ON ℓ -ADIC GALOIS POLYLOGARITHMS AND TRIPLE ℓ TH POWER RESIDUE SYMBOLS

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Abstract. The ℓ -adic Galois polylogarithm is an arithmetic function on an absolute Galois group with values in ℓ -adic numbers, which arises from Galois actions on ℓ -adic étale paths on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. In the present paper, we discuss a relationship between ℓ -adic Galois polylogarithms and triple ℓ th power residue symbols in some special cases studied in a work of Hirano and Morishita [J. Number Theory **198** (2019), 211–238]. We show that a functional equation of ℓ -adic Galois polylogarithms by Nakamura and Wojtkowiak [Non-abelian Fundamental Groups and Iwasawa Theory. Cambridge University Press, 2012, pp. 258–310] implies a reciprocity law of triple ℓ th power residue symbols.

0. Introduction

Let K be a number field, and $\overline{K}$ its algebraic closure in the complex number field $\mathbb{C}$. For a prime number ℓ , let $\zeta_\ell := \exp(2\pi\sqrt{-1}/\ell)$ be a primitive ℓ th root of unity in $\overline{K}$.

Choose a K -rational point z of $\mathbb{P}_K^1 \setminus \{0, 1, \infty\}$. For any prime number ℓ , the absolute Galois group $G_K := \text{Gal}(\overline{K}/K)$ acts on the ℓ -adic étale path space $\pi_1^\ell(\mathbb{P}_K^1 \setminus \{0, 1, \infty\}; \overrightarrow{01}, \bar{z})$, where $\overrightarrow{01}$ is the standard K -rational tangential base point [De, Ih, Na]. In [Wo], for a fixed ℓ -adic étale path $\gamma \in \pi_1^\ell(\mathbb{P}_K^1 \setminus \{0, 1, \infty\}; \overrightarrow{01}, \bar{z})$, Wojtkowiak introduced an arithmetic function

$$\ell i_n^{(\ell)}(z, \gamma) : G_K \rightarrow \mathbb{Q}_\ell$$

(for $n = 2, 3, 4, \dots$) valued in the ℓ -adic number field $\mathbb{Q}_\ell$, called the n th ℓ -adic Galois polylogarithm, defined as a certain coefficient in the ℓ -adic Magnus expansion of the loop

$$\mathfrak{f}_\gamma^{(\ell)}(\sigma) := \gamma \cdot \sigma(\gamma)^{-1} \in \pi_1^\ell(\mathbb{P}_K^1 \setminus \{0, 1, \infty\}, \overrightarrow{01}) \quad (\sigma \in G_K).$$

On the other hand, following the analogy between knots and primes, Morishita introduced the mod ℓ Milnor invariant $\mu_\ell(123) \in \mathbb{Z}/\ell\mathbb{Z}$ for certain prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ of $\mathbb{Q}(\zeta_\ell)$ for $\ell = 2, 3$, as an arithmetic analog of the Milnor invariant of links [Mo, AMM]. As a result, the *triple ℓ th power residue symbol* is defined by

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_\ell := \zeta_\ell^{\mu_\ell(123)},$$

which controls the decomposition law of $\mathfrak{p}_3$ in a certain nilpotent extension $R_{\mathfrak{p}_1, \mathfrak{p}_2}^{(\ell)}/\mathbb{Q}(\zeta_\ell)$.

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In the present paper, we relate $[p_1, p_2, p_3]_\ell$ to $\ell i_n^{(\ell)}(z, \gamma)$ for $\ell = 2, 3$ as follows.

MAIN FORMULA. (Naive form) For $\ell \in \{2, 3\}$, we have

$$[p_1, p_2, p_3]_\ell = \pm \zeta_\ell^{-\ell i_2^{(\ell)}(z, \gamma)(\sigma)}, \quad (0.0.1)$$

where K, z, γ, σ are suitably chosen to satisfy certain conditions depending on the triple of primes $\{p_1, p_2, p_3\}$. (See Theorem 2.3 for more details.)

Moreover, as a consequence of (0.0.1), we derive a reciprocity law of the triple symbol $[p_1, p_2, p_3]_\ell$ due to Rédei [Ré] and Amano–Mizusawa–Morishita [AMM] in the form

$$[p_1, p_2, p_3]_\ell \cdot [p_2, p_1, p_3]_\ell = 1 \quad (\ell = 2, 3) \quad (0.0.2)$$

from a functional equation between $\ell i_2^{(\ell)}(z, \gamma)$ and $\ell i_2^{(\ell)}(1 - z, \gamma')$ due to Nakamura and Wojtkowiak [NW2]. (See Corollary 2.9 for details.) Thus, by using a functional equation of ℓ -adic Galois polylogarithms, we have another proof of a reciprocity law of triple ℓ th power residue symbols. This fact is an indication that the Galois action mentioned at the beginning of this introduction has abundant arithmetic information.

1. Preliminaries

1.1. ℓ -adic Galois polylogarithms

In this section, we recall the definition and some properties of ℓ -adic Galois polylogarithms. Fix any prime number ℓ . Let K be a sub-field of $\mathbb{C}$, $\overline{K}$ a fixed algebraic closure of K , and $G_K := \text{Gal}(\overline{K}/K)$ the absolute Galois group of K with respect to $\overline{K}$. Fix an embedding $\overline{K} \hookrightarrow \mathbb{C}$. For any positive integer m , let $\zeta_{\ell^m} := \exp(2\pi\sqrt{-1}/\ell^m)$ be a fixed primitive ℓ^m th root of unity in $\overline{K}$. Let

$$X := \mathbb{P}_K^1 \setminus \{0, 1, \infty\}$$

be a projective line minus three points over K , $X_{\overline{K}} := X \times_K \overline{K}$ the base change of $X \rightarrow \text{Spec } K$ via $K \hookrightarrow \overline{K}$, and $X^{\text{an}} = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ the complex analytic space associated to the base change of $X_{\overline{K}}$ via $\overline{K} \hookrightarrow \mathbb{C}$.

First, we choose a K -rational point

$$z \in X(K).$$

As appropriate, we regard z as a point on X^{an} by the fixed embeddings $K \hookrightarrow \overline{K}$ and $\overline{K} \hookrightarrow \mathbb{C}$. We denote by $\pi_1^{\text{top}}(X^{\text{an}}; \overrightarrow{01}, z)$ the set of homotopy classes of piecewise smooth topological paths on X^{an} from the unit tangent vector $\overrightarrow{01}$ to z ; and by $\pi_1^{\text{top}}(X^{\text{an}}, \overrightarrow{01})$ the topological fundamental group of X^{an} with the base point $\overrightarrow{01}$. The group $\pi_1^{\text{top}}(X^{\text{an}}, \overrightarrow{01})$ is a free group of rank two generated by the homotopy classes of $\{l_0, l_1\}$ in Figure 1:

$$\pi_1^{\text{top}}(X^{\text{an}}, \overrightarrow{01}) = \langle l_0, l_1 \rangle.$$

Fix a homotopy class

$$\gamma \in \pi_1^{\text{top}}(X^{\text{an}}; \overrightarrow{01}, z).$$

FIGURE 1. Figure of X^{an} .

Let $\bar{z} : \text{Spec } \bar{K} \rightarrow X_{\bar{K}}$ be the base change of $z : \text{Spec } K \rightarrow X$ via $K \hookrightarrow \bar{K}$. We denote by $\pi_1^\ell(X_{\bar{K}}; \vec{01}, \bar{z})$ the profinite set of ℓ -adic étale paths on $X_{\bar{K}}$ from the standard K -rational tangential base point $\vec{01}$ to $\bar{z}$; and by $\pi_1^\ell(X_{\bar{K}}, *)$ the pro- ℓ étale fundamental group of $X_{\bar{K}}$ with base point $*$ in $\{\vec{01}, \bar{z}\}$. By using the comparison maps induced by the fixed embedding $\bar{K} \hookrightarrow \mathbb{C}$, we regard homotopy classes $l_0, l_1 \in \pi_1^{\text{top}}(X^{\text{an}}, \vec{01})$, $\gamma \in \pi_1^{\text{top}}(X^{\text{an}}; \vec{01}, z)$ as ℓ -adic étale paths

$$l_0, l_1 \in \pi_1^\ell(X_{\bar{K}}, \vec{01}), \quad \gamma \in \pi_1^\ell(X_{\bar{K}}; \vec{01}, \bar{z}).$$

Then $\pi_1^\ell(X_{\bar{K}}, \vec{01})$ is a free pro- ℓ group topologically generated by $\{l_0, l_1\}$:

$$\pi_1^\ell(X_{\bar{K}}, \vec{01}) = \overline{\langle l_0, l_1 \rangle}.$$

Next, we focus on the Galois action

$$G_K \rightarrow \text{Aut}(\pi_1^\ell(X_{\bar{K}}; \vec{01}, \bar{z}))$$

defined by $\sigma(p) := s_{\vec{01}}(\sigma) \cdot p \cdot s_{\bar{z}}(\sigma)$ for $\sigma \in G_K$ and $p \in \pi_1^\ell(X_{\bar{K}}; \vec{01}, \bar{z})$, where $s_* : G_K \rightarrow \pi_1^\ell(X_{\bar{K}}, *)$ is a canonical homomorphism induced by a geometric point $*$ in $\{\vec{01}, \bar{z}\}$ on $X_{\bar{K}}$ and paths are composed from left to right. Consider the continuous 1-cocycle

$$f_\gamma^{(\ell)} : G_K \rightarrow \pi_1^\ell(X_{\bar{K}}, \vec{01})$$

defined by $f_\gamma^{(\ell)}(\sigma) := \gamma \cdot \sigma(\gamma)^{-1} \in \pi_1^\ell(X_{\bar{K}}, \vec{01})$. To understand clearly the behavior of $f_\gamma^{(\ell)}$, we use the ℓ -adic Magnus embedding

$$E : \pi_1^\ell(X_{\bar{K}}, \vec{01}) \hookrightarrow \mathbb{Q}_\ell \langle\langle e_0, e_1 \rangle\rangle$$

defined by $E(l_0) = \exp(e_0)$, $E(l_1) = \exp(e_1)$, where $\mathbb{Q}_\ell \langle\langle e_0, e_1 \rangle\rangle$ is the $\mathbb{Q}_\ell$ -algebra of formal power series over $\mathbb{Q}_\ell$ in two non-commuting variables e_0 and e_1 . The constant term of $E(f_\gamma^{(\ell)}(\sigma)) \in \mathbb{Q}_\ell \langle\langle e_0, e_1 \rangle\rangle$ is equal to one for any $\sigma \in G_K$, so we can consider the Lie formal power series $\log(E(f_\gamma^{(\ell)}(\sigma)))^{-1} \in \text{Lie}(\langle\langle e_0, e_1 \rangle\rangle) \subset \mathbb{Q}_\ell \langle\langle e_0, e_1 \rangle\rangle$, where $\text{Lie}(\langle\langle e_0, e_1 \rangle\rangle)$ is the complete free Lie algebra generated by e_0 and e_1 .

Now we shall introduce a certain function on G_K which quantifies the loop $f_\gamma^{(\ell)}(\sigma) \in \pi_1^\ell(X_{\bar{K}}, \vec{01})$ ($\sigma \in G_K$) as a ‘polylogarithm’ with values in ℓ -adic numbers. To introduce it,

we need the following preparations. Let $\delta \in \pi_1^{\text{top}}(X^{\text{an}}; \overrightarrow{01}, \overrightarrow{10})$ be a homotopy class of the canonical path on X^{an} as in Figure 1, and

$$\gamma' := \delta \cdot \varphi(\gamma) \in \pi_1^{\text{top}}(X^{\text{an}}; \overrightarrow{01}, 1 - z),$$

where $\varphi \in \text{Aut}(X^{\text{an}})$ given by $\varphi(*) = 1 - *$. We will choose $z^{1/n}, (1 - z)^{1/n}, (1 - \zeta_n^a z^{1/n})^{1/m}$ ($n, m \in \mathbb{N}, a \in \mathbb{Z}$) as the specific n th power roots determined by $\gamma \in \pi_1^{\text{top}}(X^{\text{an}}; \overrightarrow{01}, z)$ (see [NW1] for details). Let

$$\rho_{z,\gamma} : G_K \rightarrow \mathbb{Z}_\ell \quad (\text{respectively } \rho_{1-z,\gamma'} : G_K \rightarrow \mathbb{Z}_\ell)$$

be the Kummer 1-cocycle along γ (respectively γ') defined by $\sigma(z^{1/\ell^n}) = \zeta_{\ell^n}^{\rho_{z,\gamma}(\sigma)} z^{1/\ell^n}$ (respectively $\sigma((1 - z)^{1/\ell^n}) = \zeta_{\ell^n}^{\rho_{1-z,\gamma'}(\sigma)} (1 - z)^{1/\ell^n}$) for $\sigma \in G_K$. Denote by $\chi : G_K \rightarrow \mathbb{Z}_\ell^\times$ the ℓ -adic cyclotomic character defined by $\sigma(\zeta_{\ell^n}) = \zeta_{\ell^n}^{\chi(\sigma)}$ for $\sigma \in G_K$.

Definition 1. (ℓ -adic Galois polylogarithm function [NW1, Wo, §11]) We define a function $\ell i_n^{(\ell)}(z, \gamma) : G_K \rightarrow \mathbb{Q}_\ell$ ($n \geq 2$) as a coefficient of $\text{ad}(e_0)^{n-1}(e_1)$ in the following Lie expression of $\log(E(\mathfrak{f}_\gamma^{(\ell)}(\sigma)))^{-1}$ for any $\sigma \in G_K$:

$$\begin{aligned} \log(E(\mathfrak{f}_\gamma^{(\ell)}(\sigma)))^{-1} &\equiv \rho_{z,\gamma}(\sigma)e_0 + \rho_{1-z,\gamma'}(\sigma)e_1 \\ &\quad + \sum_{n=2}^{\infty} \ell i_n^{(\ell)}(z, \gamma)(\sigma) \text{ad}(e_0)^{n-1}(e_1) \pmod{I_{e_1}}, \end{aligned}$$

where I_{e_1} denotes the ideal generated by Lie monomials involving e_1 at least twice. This function

$$\ell i_n^{(\ell)}(z, \gamma) : G_K \rightarrow \mathbb{Q}_\ell \quad (n \geq 2)$$

is called the n th ℓ -adic Galois polylogarithm function associated to $\gamma \in \pi_1^{\text{top}}(X^{\text{an}}; \overrightarrow{01}, z)$. We shall also define

$$\ell i_0^{(\ell)}(z, \gamma) := \rho_{z,\gamma}, \quad \ell i_1^{(\ell)}(z, \gamma) := \rho_{1-z,\gamma'}.$$

Here we shall introduce a certain character on G_K which generalizes the so-called Soulé character.

Definition 2. (ℓ -adic Galois polylogarithmic character [NW1]) For any integer $m \geq 1$, we define $\tilde{\chi}_m^{z,\gamma} : G_K \rightarrow \mathbb{Z}_\ell$ by the following Kummer properties:

$$\zeta_{\ell^n}^{\tilde{\chi}_m^{z,\gamma}(\sigma)} = \sigma \left(\prod_{i=0}^{\ell^n-1} (1 - \zeta_{\ell^n}^{\chi(\sigma)^{-1}i} z^{1/\ell^n})^{i^{m-1}/\ell^n} \right) / \prod_{i=0}^{\ell^n-1} (1 - \zeta_{\ell^n}^{i+\rho_{z,\gamma}(\sigma)} z^{1/\ell^n})^{i^{m-1}/\ell^n} \quad (n \geq 1).$$

This function

$$\tilde{\chi}_m^{z,\gamma} : G_K \rightarrow \mathbb{Z}_\ell \quad (m \geq 1),$$

valued in the ring $\mathbb{Z}_\ell$ of ℓ -adic integers, is called the m th ℓ -adic Galois polylogarithmic character associated to $\gamma \in \pi_1^{\text{top}}(X^{\text{an}}; \overrightarrow{01}, z)$.

In fact, ℓ -adic Galois polylogarithmic characters describe values of the ℓ -adic Galois polylogarithm function.

THEOREM 1.1. (Explicit formula [NW1, Corollary]) *For each $\sigma \in G_K$, the quantity $\ell i_n^{(\ell)}(z, \gamma)(\sigma)$ is explicitly described by ℓ -adic Galois polylogarithmic characters as follows:*

$$\ell i_n^{(\ell)}(z, \gamma)(\sigma) = (-1)^{n+1} \sum_{k=0}^{n-1} \frac{B_k}{k!} (-\rho_{z,\gamma}(\sigma))^k \frac{\tilde{\chi}_{n-k}^{z,\gamma}(\sigma)}{(n-k-1)!} \quad (n \geq 1),$$

where B_k denotes the k th Bernoulli number.

One reason for the name ‘ ℓ -adic Galois polylogarithm’ is that ℓ -adic Galois polylogarithm functions/polylogarithmic characters satisfy some typical functional equations analogous to functional equations of the classical polylogarithm [NW2, Ch. 6]. The following functional equation is one example of them.

THEOREM 1.2. (A functional equation [NW2, Ch. 6, (6.14)]) *The second ℓ -adic Galois polylogarithm function satisfies the following functional equation. For any $\sigma \in G_K$,*

$$\ell i_2^{(\ell)}(z, \gamma)(\sigma) + \ell i_2^{(\ell)}(1-z, \gamma')(\sigma) = \ell i_2^{(\ell)}(\vec{10}, \delta)(\sigma).$$

By Theorem 1.1, this equation is equivalent to the following functional equation of the second ℓ -adic Galois polylogarithmic character. For any $\sigma \in G_K$,

$$\tilde{\chi}_2^{z,\gamma}(\sigma) + \tilde{\chi}_2^{1-z,\gamma'}(\sigma) + \rho_{z,\gamma}(\sigma)\rho_{1-z,\gamma'}(\sigma) = \frac{1}{24}(\chi(\sigma)^2 - 1).$$

Remark 3. The latter functional equation in Theorem 1.2 is an ℓ -adic Galois analog of the functional equation

$$Li_2(z) + Li_2(1-z) + \log(z)\log(1-z) = \frac{\pi^2}{6},$$

where $Li_2(z)$ denotes the classical dilogarithm function.

1.2. Triple ℓ th power residue symbols for $\ell = 2, 3$

The triple ℓ th power residue symbol is defined at present for $\ell = 2, 3$ in [Mo] and [AMM]. In this section, following [HM, Section 4], [Mo] and [AMM], we recall the definition and some properties of triple ℓ th power residue symbols for $\ell = 2, 3$.

1.2.1. Case of $\ell = 2$. Let p_1, p_2 be distinct prime numbers which satisfy

$$p_i \equiv 1 \pmod{4} \quad (i = 1, 2), \quad \left(\frac{p_i}{p_j} \right) = 1 \quad (1 \leq i \neq j \leq 2). \quad (1.2.1)$$

By (1.2.1), there exist integers x, y, w satisfying the following conditions [Am, Lemma 1.1]:

$$\begin{aligned} x^2 - p_1 y^2 - p_2 w^2 &= 0, \\ \gcd(x, y, w) &= 1, \quad y \equiv 0 \pmod{2}, \quad x - y \equiv 1 \pmod{4}. \end{aligned} \quad (1.2.2)$$

Note that the triple (x, y, w) is not unique. For such a pair (x, y) , we let

$$\theta_{p_1, p_2}^{(2)} := x + \sqrt{p_1} y. \quad (1.2.3)$$

Moreover, we set

$$R^{(2)}(= R_{p_1, p_2}^{(2)}) := \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\theta_{p_1, p_2}^{(2)}}) \subset \mathbb{C}, \quad (1.2.4)$$

$$K^{(2)}(= K_{p_1, p_2}^{(2)}) := \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}). \quad (1.2.5)$$

THEOREM 1.3. [**Am**, Theorem 1.2, Corollary 1.5] *The field $R^{(2)}$ is a finite Galois extension of $\mathbb{Q}$ in $\mathbb{C}$ which satisfies the following properties:*

(i) *The Galois group $\text{Gal}(R^{(2)}/\mathbb{Q})$ is isomorphic to the Heisenberg group*

$$H_3(\mathbb{Z}/2\mathbb{Z}) := \left\{ \left(\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \middle| * \in \mathbb{Z}/2\mathbb{Z} \right\}.$$

(Note that this group $H_3(\mathbb{Z}/2\mathbb{Z})$ is isomorphic to the dihedral group D_8 of order eight.)

- (ii) *Prime numbers ramified in $R^{(2)}/\mathbb{Q}$ are only p_1, p_2 with ramification index 2.*
- (iii) *The field $R^{(2)}$ is independent of the choice of the triple (x, y, w) . Hence, $R^{(2)}/\mathbb{Q}$ depends only on the pair $\{p_1, p_2\}$.*

THEOREM 1.4. (An arithmetic characterization of $R^{(2)}$ [**Am**, Theorem 2.1]) *Let p_1, p_2 be distinct prime numbers satisfying (1.2.1). For a number field $L \subset \mathbb{C}$, the following conditions are equivalent:*

- (1) *L is the field $R^{(2)}$;*
- (2) *$L/\mathbb{Q}$ is a Galois extension in which only prime numbers p_1, p_2 are ramified with ramification index 2 and whose Galois group is isomorphic to the Heisenberg group $H_3(\mathbb{Z}/2\mathbb{Z})$.*

Here we take another prime number p_3 satisfying

$$p_3 \equiv 1 \pmod{4}, \quad \left(\frac{p_i}{p_j} \right) = 1 \quad (1 \leq i \neq j \leq 3). \quad (1.2.6)$$

Note that p_3 is unramified in $R^{(2)}/\mathbb{Q}$ by Theorem 1.3(ii). Then we introduce an arithmetic symbol which controls the decomposition of p_3 in the nilpotent extension $R^{(2)}/\mathbb{Q}$.

Definition 4. (Triple quadratic residue symbol [**Mo**, Section 8.4]) For prime ideals $(p_1), (p_2), (p_3)$ of $\mathbb{Z}$, where prime numbers p_1, p_2, p_3 satisfy (1.2.1) and (1.2.6), the *triple quadratic residue symbol* is defined by

$$[(p_1), (p_2), (p_3)]_2 := (-1)^{\mu_2(123)} \in \{1, -1\},$$

where $\mu_2(123) \in \mathbb{Z}/2\mathbb{Z}$ is the mod 2 Milnor invariant for the prime numbers p_1, p_2, p_3 . See [**Mo**, Section 8.4] for the detailed account of $\mu_2(123)$.

Let $\tilde{\mathfrak{p}}_i$ be a prime ideal of $K^{(2)}$ above p_i . By (1.2.1) and (1.2.6), the primes p_1, p_2, p_3 are completely decomposed or ramified in $K^{(2)}/\mathbb{Q}$ as follows.

THEOREM 1.5. [Mo, Section 8.4, Theorem 8.25] *Let $\sigma_{p_3} := \text{Frob}_{\tilde{\mathfrak{p}}_3} \in \text{Gal}(R^{(2)}/K^{(2)})$ be the Frobenius substitution of $\tilde{\mathfrak{p}}_3$ in $R^{(2)}/K^{(2)}$. Then we have*

$$[(p_1), (p_2), (p_3)]_2 = \frac{\sigma_{p_3}(\sqrt{\theta_{p_1, p_2}^{(2)}})}{\sqrt{\theta_{p_1, p_2}^{(2)}}}.$$

In particular, $[(p_1), (p_2), (p_3)]_2 = 1$ if and only if p_3 is completely decomposed in $R^{(2)}/\mathbb{Q}$.

Remark 1.6. The right side of the equation in Theorem 1.5 is the Rédei symbol introduced by Rédei in [Ré]:

$$[p_1, p_2, p_3]_{\text{Rédei}} := \frac{\sigma_{p_3}(\sqrt{\theta_{p_1, p_2}^{(2)}})}{\sqrt{\theta_{p_1, p_2}^{(2)}}}.$$

That is, Theorem 1.5 means that the triple quadratic residue symbol is equal to the Rédei symbol:

$$[(p_1), (p_2), (p_3)]_2 = [p_1, p_2, p_3]_{\text{Rédei}}.$$

In [Ré], Rédei proved the following reciprocity law of the triple symbol. In [Am], Amano gave another simple proof of it.

THEOREM 1.7. (Reciprocity law of triple quadratic residue symbols [Ré, Am]) *Let $\rho \in S_3$ be any permutation of the set $\{1, 2, 3\}$. Then*

$$[(p_1), (p_2), (p_3)]_2 \cdot [(p_{\rho(1)}), (p_{\rho(2)}), (p_{\rho(3)})]_2 = 1,$$

that is, $[(p_1), (p_2), (p_3)]_2 = [(p_{\rho(1)}), (p_{\rho(2)}), (p_{\rho(3)})]_2$.

1.2.2. *Case of $\ell = 3$.* In this section, we essentially follow [HM, Section 4.2] for various assumptions. Let $k := \mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$ be the Eisenstein field, where $\zeta_3 := \exp(2\pi\sqrt{-1}/3) = (-1 + \sqrt{-3})/2$. Let $\mathfrak{p}_i = (p_i)$ ($i = 1, 2$) be distinct prime ideals of k which satisfy

$$N\mathfrak{p}_i \equiv 1 \pmod{9} \ (i = 1, 2), \quad \left(\frac{p_i}{p_j}\right)_3 = 1 \quad (1 \leq i \neq j \leq 2). \quad (1.2.7)$$

Following [AMM, Corollary 5.9] and [HM, Section 4.2], we assume that

$$\text{each } p_i \text{ is an associate of a rational prime number in } k. \quad (1.2.8)$$

There is an ambiguity of the choice of p_i up to units $\mathbb{Z}[\zeta_3]^\times = \{\pm\zeta_3^m \mid m = 0, 1, 2\}$, but we can take it uniquely by posing the following condition (cf. [AMM, Lemma 1.1]):

$$p_i \equiv 1 \pmod{(3\sqrt{-3})}. \quad (1.2.9)$$

We fix such a prime element $p_i \in \mathbb{Z}[\zeta_3]$. We set $K_i := k(\sqrt[3]{p_i})$. The field K_i is a cyclic extension of degree three over k in which only $\mathfrak{p}_i$ is ramified (cf. [AMM, Theorem 3.5]). Let ϕ be a generator of $\text{Gal}(K_1/k)$ determined by $\phi(\sqrt[3]{p_1}) = \zeta_3 \sqrt[3]{p_1}$. By (1.2.7) and (1.2.9), there exist algebraic integers

$$\alpha_{p_1, p_2} \in \mathcal{O}_{K_1}, \quad w \in \mathbb{Z}[\zeta_3] \quad (1.2.10)$$

together with prime ideals $\mathfrak{P}, \mathfrak{B}$ of K_1 which satisfy the following conditions [AMM, Proposition 5.6]:

$$\begin{aligned} N_{K_1/k}(\alpha_{p_1, p_2}) &= p_2 w^3, \\ (\alpha_{p_1, p_2}) &= \mathfrak{P}^e \mathfrak{B}^f, \quad (e, 3) = 1, \quad (\mathfrak{B}, 3) = 1, \quad f \equiv 0 \pmod{3}. \end{aligned} \quad (1.2.11)$$

Note that α_{p_1, p_2} is not unique. For such an $\alpha_{p_1, p_2} \in \mathcal{O}_{K_1}$, we let

$$\theta_{p_1, p_2}^{(3)} := \phi(\alpha_{p_1, p_2})(\phi^2(\alpha_{p_1, p_2}))^2. \quad (1.2.12)$$

Moreover, we set

$$R^{(3)}(= R_{p_1, p_2}^{(3)}) := k\left(\sqrt[3]{p_1}, \sqrt[3]{p_2}, \sqrt[3]{\theta_{p_1, p_2}^{(3)}}\right) \subset \mathbb{C}, \quad (1.2.13)$$

$$K^{(3)}(= K_{p_1, p_2}^{(3)}) := k(\sqrt[3]{p_1}, \sqrt[3]{p_2}). \quad (1.2.14)$$

By using the assumption (1.2.8), we obtain the following theorem.

THEOREM 1.8. [AMM, Theorem 5.11, Corollary 5.12] *The field $R^{(3)}$ is a finite Galois extension of k in $\mathbb{C}$ which the following properties hold:*

(i) *The Galois group $\text{Gal}(R^{(3)}/k)$ is isomorphic to the Heisenberg group*

$$H_3(\mathbb{Z}/3\mathbb{Z}) := \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \mid * \in \mathbb{Z}/3\mathbb{Z} \right\}.$$

(ii) *Prime ideals ramified in $R^{(3)}/k$ are only $\mathfrak{p}_1, \mathfrak{p}_2$ with ramification index 3.*

(iii) *The field $R^{(3)}$ is independent of the choice of $\alpha_{p_1, p_2} \in \mathcal{O}_{K_1}$. Hence, $R^{(3)}/k$ depends only on the pair $\{\mathfrak{p}_1, \mathfrak{p}_2\}$.*

THEOREM 1.9. (An arithmetic characterization of $R^{(3)}$; cf. [AMM, Corollary 5.12]) *Let $\mathfrak{p}_1 = (p_1)$, $\mathfrak{p}_2 = (p_2)$ be distinct prime ideals of k satisfying (1.2.7), (1.2.8) and (1.2.9). For a finite extension field L of k in $\mathbb{C}$, the following conditions are equivalent:*

- (1) *L is the field $R^{(3)}$;*
- (2) *L/k is a Galois extension in which only primes $\mathfrak{p}_1, \mathfrak{p}_2$ are ramified with ramification index 3 and whose Galois group is isomorphic to the Heisenberg group $H_3(\mathbb{Z}/3\mathbb{Z})$.*

Here we take another prime ideal $\mathfrak{p}_3 = (p_3)$ of k satisfying

$$N\mathfrak{p}_3 \equiv 1 \pmod{9}, \quad \left(\frac{p_i}{p_j} \right)_3 = 1 \quad (1 \leq i \neq j \leq 3). \quad (1.2.15)$$

Note that $\mathfrak{p}_3$ is unramified in $R^{(3)}/k$ by Theorem 1.8(ii). Then we introduce an arithmetic symbol which controls the decomposition of $\mathfrak{p}_3$ in the nilpotent extension $R^{(3)}/k$.

Definition 5. (Triple cubic residue symbol [AMM, Definition 6.2]) For a triple of primes $(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$ of k satisfying (1.2.7) and (1.2.15), the *triple cubic residue symbol* is defined by

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3 := \zeta_3^{\mu_3(123)} \in \{1, \zeta_3, \zeta_3^{-1}\},$$

where $\mu_3(123) \in \mathbb{Z}/3\mathbb{Z}$ is the mod 3 Milnor invariant for the primes $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$. See [AMM, (2.3) of Ch. 2, Theorem 4.4] for the detailed account of $\mu_3(123)$.

Let $\tilde{\mathfrak{p}}_i$ be a prime ideal of $K^{(3)}$ above $\mathfrak{p}_i$. By (1.2.7) and (1.2.15), the primes $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ are completely decomposed or ramified in $K^{(3)}/k$ as follows.

$$\begin{array}{cccc}
 R^{(3)} = K^{(3)}(\sqrt[3]{\theta_{\mathfrak{p}_1, \mathfrak{p}_2}^{(3)}}) & & & \\
 \left| \right. & & & \\
 K^{(2)} = k(\sqrt[3]{p_1}, \sqrt[3]{p_2}) & \tilde{\mathfrak{p}}_1 & \tilde{\mathfrak{p}}_2 & \tilde{\mathfrak{p}}_3 \\
 \left| \right. & \left| \text{completely} \right. & \left| \text{ramified} \right. & \\
 & \text{decomposed} & & \\
 K_1 = k(\sqrt[3]{p_1}) & \tilde{\mathfrak{p}}_1 \cap K_1 & \tilde{\mathfrak{p}}_2 \cap K_1 & \text{completely} \\
 \left| \right. & \left| \text{ramified} \right. & \left| \text{completely} \right. & \text{decomposed} \\
 & & \text{decomposed} & \\
 k = \mathbb{Q}(\zeta_3) & \mathfrak{p}_1 & \mathfrak{p}_2 & \mathfrak{p}_3
 \end{array}$$

THEOREM 1.10. [AMM, Theorem. 6.3] *Let $\sigma_{\mathfrak{p}_3} := \text{Frob}_{\tilde{\mathfrak{p}}_3} \in \text{Gal}(R^{(3)}/K^{(3)})$ be the Frobenius substitution of $\tilde{\mathfrak{p}}_3$ in $R^{(3)}/K^{(3)}$. Then we have*

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3 = \frac{\sigma_{\mathfrak{p}_3} \left(\sqrt[3]{\theta_{\mathfrak{p}_1, \mathfrak{p}_2}^{(3)}} \right)}{\sqrt[3]{\theta_{\mathfrak{p}_1, \mathfrak{p}_2}^{(3)}}}.$$

In particular, $[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3 = 1$ if and only if $\mathfrak{p}_3$ is completely decomposed in $R^{(3)}/k$.

THEOREM 1.11. (A reciprocity law of triple cubic residue symbols [AMM, Proposition 6.5])
We have

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3 \cdot [\mathfrak{p}_2, \mathfrak{p}_1, \mathfrak{p}_3]_3 = 1,$$

that is, $[\mathfrak{p}_2, \mathfrak{p}_1, \mathfrak{p}_3]_3 = [\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3^{-1}$.

Hereafter, following Hirano and Morishita [HM, Section 4.2], we shall restrict ourselves to the case with the following assumption.

Assumption (A). The pair (p_1, p_2) has $\alpha_{p_1, p_2} \in \mathcal{O}_{K_1}$ in (1.2.10) and (1.2.11) which can be given in the form $\alpha_{p_1, p_2} = x + y\sqrt[3]{p_1}$ ($x, y \in k$).

Under the above assumption (A), the conditions (1.2.11) and (1.2.12) are equivalent to

$$x^3 + p_1 y^3 = p_2 w^3, \quad (1.2.16)$$

$$\theta_{p_1, p_2}^{(3)} = (x + \zeta_3 y \sqrt[3]{p_1})(x + \zeta_3^2 y \sqrt[3]{p_1})^2. \quad (1.2.17)$$

These equations will play an important role in the next section.

2. Triple ℓ th power residue symbols and ℓ -adic Galois polylogarithms

In this section, we interpret triple ℓ th power residue symbols in terms of ℓ -adic Galois polylogarithms for $\ell = 2, 3$. As a result, we derive a reciprocity law of triple ℓ th power residue symbols from a functional equation of ℓ -adic Galois polylogarithms.

2.1. Main formula

Let $\ell \in \{2, 3\}$. Let

$$k := \begin{cases} \mathbb{Q} & (\text{if } \ell = 2), \\ \mathbb{Q}(\zeta_3) & (\text{if } \ell = 3), \end{cases}$$

and

$$K := \mathbb{Q}(\zeta_\ell)(\sqrt[\ell]{p_1}, \sqrt[\ell]{p_2}), \quad (2.1.1)$$

where $\zeta_\ell = \exp(2\pi\sqrt{-1}/\ell)$ is a fixed primitive ℓ th root of unity in $\overline{K} \subset \mathbb{C}$. We set

$$p_i \in \mathbb{Z}[\zeta_\ell] \quad (i = 1, 2, 3), \quad x, y, w \in k,$$

$$\theta_{p_1, p_2}^{(\ell)}, R_{p_1, p_2}^{(\ell)}, K_{p_1, p_2}^{(\ell)}$$

as in Section 1.2.1 for $\ell = 2$ and as in Section 1.2.2 with the assumption (A) for $\ell = 3$. Note that $K = K_{p_1, p_2}^{(\ell)}$ by (1.2.5) and (1.2.14).

Hence, by (1.2.3), (1.2.17) and (1.2.2), (1.2.16), we have

$$x^\ell - (-y)^\ell p_1 = w^\ell p_2, \quad (2.1.2)$$

$$\theta_{p_1, p_2}^{(\ell)} = \prod_{i=0}^{\ell-1} (x + \zeta_\ell^i y \sqrt[\ell]{p_1})^i. \quad (2.1.3)$$

For the prime element $p_i \in \mathbb{Z}[\zeta_\ell]$ ($i = 1, 2, 3$), we denote by

$$\mathfrak{p}_i = (p_i)$$

the prime ideal of k generated by p_i . For the triple $(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$ of primes of k , the triple ℓ th power residue symbol $[p_1, p_2, p_3]_\ell$ is defined as discussed in Section 1.2.

Moreover, we choose

$$z := p_1 \left(-\frac{y}{x} \right)^\ell. \quad (2.1.4)$$

Since $z \in K \setminus \{0, 1\}$, we regard z as a K -rational point of $\mathbb{P}_K^1 \setminus \{0, 1, \infty\}$. Let

$$\tilde{\sigma}_{\mathfrak{p}_3} \in \text{Gal}(\overline{K}/K) \quad (2.1.5)$$

be an extension of the Frobenius substitution $\sigma_{\mathfrak{p}_3} := \text{Frob}_{\tilde{\mathfrak{p}_3}} \in \text{Gal}(R_{p_1, p_2}^{(\ell)}/K)$, where $\tilde{\mathfrak{p}_3}$ is a prime ideal of K above $\mathfrak{p}_3$. Let $\tilde{z} : \text{Spec } \overline{K} \rightarrow \mathbb{P}_{\overline{K}}^1 \setminus \{0, 1, \infty\}$ be the base change of $z : \text{Spec } K \rightarrow X$ via $K \hookrightarrow \overline{K}$. Fix a homotopy class

$$\gamma \in \pi_1^{\text{top}}(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}; \overrightarrow{01}, z)$$

of a piecewise smooth topological path on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ from $\overrightarrow{01}$ to z . Then, the second ℓ -adic Galois polylogarithms $\ell i_2^{(\ell)}(z, \gamma) : G_K \rightarrow \mathbb{Q}_\ell$, $\tilde{\chi}_2^{z, \gamma} : G_K \rightarrow \mathbb{Z}_\ell$ are defined as discussed in Section 1.1.

PROPOSITION 2.1. *Let the notation and assumptions be as above. For any $\tau \in G_K$, the value $\ell i_2^{(\ell)}(z, \gamma)(\tau) \bmod \ell$, together with $\tilde{\chi}_2^{z, \gamma}(\tau) \bmod \ell$, is independent of the choice of γ .*

Proof. Let $\tau \in G_K$. By (2.1.1), we have $\chi(\tau) \equiv 1$, $\rho_{z, \gamma}(\tau) \equiv 0 \bmod \ell$. Hence, it follows from Definition 2 that

$$\zeta_\ell^{\tilde{\chi}_2^{z, \gamma}(\tau)} = \tau \left(\prod_{i=0}^{\ell-1} (1 - \zeta_\ell^i z^{1/\ell})^{i/\ell} \right) / \prod_{i=0}^{\ell-1} (1 - \zeta_\ell^i z^{1/\ell})^{i/\ell}. \quad (2.1.6)$$

Let $\gamma_0, \gamma_1 \in \pi_1^{\text{top}}(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}; \overrightarrow{01}, z)$. For $\epsilon \in \{0, 1\}$, we choose $z_\epsilon^{1/\ell}$, $(1 - \zeta_\ell^i z_\epsilon^{1/\ell})$ as the specific ℓ th power roots determined by γ_ϵ (cf. [NW1]).

First, in order to prove that $\tilde{\chi}_2^{z, \gamma}(\tau) \bmod \ell$ is independent of the choice of γ , it suffices to show that

$$\zeta_\ell^{\tilde{\chi}_2^{z, \gamma_0}(\tau)} = \zeta_\ell^{\tilde{\chi}_2^{z, \gamma_1}(\tau)} \quad (2.1.7)$$

by comparing the right-hand side of (2.1.6) for $\gamma = \gamma_0, \gamma_1$. We now show (2.1.7). Let

$$A_\epsilon := \prod_{i=0}^{\ell-1} (1 - \zeta_\ell^i z_\epsilon^{1/\ell})^{i/\ell} \quad (\epsilon \in \{0, 1\}).$$

Take $s \in \mathbb{Z}/\ell\mathbb{Z}$ such that $z_1^{1/\ell} = \zeta_\ell^{-s} \cdot z_0^{1/\ell}$. For each $i \in \mathbb{Z}/\ell\mathbb{Z}$, there exists $t_i \in \mathbb{Z}/\ell\mathbb{Z}$ such that $(1 - \zeta_\ell^i z_1^{1/\ell})^{1/\ell} = \zeta_\ell^{t_i} (1 - \zeta_\ell^{i-s} z_0^{1/\ell})^{1/\ell}$ since $(1 - \zeta_\ell^i z_1^{1/\ell}) = (1 - \zeta_\ell^{i-s} z_0^{1/\ell})$. Then, we compute A_1/A_0 as follows:

$$\frac{A_1}{A_0} = \frac{\prod_{i=0}^{\ell-1} (1 - \zeta_\ell^i z_0^{1/\ell})^{i/\ell} \cdot \zeta_\ell^{\sum_{i=0}^{\ell-1} i t_i}}{\prod_{i=0}^{\ell-1} (1 - \zeta_\ell^i z_0^{1/\ell})^{i/\ell}}$$

$$\begin{aligned}
&= \frac{\prod_{j=0}^{\ell-1} (1 - \zeta_\ell^j z_0^{1/\ell})^{(j+s)/\ell}}{\prod_{i=0}^{\ell-1} (1 - \zeta_\ell^i z_0^{1/\ell})^{i/\ell}} \cdot \zeta_\ell^{\sum_{i=0}^{\ell-1} i t_i} \\
&= \left(\prod_{j=0}^{\ell-1} (1 - \zeta_\ell^j z_0^{1/\ell})^{1/\ell} \right)^s \cdot \zeta_\ell^{\sum_{i=0}^{\ell-1} i t_i}.
\end{aligned}$$

Since

$$\prod_{j=0}^{\ell-1} (1 - \zeta_\ell^j z_0^{1/\ell}) = 1 - z = \left(\frac{w}{x} \right)^\ell p_2 \quad \text{and} \quad K = \mathbb{Q}(\zeta_\ell)(\sqrt[\ell]{p_1}, \sqrt[\ell]{p_2})$$

by (2.1.1), (2.1.2), and (2.1.4), we obtain $A_1/A_0 \in K$, hence

$$\zeta_\ell^{\tilde{\chi}_2^{z, \gamma_0}(\tau)} = \frac{\tau(A_0)}{A_0} = \frac{\tau(A_1)}{A_1} = \zeta_\ell^{\tilde{\chi}_2^{z, \gamma_1}(\tau)}.$$

This completes the proof of (2.1.7).

Moreover, by Theorem 1.1 and (2.1.1) and (2.1.5), we have

$$\ell i_2^{(\ell)}(z, \gamma)(\tau) \equiv -\tilde{\chi}_2^{z, \gamma}(\tau) \pmod{\ell}; \quad (2.1.8)$$

therefore, $\ell i_2^{(\ell)}(z, \gamma)(\tau) \pmod{\ell}$ is also independent of the choice of γ . $\square$

Definition 2.2. Let the notation and assumptions be as above. Based on Proposition 2.1, for $\tau \in G_K$, we let

$$\begin{aligned}
\ell i_2^{(\ell)}(z)(\tau) \pmod{\ell} &:= \ell i_2^{(\ell)}(z, \gamma)(\tau) \pmod{\ell}, \\
\tilde{\chi}_2^z(\tau) \pmod{\ell} &:= \tilde{\chi}_2^{z, \gamma}(\tau) \pmod{\ell},
\end{aligned}$$

that is,

$$\zeta_\ell^{\ell i_2^{(\ell)}(z)(\tau)} := \zeta_\ell^{\ell i_2^{(\ell)}(z, \gamma)(\tau)}, \quad \zeta_\ell^{\tilde{\chi}_2^z(\tau)} := \zeta_\ell^{\tilde{\chi}_2^{z, \gamma}(\tau)}.$$

Now we shall describe the triple symbol $[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_\ell$ by a special value of the second ℓ -adic Galois polylogarithm.

THEOREM 2.3. *Let the notation and assumptions be as above. For $\ell \in \{2, 3\}$, we have*

$$\begin{aligned}
[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_\ell &= \frac{\tilde{\sigma}_{\mathfrak{p}_3}(x^{(\ell-1)/2})}{x^{(\ell-1)/2}} \cdot \zeta_\ell^{\tilde{\chi}_2^z(\tilde{\sigma}_{\mathfrak{p}_3})} \\
&= \frac{\tilde{\sigma}_{\mathfrak{p}_3}(x^{(\ell-1)/2})}{x^{(\ell-1)/2}} \cdot \zeta_\ell^{-\ell i_2^{(\ell)}(z)(\tilde{\sigma}_{\mathfrak{p}_3})}.
\end{aligned}$$

Proof. Let $\ell \in \{2, 3\}$. We compute the triple symbol $[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_\ell$ as follows:

$$\begin{aligned}
[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_\ell &= \sigma_{\mathfrak{p}_3} \left(\sqrt[\ell]{\theta_{\mathfrak{p}_1, \mathfrak{p}_2}^{(\ell)}} \right) / \sqrt[\ell]{\theta_{\mathfrak{p}_1, \mathfrak{p}_2}^{(\ell)}} \quad (\text{by Theorem 1.5, Theorem 1.10}) \\
&= \tilde{\sigma}_{\mathfrak{p}_3} \left(\sqrt[\ell]{\theta_{\mathfrak{p}_1, \mathfrak{p}_2}^{(\ell)}} \right) / \sqrt[\ell]{\theta_{\mathfrak{p}_1, \mathfrak{p}_2}^{(\ell)}} \\
&= \tilde{\sigma}_{\mathfrak{p}_3} \left(\prod_{i=0}^{\ell-1} (x + \zeta_\ell^i y \sqrt[\ell]{p_1})^{i/\ell} \right) / \prod_{i=0}^{\ell-1} (x + \zeta_\ell^i y \sqrt[\ell]{p_1})^{i/\ell} \quad (\text{by (2.1.3)}) \\
&= \frac{\tilde{\sigma}_{\mathfrak{p}_3} \left(\prod_{i=0}^{\ell-1} x^{i/\ell} \right)}{\prod_{i=0}^{\ell-1} x^{i/\ell}} \cdot \frac{\tilde{\sigma}_{\mathfrak{p}_3} \left(\prod_{i=0}^{\ell-1} \left(1 + \zeta_\ell^i \frac{y}{x} p_1^{1/\ell} \right)^{i/\ell} \right)}{\prod_{i=0}^{\ell-1} \left(1 + \zeta_\ell^i \frac{y}{x} p_1^{1/\ell} \right)^{i/\ell}} \\
&= \frac{\tilde{\sigma}_{\mathfrak{p}_3} (x^{(\ell-1)/2})}{x^{(\ell-1)/2}} \cdot \frac{\tilde{\sigma}_{\mathfrak{p}_3} \left(\prod_{i=0}^{\ell-1} \left(1 + \zeta_\ell^i \frac{y}{x} p_1^{1/\ell} \right)^{i/\ell} \right)}{\prod_{i=0}^{\ell-1} \left(1 + \zeta_\ell^i \frac{y}{x} p_1^{1/\ell} \right)^{i/\ell}}.
\end{aligned}$$

Since $z = p_1(-y/x)^\ell$ by (2.1.4), the second factor of the above last right-hand side is equal to

$$\frac{\tilde{\sigma}_{\mathfrak{p}_3} \left(\prod_{i=0}^{\ell-1} (1 - \zeta_\ell^i z^{1/\ell})^{i/\ell} \right)}{\prod_{i=0}^{\ell-1} (1 - \zeta_\ell^i z^{1/\ell})^{i/\ell}} = \zeta_\ell^{\tilde{\chi}_2^z(\tilde{\sigma}_{\mathfrak{p}_3})} \quad (\text{by (2.1.6)}).$$

Therefore, by combining the above formulas and (2.1.8), we obtain

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_\ell = \frac{\tilde{\sigma}_{\mathfrak{p}_3} (x^{(\ell-1)/2})}{x^{(\ell-1)/2}} \cdot \zeta_\ell^{\tilde{\chi}_2^z(\tilde{\sigma}_{\mathfrak{p}_3})} = \frac{\tilde{\sigma}_{\mathfrak{p}_3} (x^{(\ell-1)/2})}{x^{(\ell-1)/2}} \cdot \zeta_\ell^{-\ell i_2^{(\ell)}(z)(\tilde{\sigma}_{\mathfrak{p}_3})}. \quad \square$$

COROLLARY 2.4. (Case of $\ell = 2$) *Let the notation and assumptions be as above. Then we have*

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_2 = (-1)^{\rho_x(\tilde{\sigma}_{\mathfrak{p}_3}) - \ell i_2^{(2)}(z)(\tilde{\sigma}_{\mathfrak{p}_3})},$$

where the value $\rho_x(\tilde{\sigma}_{\mathfrak{p}_3}) \in \mathbb{Z}/2\mathbb{Z}$ is defined by $\tilde{\sigma}_{\mathfrak{p}_3}(\sqrt{x})/\sqrt{x} = (-1)^{\rho_x(\tilde{\sigma}_{\mathfrak{p}_3})}$. Hence, we obtain

$$\mu_2(123) = \rho_x(\tilde{\sigma}_{\mathfrak{p}_3}) - \ell i_2^{(2)}(z)(\tilde{\sigma}_{\mathfrak{p}_3}) \pmod{2}.$$

Proof. The assertion follows from Theorem 2.3 and Definition 4. $\square$

COROLLARY 2.5. (Case of $\ell = 3$) *Let the notation and assumptions be as above. Then we have*

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3 = \zeta_3^{-\ell i_2^{(3)}(z)(\tilde{\sigma}_{\mathfrak{p}_3})}.$$

Hence, we obtain

$$\mu_3(123) = -\ell i_2^{(3)}(z)(\tilde{\sigma}_{\mathfrak{p}_3}) \pmod{3}.$$

Proof. The assertion follows from Theorem 2.3 and Definition 5. $\square$

2.2. Deriving a reciprocity law

Let the notation and assumptions be as in the previous section. Note that $\gamma' = \delta \cdot \varphi(\gamma) \in \pi_1^{\text{top}}(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}; \overrightarrow{01}, 1-z)$ is as in Section 1.1.

PROPOSITION 2.6. *For any $\tau \in G_K$, the value $\ell i_2^{(\ell)}(1-z, \gamma')(\tau) \bmod \ell$, together with $\tilde{\chi}_2^{1-z, \gamma'}(\tau) \bmod \ell$, is independent of the choice of γ .*

Proof. The proof can be done in the same way as the proof of Proposition 2.1. $\square$

Definition 2.7. Based on Proposition 2.6, for $\tau \in G_K$, we let

$$\begin{aligned} \ell i_2^{(\ell)}(1-z)(\tau) \bmod \ell &:= \ell i_2^{(\ell)}(1-z, \gamma')(\tau) \bmod \ell, \\ \tilde{\chi}_2^{1-z}(\tau) \bmod \ell &:= \tilde{\chi}_2^{1-z, \gamma'}(\tau) \bmod \ell, \end{aligned}$$

that is,

$$\zeta_\ell^{\ell i_2^{(\ell)}(1-z)(\tau)} := \zeta_\ell^{\ell i_2^{(\ell)}(1-z, \gamma')(\tau)}, \quad \zeta_\ell^{\tilde{\chi}_2^{1-z}(\tau)} := \zeta_\ell^{\tilde{\chi}_2^{1-z, \gamma'}(\tau)}.$$

Firstly, to derive a reciprocity law of triple ℓ th power residue symbols, we describe the triple symbol $[\mathfrak{p}_2, \mathfrak{p}_1, \mathfrak{p}_3]_\ell$ by the second ℓ -adic Galois polylogarithmic character.

THEOREM 2.8. *For $\ell \in \{2, 3\}$, we have*

$$[\mathfrak{p}_2, \mathfrak{p}_1, \mathfrak{p}_3]_\ell = \frac{\tilde{\sigma}_{\mathfrak{p}_3}(x^{(\ell-1)/2})}{x^{(\ell-1)/2}} \cdot \zeta_\ell^{\tilde{\chi}_2^{1-z}(\tilde{\sigma}_{\mathfrak{p}_3})}.$$

Proof. Let $\ell \in \{2, 3\}$. Since $x^\ell - (-y)^\ell p_1 = w^\ell p_2 \iff x^\ell - w^\ell p_2 = (-y)^\ell p_1$ by (2.1.2), we can take

$$\theta_{p_2, p_1}^{(\ell)} = \prod_{i=0}^{\ell-1} (x - \zeta_\ell^i w \sqrt[\ell]{p_2})^i \quad (2.2.1)$$

by replacing p_1, p_2 , and y in (2.1.3) with p_2, p_1 , and $-w$. As with Theorem 2.3, we compute the triple symbol $[\mathfrak{p}_2, \mathfrak{p}_1, \mathfrak{p}_3]_\ell$ as follows:

$$\begin{aligned} [\mathfrak{p}_2, \mathfrak{p}_1, \mathfrak{p}_3]_\ell &= \sigma_{\mathfrak{p}_3} \left(\sqrt[\ell]{\theta_{p_2, p_1}^{(\ell)}} \right) / \sqrt[\ell]{\theta_{p_2, p_1}^{(\ell)}} \quad (\text{by Theorem 1.5, Theorem 1.10}) \\ &= \sigma_{\mathfrak{p}_3} \left(\prod_{i=0}^{\ell-1} (x - \zeta_\ell^i w \sqrt[\ell]{p_2})^{i/\ell} \right) / \prod_{i=0}^{\ell-1} (x - \zeta_\ell^i w \sqrt[\ell]{p_2})^{i/\ell} \quad (\text{by (2.2.1)}) \\ &= \tilde{\sigma}_{\mathfrak{p}_3} \left(\prod_{i=0}^{\ell-1} (x - \zeta_\ell^i w \sqrt[\ell]{p_2})^{i/\ell} \right) / \prod_{i=0}^{\ell-1} (x - \zeta_\ell^i w \sqrt[\ell]{p_2})^{i/\ell} \\ &= \frac{\tilde{\sigma}_{\mathfrak{p}_3} \left(\prod_{i=0}^{\ell-1} x^{i/\ell} \right)}{\prod_{i=0}^{\ell-1} x^{i/\ell}} \cdot \frac{\tilde{\sigma}_{\mathfrak{p}_3} \left(\prod_{i=0}^{\ell-1} \left(1 - \zeta_\ell^i \frac{w}{x} p_2^{1/\ell} \right)^{i/\ell} \right)}{\prod_{i=0}^{\ell-1} \left(1 - \zeta_\ell^i \frac{w}{x} p_2^{1/\ell} \right)^{i/\ell}} \end{aligned}$$

$$= \frac{\tilde{\sigma}_{\mathfrak{p}_3}(x^{(\ell-1)/2})}{x^{(\ell-1)/2}} \cdot \frac{\tilde{\sigma}_{\mathfrak{p}_3}\left(\prod_{i=0}^{\ell-1}\left(1 - \zeta_\ell^i \frac{w}{x} p_2^{1/\ell}\right)^{i/\ell}\right)}{\prod_{i=0}^{\ell-1}\left(1 - \zeta_\ell^i \frac{w}{x} p_2^{1/\ell}\right)^{i/\ell}}.$$

Since

$$1 - z = \frac{x^\ell - (-y)^\ell p_1}{x^\ell} = \frac{w^\ell}{x^\ell} p_2$$

by (2.1.2), the second factor of the above last right-hand side is equal to

$$\frac{\tilde{\sigma}_{\mathfrak{p}_3}\left(\prod_{i=0}^{\ell-1}(1 - \zeta_\ell^i (1 - z)^{1/\ell})^{i/\ell}\right)}{\prod_{i=0}^{\ell-1}(1 - \zeta_\ell^i (1 - z)^{1/\ell})^{i/\ell}} = \zeta_\ell^{\tilde{\chi}_2^{1-z}(\tilde{\sigma}_{\mathfrak{p}_3})} \quad (\text{by Definition 2, (2.1.1), (2.1.5)}).$$

Therefore we obtain the assertion of the theorem. $\square$

Now, we derive a reciprocity law of triple ℓ th power residue symbols from the functional equation of ℓ -adic Galois polylogarithms introduced in Theorem 1.2.

COROLLARY 2.9. (A reciprocity law) *Let the notation and assumptions be as above. For $\ell \in \{2, 3\}$, we have*

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_\ell \cdot [\mathfrak{p}_2, \mathfrak{p}_1, \mathfrak{p}_3]_\ell = 1.$$

Proof. By combining Theorem 2.3 and Theorem 2.8,

$$\begin{aligned} [\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_\ell \cdot [\mathfrak{p}_2, \mathfrak{p}_1, \mathfrak{p}_3]_\ell &= \begin{cases} \frac{\tilde{\sigma}_{\mathfrak{p}_3}(\sqrt{x})}{\sqrt{x}} (-1)^{\tilde{\chi}_2^z(\tilde{\sigma}_{\mathfrak{p}_3})} \cdot \frac{\tilde{\sigma}_{\mathfrak{p}_3}(\sqrt{x})}{\sqrt{x}} (-1)^{\tilde{\chi}_2^{1-z}(\tilde{\sigma}_{\mathfrak{p}_3})} & (\text{if } \ell = 2) \\ \zeta_3^{\tilde{\chi}_2^z(\tilde{\sigma}_{\mathfrak{p}_3})} \cdot \zeta_3^{\tilde{\chi}_2^{1-z}(\tilde{\sigma}_{\mathfrak{p}_3})} & (\text{if } \ell = 3) \end{cases} \\ &= \begin{cases} (-1)^{\tilde{\chi}_2^z(\tilde{\sigma}_{\mathfrak{p}_3}) + \tilde{\chi}_2^{1-z}(\tilde{\sigma}_{\mathfrak{p}_3})} & (\text{if } \ell = 2) \\ \zeta_3^{\tilde{\chi}_2^z(\tilde{\sigma}_{\mathfrak{p}_3}) + \tilde{\chi}_2^{1-z}(\tilde{\sigma}_{\mathfrak{p}_3})} & (\text{if } \ell = 3) \end{cases} \\ &= \zeta_\ell^{\tilde{\chi}_2^z(\tilde{\sigma}_{\mathfrak{p}_3}) + \tilde{\chi}_2^{1-z}(\tilde{\sigma}_{\mathfrak{p}_3})} \\ &= \zeta_\ell^{\tilde{\chi}_2^{z, \gamma}(\tilde{\sigma}_{\mathfrak{p}_3}) + \tilde{\chi}_2^{1-z, \gamma'}(\tilde{\sigma}_{\mathfrak{p}_3})}. \end{aligned}$$

By the functional equation in Theorem 1.2, the above last right-hand side is equal to

$$\zeta_\ell^{-\rho_{z, \gamma}(\tilde{\sigma}_{\mathfrak{p}_3}) \rho_{1-z, \gamma'}(\tilde{\sigma}_{\mathfrak{p}_3}) + \frac{1}{24}(\chi(\tilde{\sigma}_{\mathfrak{p}_3})^2 - 1)} = 1 \quad (\text{by (2.1.1), (2.1.5)}).$$

This completes the proof. $\square$

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Appendix A. Examples: case of $\ell = 3$ with assumption (A)

In this appendix, we present examples of $[p_1, p_2, p_3]_3$ and $\ell i_2^{(3)}(z)(\tilde{\sigma}_{p_3}) \bmod 3$ for some pairs (p_1, p_2) which have α_{p_1, p_2} satisfying the assumption (A) in Section 1.2.2. The rational primes p which satisfy $p \equiv 1 \bmod 9$ and $1 \leq -p \leq 1000$ are the following 29 numbers:

$$\mathbf{L} := \left\{ \begin{array}{l} -17, -53, -71, -89, -107, -179, -197, -233, -251, -269, \\ -359, -431, -449, -467, -503, -521, -557, -593, -647, -683, \\ -701, -719, -773, -809, -827, -863, -881, -953, -971 \end{array} \right\}. \quad (\text{A.0.1})$$

For any pair (p_1, p_2) of distinct rational primes in $\mathbf{L}$, prime ideals $\mathfrak{p}_1 = (p_1)$, $\mathfrak{p}_2 = (p_2)$ of $\mathbb{Q}(\zeta_3)$ satisfy the conditions (1.2.7), (1.2.8) and (1.2.9).

In [AMM, Example 6.4], Amano showed that one can take $\alpha_{-17, -53} = 8 - 3\sqrt[3]{17}$ satisfying (A) in the case where $(p_1, p_2) = (-17, -53)$, and gave values of $[p_1, p_2, p_3]_3$ for $p_3 = -71, -89, -107, -179, -197$. According to [HM, Example 4.2.15], Mizusawa found other pairs $(p_1, p_2) = (-17, -467), (-107, -449), (-431, -233)$ which have $\alpha_{p_1, p_2} \in O_{K_1}$ with the assumption (A). For example, one can take

$$\alpha_{-17, -469} = 6 - 9\sqrt[3]{-17},$$

$$\alpha_{-107, -449} = -24 - 5\sqrt[3]{-107},$$

$$\alpha_{-431, -233} = -68 - 9\sqrt[3]{-431},$$

respectively, for these cases.

Let us now give new examples. Consider the case where $(p_1, p_2) = (-17, -593)$. In this case, we can take

$$x = 9, \quad y = 2, \quad w = -1 \quad (\text{A.0.2})$$

as a solution of (1.2.16) and

$$\alpha_{p_1, p_2} (= \alpha_{-17, -593}) = 9 + 2\sqrt[3]{-17}$$

satisfying assumption (A). Hence

$$\theta_{p_1, p_2}^{(3)} (= \theta_{-17, -593}^{(3)}) = (9 + 2\zeta_3\sqrt[3]{-17})(9 + 2\zeta_3^2\sqrt[3]{-17})^2.$$

Moreover, let $\mathfrak{p}_3 = (p_3)$ be a prime ideal of $\mathbb{Q}(\zeta_3)$ which satisfies (1.2.15). Then,

$$\begin{aligned} [p_1, p_2, p_3]_3 &= \frac{\sigma_{\mathfrak{p}_3} \left(\sqrt[3]{\theta_{p_1, p_2}^{(3)}} \right)}{\sqrt[3]{\theta_{p_1, p_2}^{(3)}}} \quad (\text{by Theorem 1.10}) \\ &\equiv \theta_{p_1, p_2}^{(3)} (p_3^2 - 1)^{1/3} \bmod \tilde{\mathfrak{p}}_3, \end{aligned}$$

where $\tilde{\mathfrak{p}}_3$ is a prime ideal of $K_{p_1, p_2}^{(3)}$ above $\mathfrak{p}_3$. Since $\theta_{p_1, p_2}^{(3)} (p_3^2 - 1)^{1/3} \in K_1 = \mathbb{Q}(\zeta_3, \sqrt[3]{p_1})$,

$$[p_1, p_2, p_3]_3 \equiv \theta_{p_1, p_2}^{(3)} (p_3^2 - 1)^{1/3} \bmod \tilde{\mathfrak{p}}_3 \cap K_1.$$

TABLE A.1. Table of $[p_1, p_2, p_3]_3$, $[p_2, p_1, p_3]_3$, $\ell i_2^{(3)}(z)(\tilde{\sigma}_{p_3}) \bmod 3$ and $\ell i_2^{(3)}(1-z)(\tilde{\sigma}_{p_3}) \bmod 3$ for $(p_1, p_2) = (-17, -593)$ and $p_3 \in \mathbf{L} \setminus \{p_1, p_2\}$.

p_3	$[p_1, p_2, p_3]_3$	$[p_2, p_1, p_3]_3$	$\ell i_2^{(3)}(z)(\tilde{\sigma}_{p_3}) \bmod 3$	$\ell i_2^{(3)}(1-z)(\tilde{\sigma}_{p_3}) \bmod 3$
-53	ζ_3	ζ_3^{-1}	-1	1
-71	ζ_3^{-1}	ζ_3	1	-1
-89	ζ_3	ζ_3^{-1}	-1	1
-107	ζ_3^{-1}	ζ_3	1	-1
-179	ζ_3	ζ_3^{-1}	-1	1
-197	ζ_3	ζ_3^{-1}	-1	1
-233	1	1	0	0
-251	ζ_3	ζ_3^{-1}	-1	1
-269	ζ_3	ζ_3^{-1}	-1	1
-359	ζ_3^{-1}	ζ_3	1	-1
-431	ζ_3^{-1}	ζ_3	1	-1
-449	1	1	0	0
-467	ζ_3	ζ_3^{-1}	-1	1
-503	ζ_3^{-1}	ζ_3	1	-1
-521	ζ_3	ζ_3^{-1}	-1	1
-557	ζ_3	ζ_3^{-1}	-1	1
-647	ζ_3^{-1}	ζ_3	1	-1
-683	ζ_3^{-1}	ζ_3	1	-1
-701	ζ_3^{-1}	ζ_3	1	-1
-719	ζ_3^{-1}	ζ_3	1	-1
-773	ζ_3^{-1}	ζ_3	1	-1
-809	ζ_3^{-1}	ζ_3	1	-1
-827	1	1	0	0
-863	ζ_3	ζ_3^{-1}	-1	1
-881	ζ_3^{-1}	ζ_3	1	-1
-953	ζ_3	ζ_3^{-1}	-1	1
-971	1	1	0	0

Therefore, we obtain the following test: for $c = 0, 1, -1$,

$$[p_1, p_2, p_3]_3 = \zeta_3^c \iff N_{K_1/\mathbb{Q}}(\theta_{p_1, p_2}^{(3)} (p_3^2 - 1)^{1/3} - \zeta_3^c) \equiv 0 \bmod p_3. \quad (\text{A.0.3})$$

On the other hand, we can take $\alpha_{p_2, p_1} (= \alpha_{-593, -17}) = 9 + \sqrt[3]{-593}$ satisfying assumption (A). Hence $\theta_{p_2, p_1}^{(3)} (= \theta_{-593, -17}^{(3)}) = (9 + \zeta_3 \sqrt[3]{-593})(9 + \zeta_3^2 \sqrt[3]{-593})^2$. By replacing $\theta_{p_1, p_2}^{(3)}$ (respectively K_1) with $\theta_{p_2, p_1}^{(3)}$ (respectively $K_2 = \mathbb{Q}(\zeta_3, \sqrt[3]{p_2})$) in (A.0.3), we obtain the following test: for $c = 0, 1, -1$,

$$[p_2, p_1, p_3]_3 = \zeta_3^c \iff N_{K_2/\mathbb{Q}}(\theta_{p_2, p_1}^{(3)} (p_3^2 - 1)^{1/3} - \zeta_3^c) \equiv 0 \bmod p_3. \quad (\text{A.0.4})$$

Checking the right-hand condition of (A.0.3) and (A.0.4) by PARI/GP, we can compute $[p_1, p_2, p_3]_3$ and $[p_2, p_1, p_3]_3$. Furthermore, by combining with Theorem 2.3 and

TABLE A.2. Table of $[p_1, p_2, p_3]_3$ and $\ell i_2^{(3)}(z)(\tilde{\sigma}_{p_3}) \bmod 3$ for the cases of $\{p_1, p_2, p_3\} = \{-17, -53, -431\}, \{-17, -557, -773\}, \{-17, -593, -773\}$.

(p_1, p_2)	$\alpha_{p_1, p_2} =$ $x + y \sqrt[3]{p_1}$	$z = -p_1 y^3 / x^3$	p_3	$[p_1, p_2, p_3]_3$	$\ell i_2^{(3)}(z)(\tilde{\sigma}_{p_3})$ $\bmod 3$
$(-17, -53)$	$8 + 3\sqrt[3]{-17}$	$459/512$	-431	1	0
$(-53, -17)$	$8 + \sqrt[3]{-53}$	$53/512$	-431	1	0
$(-17, -431)$	$31 + 15\sqrt[3]{-17}$	$57375/29791$	-51	1	0
$(-431, -53)$	$10 + 3\sqrt[3]{-431}$	$11637/1000$	-17	1	0
$(-53, -431)$	$10 - \sqrt[3]{-53}$	$-53/1000$	-17	1	0
$(-431, -17)$	$31 - 4\sqrt[3]{-431}$	$-27584/29791$	-53	1	0
$(-17, -557)$	$-42 - 16\sqrt[3]{-17}$	$8704/9261$	-773	ζ_3	-1
$(-557, -17)$	$-42 - 2\sqrt[3]{-557}$	$557/9261$	-773	ζ_3^{-1}	1
$(-17, -773)$	$-23 + 8\sqrt[3]{-17}$	$-8704/12167$	-557	ζ_3^{-1}	1
$(-773, -557)$	$-6 - \sqrt[3]{-773}$	$773/216$	-17	ζ_3^{-1}	1
$(-557, -773)$	$-6 + \sqrt[3]{-557}$	$-557/216$	-17	ζ_3	-1
$(-773, -17)$	$-23 - 3\sqrt[3]{-773}$	$20871/12167$	-557	ζ_3	-1
$(-17, -593)$	$9 + 2\sqrt[3]{-17}$	$136/729$	-773	ζ_3^{-1}	1
$(-593, -17)$	$9 + \sqrt[3]{-593}$	$593/729$	-773	ζ_3	-1
$(-17, -773)$	$-23 + 8\sqrt[3]{-17}$	$-8704/12167$	-593	ζ_3	-1
$(-773, -593)$	$-55 - 6\sqrt[3]{-773}$	$166968/166375$	-17	ζ_3	-1
$(-593, -773)$	$-55 + \sqrt[3]{-593}$	$-593/166375$	-17	ζ_3^{-1}	1
$(-773, -17)$	$-23 - 3\sqrt[3]{-773}$	$20871/12167$	-593	ζ_3^{-1}	1

Theorem 2.8, we can also compute $\ell i_2^{(3)}(z)(\tilde{\sigma}_{p_3}) \bmod 3$ and $\ell i_2^{(3)}(1 - z)(\tilde{\sigma}_{p_3}) \bmod 3$, where

$$z\left(-p_1 \frac{y^3}{x^3}\right) = \frac{136}{729}.$$

Consequently, for $p_3 \in \mathbf{L} \setminus \{-17, -593\}$, we get Table A.1. Thus, we can be assured that the reciprocity law $[p_1, p_2, p_3]_3 \cdot [p_2, p_1, p_3]_3 = 1$ of Theorem 1.11 holds.

Let us examine more general behaviors of

$$[p_{\rho(1)}, p_{\rho(2)}, p_{\rho(3)}]_3,$$

where $\rho \in S_3$ is any permutation of the set $\{1, 2, 3\}$, in the cases

$$\{p_1, p_2, p_3\} = \{-17, -53, -431\}, \{-17, -557, -773\}, \{-17, -593, -773\}.$$

Finding a solution of (1.2.16) and checking the test (A.0.3) for each case, we can compute $[p_{\rho(1)}, p_{\rho(2)}, p_{\rho(3)}]_3$. Furthermore, by combining with Corollary 2.5, we can also compute $\ell i_2^{(3)}(z)(\tilde{\sigma}_{p_3}) \bmod 3$. Consequently, we get Table A.2. Based on Table A.2, it may be plausible to expect that

$$[p_{\rho(1)}, p_{\rho(2)}, p_{\rho(3)}]_3 = [p_1, p_2, p_3]_3^{\text{sgn}(\rho)}, \quad (\text{A.0.5})$$

where $\text{sgn}(\rho) \in \{1, -1\}$ is the signature of $\rho \in S_3$, for (p_1, p_2, p_3) satisfying the conditions (1.2.7), (1.2.8), (1.2.9) and (1.2.15).

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