we obtain the claim, using $-1 = e_8(4r)$. $\square$

We are now ready to prove Theorem 1.1.

Let ρ_k be a representation of $\pi_1(L(p, q)) = \mathbb{Z}/p\mathbb{Z}$ to $\mathrm{SL}_2\mathbb{C}$ such that the eigenvalues of $\rho_k(g)$ for a generator g of $\pi_1(L(p, q)) = \mathbb{Z}/p\mathbb{Z}$ are $e^{\pm 2\pi i k/p}$.

THEOREM 1.1. *Let p and q be coprime integers with $p \geq 3$ and $p > q > 0$. Then, the quantum invariant $\tau_r(L(p, q))$ of $L(p, q)$ for odd r is expanded as $r \rightarrow \infty$ in the form*

$$\tau_r(L(p, q)) \sim e^{\pi i/2} e^{(\pi i/4)(n_0(p, q) - 3r)r} e^{(\pi i/2)(-1)^{(r-1)/2} n_1(p, q)} c(p)r \\ \times \begin{cases} \sum_{\substack{1 \leq k < p \\ k \text{ is even}}} e^{\pi i \mathrm{CS}(L(p, q); ad \circ \rho_k)r} \omega(L(p, q); \rho_k) & \text{if } p \equiv 0(8), \\ \sum_{\substack{1 \leq k < p \\ k \text{ is odd}}} e^{\pi i \mathrm{CS}(L(p, q); ad \circ \rho_k)r} \omega(L(p, q); \rho_k) & \text{otherwise,} \end{cases}$$

where $n_0(p, q)$ and $n_1(p, q)$ are defined in Proposition 3.6 and $p/q = [m_1, \dots, m_t]$ with $m_i \geq 2$,

$$c(p) := \begin{cases} 1 & \text{if } p \text{ is odd,} \\ \frac{1}{2} & \text{if } p \equiv 0(4), \\ \frac{1}{2}(1 + (-1)^{p/4}) & \text{if } p \equiv 2(4), \end{cases}$$

$\mathrm{CS}(L(p, q); ad \circ \rho_k)$ is the Chern–Simons invariant, and we put

$$\omega(L(p, q); \rho_k) = \frac{1}{\pi \sqrt{p}} \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p}.$$

Further, we have that

$$\omega(L(p, q); \rho_k)^2 = \pm \frac{1}{32\pi^2} \mathrm{Tor}(L(p, q); ad \circ \rho_k),$$

where $\mathrm{Tor}(L(p, q); ad \circ \rho_k)$ is the Reidemeister torsion.

Proof. We note that, as $r \rightarrow \infty$,

$$\frac{1}{A^2 - A^{-2}} = \frac{1}{2i \sin(2\pi/r)} \sim \frac{1}{4\pi i} r.$$

From Proposition 2.9, the definition of $F(p, q, r)$ and Proposition 3.6, we have

$$\begin{aligned}\tau_r(L(p, q)) &\sim \frac{e_4(1)}{4\pi\sqrt{p}} r \left(\frac{e_8(-3r)}{2} \right)^l F(p, q, r) \\ &= \frac{e_4(1)}{\pi\sqrt{p}} e_8((n_0(p, q) - 3t)r) e_4((-1)^{(r-1)/2} n_1(p, q)) c(p) r \\ &\quad \times \begin{cases} \sum_{\substack{1 \leq k < p \\ k \text{ is even}}} e_{2p}(-qrk^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p} & \text{if } p \equiv 0(8), \\ \sum_{\substack{1 \leq k < p \\ k \text{ is odd}}} e_{2p}(-qrk^2) \sin \frac{2\pi qk}{p} \sin \frac{2\pi k}{p} & \text{otherwise.} \end{cases}\end{aligned}$$

It is known, see [9], that the Chern–Simons invariant and the Reidemeister torsion of $L(p, q)$ are given by

$$\begin{aligned}\text{CS}(L(p, q); ad \circ \rho_k) &= -\frac{qk^2}{p}, \\ \text{Tor}(L(p, q); ad \circ \rho_k) &= \pm \frac{32}{p} \sin^2 \frac{2\pi qk}{p} \sin^2 \frac{2\pi k}{p}.\end{aligned}$$

Hence, we obtain the theorem. $\square$

Appendix A

In the appendix, we prove formulas of $\varphi(U)_{t,j}$ defined in (6).

LEMMA A.1. *We have that*

$$\begin{aligned}\varphi(U)_{t,0} &= -e_{2r} \left(-\frac{1}{a_{t-1}c_{t-1}} \right) \sqrt{\frac{a_{t-1}ri}{a_t}} \\ &\quad \times \sum_{\gamma=0}^{4a_t-1} \sum_{\pm} \pm (-1)^\gamma e_{8a_t r} \left(-a_{t-1} \left\{ r\gamma + 2 \left(2k_{t+1} + 1 \pm \frac{1}{a_{t-1}} \right) \right\}^2 \right).\end{aligned}$$

Proof. We replace $2kr + 2k_t + 1 =: 2\gamma + 1$ to get

$$\begin{aligned}\varphi(U)_{t,0} &= \sum_{k_t=0}^{r-1} \left\{ e_{2r}(m_t(2k_t + 1)^2) \left\{ \sum_{\pm} \pm e_{2r}(\pm 2(2k_{t+1} + 1)(2k_t + 1)) \right\} \right. \\ &\quad \times \left. \sum_{k=0}^{a_{t-1}-1} \sum_{\pm} \pm e_{8a_{t-1}r} \left(-c_{t-1} \left\{ 4kr + 2 \left(2k_t + 1 \pm \frac{1}{c_{t-1}} \right) \right\}^2 \right) \right\}\end{aligned}$$

$$\begin{aligned}
&= e_{2r} \left(-\frac{1}{a_{t-1}c_{t-1}} \right) \sum_{\gamma=0}^{a_{t-1}r-1} e_{2r} \left(\left(m_t - \frac{c_{t-1}}{a_{t-1}} \right) (2\gamma + 1)^2 \right) \\
&\quad \times \left\{ \sum_{\pm} \pm e_{2r} \left(2 \left(2k_{t+1} + 1 \mp \frac{1}{a_{t-1}} \right) (2\gamma + 1) \right) \right. \\
&\quad \left. + \sum_{\pm} \mp e_{2r} \left(-2 \left(2k_{t+1} + 1 \pm \frac{1}{a_{t-1}} \right) (2\gamma + 1) \right) \right\}.
\end{aligned}$$

By replacing $a_{t-1}r - \gamma - 1 =: \gamma'$, the second sum can be rewritten as

$$\begin{aligned}
&\sum_{\gamma'=0}^{a_{t-1}r-1} e_{2r} \left(\left(m_t - \frac{c_{t-1}}{a_{t-1}} \right) (2(a_{t-1}r - 1 - \gamma') + 1)^2 \right) \\
&\quad \times \left\{ \sum_{\pm} \mp e_{2r} \left(-2 \left(2k_{t+1} + 1 \pm \frac{1}{a_{t-1}} \right) (2(a_{t-1}r - 1 - \gamma') + 1) \right) \right\} \\
&= \sum_{\gamma'=0}^{a_{t-1}r-1} e_{2r} \left(\left(m_t - \frac{c_{t-1}}{a_{t-1}} \right) (2\gamma' + 1)^2 \right) \\
&\quad \times \left\{ \sum_{\pm} \mp e_{2r} \left(-2 \left(2k_{t+1} + 1 \pm \frac{1}{a_{t-1}} \right) (2\gamma' + 1) \right) \right\}.
\end{aligned}$$

From Lemma 2.5(iii),

$$\begin{aligned}
&\varphi(U)_{t,0} \\
&= -2e_{2r} \left(-\frac{1}{a_{t-1}c_{t-1}} \right) e_{2r} \left(\frac{a_t}{a_{t-1}} \right) \sum_{\gamma=0}^{a_{t-1}r-1} e_{2a_{t-1}r} (4a_t \gamma^2) \\
&\quad \times \sum_{\pm} e_{2r} \left(4 \left(\frac{a_t}{a_{t-1}} + 2k_{t+1} + 1 \pm \frac{1}{a_{t-1}} \right) \gamma \right) e_{2r} \left(2 \left(2k_{t+1} + 1 \pm \frac{1}{a_{t-1}} \right) \right).
\end{aligned}$$

Applying the reciprocity formula in Lemma 2.4, we have that

$$\begin{aligned}
\varphi(U)_{t,0} &= -2e_{2r} \left(-\frac{1}{a_{t-1}c_{t-1}} \right) e_{2r} \left(\frac{a_t}{a_{t-1}} \right) \frac{1}{2} \sqrt{\frac{a_{t-1}ri}{a_t}} \\
&\quad \times \sum_{\gamma=0}^{4a_t-1} \sum_{\pm} \pm \left\{ e_{2r} \left(2 \left(2k_{t+1} + 1 \pm \frac{1}{a_{t-1}} \right) \right) \right. \\
&\quad \left. \times e_{8a_t} \left(-a_{t-1}r \left\{ \gamma + \frac{2}{r} \left(\frac{a_t}{a_{t-1}} + 2k_{t+1} + 1 \pm \frac{1}{a_{t-1}} \right) \right\}^2 \right) \right\} \\
&= -e_{2r} \left(-\frac{1}{a_{t-1}c_{t-1}} \right) \sqrt{\frac{a_{t-1}ri}{a_t}} \\
&\quad \times \sum_{\gamma=0}^{4a_t-1} \sum_{\pm} \pm (-1)^\gamma e_{8a_t r} \left(-a_{t-1} \left\{ r\gamma + 2 \left(2k_{t+1} + 1 \pm \frac{1}{a_{t-1}} \right) \right\}^2 \right).
\end{aligned}$$

Hence, we obtain the claim. $\square$

LEMMA A.2. *We have that*

$$\begin{aligned} \varphi(U)_{t,2} = & -(-1)^{m_t} e_{2r} \left(-\frac{1}{a_{t-1}c_{t-1}} \right) \sqrt{\frac{a_{t-1}ri}{a_t}} \\ & \times \sum_{\gamma=0}^{4a_t-1} \sum_{\pm} \pm e_{8a_t r} \left(-a_{t-1} \left\{ r\gamma + 2 \left(2k_{t+1} + 1 \pm \frac{1}{a_{t-1}} \right) \right\}^2 \right). \end{aligned}$$

Proof. By replacing $2kr + r + 2k_t + 1 =: 2\gamma$, we get the required formula. $\square$

LEMMA A.3. *We have that, for $j = 1, 3$,*

$$\begin{aligned} \varphi(U)_{t,j} = & e_{2r} \left(-\frac{1}{a_{t-1}c_{t-1}} \right) \sqrt{\frac{a_{t-1}ri}{a_t}} \frac{1}{2} e_{8(m_t r)} e_4((-1)^{(r-1)/2} m_t) \\ & \times \sum_{\gamma=0}^{4a_t-1} (e_4(\gamma) + e_4(-\gamma)) \sum_{\pm} \pm e_{8a_t r} \left(-a_{t-1} \left\{ r\gamma + 2 \left(2k_{t+1} + 1 \pm \frac{1}{a_{t-1}} \right) \right\}^2 \right). \end{aligned}$$

Proof. By replacing $4kr + jr + 4k_t + 2 =: 4\gamma - (-1)^{(r-j)/2}$, we obtain the required formula. $\square$

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