ON THE SECOND DUAL SPACE OF THE BANACH SPACE OF VECTOR-VALUED LITTLE LIPSCHITZ FUNCTIONS

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Abstract. Let X be a compact metric space and E be a Banach space. Then $\text{lip}(X, E)$ denotes the Banach space of all E -valued little Lipschitz functions on X . We show that $\text{lip}(X, E)^{**}$ is isometrically isomorphic to the Banach space of E^{**} -valued Lipschitz functions $\text{Lip}(X, E^{**})$ under several conditions. Moreover, we describe the isometric isomorphism from $\text{lip}(X, E)^{**}$ to $\text{Lip}(X, E^{**})$.

1. Introduction

Let X be a compact metric space with metric d , $\mathbb{K}$ be a field of real numbers or complex numbers, E be a Banach space over $\mathbb{K}$ with norm $\|\cdot\|_E$ and $0 < \alpha \leq 1$. Then $C(X, E)$ denotes the Banach space of all E -valued continuous functions on X with norm

$$\|f\|_{C(X, E)} := \sup_{x \in X} \|f(x)\|_E.$$

If $f \in C(X, E)$ satisfies the condition

$$\mathcal{L}_{X, E}(f) := \sup_{\substack{x, x' \in X \\ x \neq x'}} \frac{\|f(x) - f(x')\|_E}{d(x, x')^\alpha} < \infty,$$

then we say that f is α -Lipschitz. In particular, if $\alpha = 1$, then we say that f is Lipschitz. Then $\text{Lip}_\alpha(X, E)$ denotes the set of all E -valued α -Lipschitz functions on X . If $f \in \text{Lip}_\alpha(X, E)$ satisfies the condition

$$\lim_{d(x, x')^\alpha \rightarrow 0} \frac{\|f(x) - f(x')\|_E}{d(x, x')^\alpha} = 0,$$

then we say that f is little α -Lipschitz. In particular, if $\alpha = 1$, then we say that f is little Lipschitz. Then $\text{lip}_\alpha(X, E)$ denotes the set of all E -valued little α -Lipschitz functions on X . In the case that $E = \mathbb{K}$, we simply write $\text{Lip}_\alpha(X) := \text{Lip}_\alpha(X, \mathbb{K})$ and $\text{lip}_\alpha(X) := \text{lip}_\alpha(X, \mathbb{K})$, respectively. In the case that $\alpha = 1$, we simply write $\text{Lip}(X, E) := \text{Lip}_1(X, E)$ and $\text{lip}(X, E) := \text{lip}_1(X, E)$, respectively.

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Clearly, $\text{Lip}_\alpha(X, E)$ is a Banach space over $\mathbb{K}$ with respect to the norm

$$\|f\|_{\max} := \max\{\|f\|_{C(X, E)}, \mathcal{L}_{X, E}(f)\} \quad (f \in \text{Lip}(X, E)),$$

and $\text{lip}_\alpha(X, E)$ is a closed subspace of $\text{Lip}_\alpha(X, E)$. Moreover, if $0 < \alpha < 1$, then $\text{Lip}(X, E) \subset \text{lip}_\alpha(X, E)$.

For any Banach space B , B^* and B^{**} denote the dual space and second dual space of B , respectively. Also, b^* and b^{**} denote the elements of B^* and B^{**} , respectively.

Weaver defined in [9] the following property to get a Stone–Weierstrass type theorem.

Definition 1. (Weaver [9]) Let X be a compact metric space with metric d . We say that $\text{lip}(X)$ separates points uniformly if there exists a constant $c > 1$ such that, for any $x, x' \in X$ with $x \neq x'$, some $f \in \text{lip}(X)$ satisfies $\mathcal{L}_{X, \mathbb{K}}(f) \leq c$ and $|f(x) - f(x')| = d(x, x')$.

Bade, Curtis and Dales showed in [1] that $\text{lip}_\alpha(X)$ separates points uniformly for $0 < \alpha < 1$.

Hanin [3] and Weaver [9] showed the following theorem.

THEOREM A. (Hanin [3] and Weaver [9]) *Let X be a compact metric space. The following are equivalent:*

- (a) $\text{lip}(X)$ separates points uniformly;
- (b) there exists $b > 1$ such that, for any $g \in \text{Lip}(X)$ and any finite subset $S \subset X$, some $f \in \text{lip}(X)$ satisfies $\mathcal{L}_{X, \mathbb{K}}(f) \leq b\mathcal{L}_{X, \mathbb{K}}(g)$ and $f(x) = g(x)$ for $x \in S$;
- (c) $\text{lip}(X)^{**}$ is isometrically isomorphic to $\text{Lip}(X)$, via the mapping $\Lambda: \text{lip}(X)^{**} \rightarrow \text{Lip}(X)$ defined by

$$(\Lambda f^{**})(x) = f^{**}(\tau_x) \quad (f^{**} \in \text{lip}(X)^{**}, x \in X),$$

where τ_x is the evaluation functional at x .

Also, Johnson [4, Theorem 4.7] and Bade, Curtis and Dales [1, Theorem 3.5] showed that $\text{lip}_\alpha(X)^{**}$ is isometrically isomorphic to $\text{Lip}_\alpha(X)$.

For the vector-valued case, Johnson took up in [4] the Banach space $\text{lip}_\alpha(X, E)$ and proved the following theorem.

THEOREM B. (Johnson [4]) *Let X be a compact metric space with metric d , E be a Banach space and $0 < \alpha < 1$. If either E^* or $\text{lip}_\alpha(X)^*$ has the approximation property, then $\text{lip}_\alpha(X, E)^{**}$ is isometrically isomorphic to $\text{Lip}_\alpha(X, E^{**})$.*

In this paper, we take up the Banach space $\text{lip}(X, E)$ and get the following result.

THEOREM 1. *Let E be a Banach space and X be a compact metric space with metric d . If $\text{lip}(X)$ separates points uniformly and either E^* or $\text{lip}(X)^*$ has the approximation property, then $\text{lip}(X, E)^{**}$ is isometrically isomorphic to $\text{Lip}(X, E^{**})$. Moreover, if Q is an isometric isomorphism from $\text{lip}(X, E)^{**}$ to $\text{Lip}(X, E^{**})$, then Q is of the form*

$$((Qf^{**})(x))(e^*) = f^{**}(e^* \otimes \tau_x)$$

for any $f^{**} \in \text{lip}(X, E)^{**}$, $x \in X$ and $e^* \in E^*$.

2. Preliminaries

Throughout this section, we assume that X is a compact metric space and E is a Banach space with norm $\|\cdot\|_E$.

PROPOSITION 2.1. *Let $e^* \in E^*$ and $f \in \text{lip}(X, E)$. A function $e^* \circ f: X \rightarrow \mathbb{K}$ defined by*

$$(e^* \circ f)(x) = e^*(f(x)) \quad (x \in X)$$

satisfies $e^ \circ f \in \text{lip}(X)$ and $\|e^* \circ f\|_{\max} \leq \|e^*\|_{E^*} \|f\|_{\max}$.*

Proof. Straightforward. □

The next proposition implies that $\text{lip}(X)^*$ has the Radon–Nikodým property (see e.g. [7, Corollary 5.42]).

PROPOSITION 2.2. *If $\text{lip}(X)$ separates points uniformly, then $\text{lip}(X)^*$ is separable.*

Proof. By noting that X is a compact metric space, there exists a countable dense subset $\mathcal{C}$ of X . Since $\text{lip}(X)$ separates points uniformly, $\{\tau_x\}_{x \in \mathcal{C}}$ is a countable subset of $\text{lip}(X)^*$. Let $\mathcal{Q}$ denote the subset of $\mathbb{K}$ such that the real and imaginary parts are rational numbers. Note that $\mathcal{Q}$ is a countable dense subset of $\mathbb{K}$. Put

$$\text{L.h.}_{\mathcal{Q}}\{\tau_x\}_{x \in \mathcal{C}} := \left\{ \sum_{k=1}^N \beta_k \tau_{x_k} : N \in \mathbb{N}, \beta_k \in \mathcal{Q}, x_k \in \mathcal{C} (k = 1, \dots, N) \right\}$$

and

$$\text{L.h.}_{\mathbb{K}}\{\tau_x\}_{x \in X} := \left\{ \sum_{k=1}^N \beta_k \tau_{x_k} : N \in \mathbb{N}, \beta_k \in \mathbb{K}, x_k \in X (k = 1, \dots, N) \right\}.$$

It follows from the countability of $\{\tau_x\}_{x \in \mathcal{C}}$ and $\mathcal{Q}$ that $\text{L.h.}_{\mathcal{Q}}\{\tau_x\}_{x \in \mathcal{C}}$ is a countable subset of $\text{lip}(X)^*$. By using the density of $\mathcal{C}$ and $\mathcal{Q}$, we see that the closure of $\text{L.h.}_{\mathcal{Q}}\{\tau_x\}_{x \in \mathcal{C}}$ with respect to the norm $\|\cdot\|_{\text{lip}(X)^*}$ contains $\text{L.h.}_{\mathbb{K}}\{\tau_x\}_{x \in X}$. Since $\text{L.h.}_{\mathbb{K}}\{\tau_x\}_{x \in X}$ is a dense subset of $\text{lip}(X)^*$ (see e.g. [10, Theorem 3.3.3]), $\text{L.h.}_{\mathcal{Q}}\{\tau_x\}_{x \in \mathcal{C}}$ is a countable dense subset of $\text{lip}(X)^*$. □

The next proposition implies that the norm of $\text{lip}(X, E)^*$ is the cross norm on algebraic tensor product $E^* \otimes \text{lip}(X)^*$.

PROPOSITION 2.3. *Let $e^* \in E^*$ and $h^* \in \text{lip}(X)^*$. A functional $e^* \otimes h^*$ on $\text{lip}(X, E)$ defined by*

$$(e^* \otimes h^*)(f) = h^*(e^* \circ f) \quad (f \in \text{lip}(X, E)) \tag{1}$$

satisfies $e^ \otimes h^* \in \text{lip}(X, E)^*$ and $\|e^* \otimes h^*\|_{\text{lip}(X, E)^*} = \|e^*\|_{E^*} \|h^*\|_{\text{lip}(X)^*}$.*

Proof. Let $f \in \text{lip}(X, E)$ be arbitrary. By Proposition 2.1, we have

$$|(e^* \otimes h^*)(f)| \leq \|h^*\|_{\text{lip}(X)^*} \|e^* \circ f\|_{\max} \leq \|h^*\|_{\text{lip}(X)^*} \|e^*\|_{E^*} \|f\|_{\max}.$$

Therefore, we have $e^* \otimes h^* \in \text{lip}(X, E)^*$ and $\|e^* \otimes h^*\|_{\text{lip}(X, E)^*} \leq \|e^*\|_{E^*} \|h^*\|_{\text{lip}(X)^*}$.

Let $e \in E$ and $h \in \text{lip}(X)$ be arbitrary. Put $f(x) := h(x)e$ ($x \in X$). Then, $f \in \text{lip}(X, E)$ and $\|f\|_{\max} = \|e\|_E \|h\|_{\max}$. Since the equation $(e^*e)h = e^* \circ f$ holds on X , we have

$$\begin{aligned} |e^*e||h^*h| &= |h^*((e^*e)h)| = |h^*(e^* \circ f)| \\ &= |(e^* \otimes h^*)(f)| \leq \|e^* \otimes h^*\|_{\text{lip}(X, E)^*} \|e\|_E \|h\|_{\max}. \end{aligned}$$

Therefore, we have $\|e^*\|_{E^*} \|h^*\|_{\text{lip}(X)^*} \leq \|e^* \otimes h^*\|_{\text{lip}(X, E)^*}$ and $\|e^* \otimes h^*\|_{\text{lip}(X, E)^*} = \|e^*\|_{E^*} \|h^*\|_{\text{lip}(X)^*}$. $\square$

Finally, we prove the following proposition.

PROPOSITION 2.4. *If $\text{lip}(X)$ separates points uniformly, then $E^* \otimes \text{lip}(X)^*$ is dense in $\text{lip}(X, E)^*$.*

This proposition can be proved by tracing a method in [4]. If $\text{lip}(X)$ separates points uniformly, then we have

$$E^* \otimes \{\tau_x\}_{x \in X} \subset E^* \otimes \text{lip}(X)^* \subset \text{lip}(X, E)^*.$$

Therefore, to prove Proposition 2.4, we show that $E^* \otimes \{\tau_x\}_{x \in X}$ is dense in $\text{lip}(X, E)^*$ if $\text{lip}(X)$ separates the points uniformly.

Let us introduce some notation. Set

$$\Delta := \{(x, x') \in X \times X : x = x'\},$$

$$W := (X \times X) \setminus \Delta,$$

$$K := X \cup W,$$

so that K is a locally compact space. Let $C_0(K, E)$ denote the Banach space of E -valued continuous functions that vanish at infinity. We embed $\text{lip}(X, E)$ in $C_0(K, E)$ in the usual way:

$$\begin{aligned} \tilde{f}(x) &:= f(x) \quad (x \in X), \\ \tilde{f}(x, x') &:= \frac{f(x) - f(x')}{d(x, x')} \quad ((x, x') \in W), \\ \tilde{f}(\omega) &:= 0, \end{aligned}$$

where ω is the point at infinity. Hence $\text{lip}(X, E)$ can be considered as a subspace of $C(K_\omega, E)$, where K_ω is the one-point compactification of K .

Let Y be a compact Hausdorff space. Then $C(Y, E)^*$ is isometrically isomorphic to $M(Y, E^*)$, where $M(Y, E^*)$ is the Banach space of E^* -valued countably additive regular Borel measures on Y with finite variation (see e.g. [2, Theorem 1.7.1]).

Let $\mu \in M(X, E^*)$. A functional f_μ^* on $\text{lip}(X, E)$ defined by

$$f_\mu^* f := \int_X f(x) d\mu(x) \quad (f \in \text{lip}(X, E)) \quad (2)$$

satisfies $f_\mu^* \in \text{lip}(X, E)^*$. Set

$$\mathcal{N} := \{f_\mu^* : \mu \in M(X, E^*) \text{ and } f_\mu^* \text{ is represented as (2)}\}.$$

Clearly, if $\text{lip}(X)$ separates points uniformly, then $E^* \otimes \{\tau_x\}_{x \in X}$ is contained in $\mathcal{N}$.

To prove Proposition 2.4, we first show the following proposition.

PROPOSITION 2.5. $\mathcal{N}$ is dense in $\text{lip}(X, E)^*$.

Proof. Let $f^* \in \text{lip}(X, E)^*$ be arbitrary. By using the Hahn–Banach theorem, there exists $\mu \in M(K_\omega, E^*)$ such that

$$f^* f = \int_{K_\omega} \tilde{f}(\mathbf{x}) d\mu(\mathbf{x}) \quad (f \in \text{lip}(X, E)).$$

Fix a point $x_0 \in X$. For each $n = 1, 2, 3, \dots$, set

$$X_n := \{x \in X : d(x, x_0) \leq n\}$$

and

$$W_n := \{(x, x') \in W : x, x' \in X_n \text{ and } d(x, x') \geq 1/(n+1)\}.$$

Put $K_n := X_n \cup W_n$. Then, $\{K_n\}_{n=1}^\infty$ is an increasing sequence of compact sets whose union is K . For each $n = 1, 2, 3, \dots$, put

$$f_n^* f := \int_{K_n} \tilde{f}(\mathbf{x}) d\mu(\mathbf{x}) \quad (f \in \text{lip}(X, E)).$$

Let $f \in \text{lip}(X, E)$ be arbitrary. By noting that $\tilde{f}(\omega) = 0$ and $K \setminus K_n \subset K \subset K_\omega = K \cup \{\omega\}$, we have

$$\begin{aligned} |f^* f - f_n^* f| &= \left| \int_{K_\omega} \tilde{f}(\mathbf{x}) d\mu(\mathbf{x}) - \int_{K_n} \tilde{f}(\mathbf{x}) d\mu(\mathbf{x}) \right| \\ &= \left| \int_{K_\omega \setminus K_n} \tilde{f}(\mathbf{x}) d\mu(\mathbf{x}) \right| \leq \int_{K_\omega \setminus K_n} \|\tilde{f}(\mathbf{x})\|_E d|\mu|(\mathbf{x}) \\ &\leq \int_{(K \setminus K_n) \cup \{\omega\}} \|\tilde{f}(\mathbf{x})\|_E d|\mu|(\mathbf{x}) = \int_{K \setminus K_n} \|\tilde{f}(\mathbf{x})\|_E d|\mu|(\mathbf{x}) \\ &\leq |\mu|(K \setminus K_n) \left(\sup_{\mathbf{x} \in K \setminus K_n} \|\tilde{f}(\mathbf{x})\|_E \right) \leq |\mu|(K \setminus K_n) \left(\sup_{\mathbf{x} \in K_\omega} \|\tilde{f}(\mathbf{x})\|_E \right) \\ &\leq |\mu|(K \setminus K_n) \|f\|_{\max}, \end{aligned}$$

where $|\mu|$ is the variation of μ . Hence we have

$$\|f_n^* - f^*\|_{\text{lip}(X, E)^*} \leq |\mu|(K \setminus K_n).$$

Since $|\mu|$ is countably additive, we have $|\mu|(K \setminus K_n) = |\mu|(K) - |\mu|(K_n)$ ($n = 1, 2, 3, \dots$). By noting that $|\mu|$ is finite, that $\bigcup_{n=1}^\infty K_n = K$ and that $\{K_n\}_{n=1}^\infty$ is increasing, we have $|\mu|(K \setminus K_n) \rightarrow 0$ as $n \rightarrow \infty$. Therefore, we have $\|f_n^* - f^*\|_{\text{lip}(X, E)^*} \rightarrow 0$ as $n \rightarrow \infty$. We next show that $f_n^* \in \mathcal{N}$. For any $f \in \text{lip}(X, E)$, we have

$$\begin{aligned} f_n^* f &= \int_{X_n} f(x) d\mu(x) + \int_{W_n} \tilde{f}(x, x') d\mu(x, x') \\ &= \int_{X_n} f(x) d\mu(x) + \int_{W_n} \frac{f(x)}{d(x, x')} d\mu(x, x') - \int_{W_n} \frac{f(x')}{d(x, x')} d\mu(x, x'). \end{aligned}$$

Since $d(x, x')$ is bounded away from zero on W_n , there exist $\mu_1, \mu_2 \in M(X, E^*)$ with support contained in X_n such that

$$\int_{W_n} \frac{f(x)}{d(x, x')} d\mu(x, x') = \int_{X_n} f(x) d\mu_1(x)$$

and

$$\int_{W_n} \frac{f(x')}{d(x, x')} d\mu(x, x') = \int_{X_n} f(x) d\mu_2(x).$$

Hence we have

$$f_n^* f = \int_{X_n} f(x) d(\mu + \mu_1 - \mu_2)(x).$$

Therefore, we have $f_n^* \in \mathcal{N}$. □

Next, we prove the following proposition.

PROPOSITION 2.6. *If $\text{lip}(X)$ separates points uniformly, then $E^* \otimes \{\tau_x\}_{x \in X}$ is dense in $\mathcal{N}$.*

Proof. Let $f_\mu^* \in \mathcal{N}$ and $\gamma > 0$ be arbitrary, and let $\text{supp } \mu$ denote the support of μ . By noting that X is compact and that $\text{supp } \mu$ is closed in X , there exist $x'_1, \dots, x'_N \in \text{supp } \mu$ such that $\text{supp } \mu \subset \bigcup_{j=1}^N B_j$, where $B_j := \{x \in X : d(x, x'_j) < \gamma/(4|\mu|(\text{supp } \mu))\}$. Set

$$\begin{aligned} A_1 &:= B_1, \\ A_j &:= B_j \setminus \bigcup_{i=1}^{j-1} B_i \quad (j = 2, \dots, N). \end{aligned}$$

Then $\{A_j\}_{j=1}^N$ is a disjoint collection of Borel sets such that $\text{supp } \mu \subset \bigcup_{j=1}^N A_j$ and $A_j \subset B_j$ ($j = 1, \dots, N$). We may assume that $A_j \neq \emptyset$ ($j = 1, \dots, N$) without loss of generality. Then, for each $j = 1, \dots, N$, we can choose $x_j \in A_j$. Set

$$f^* := \sum_{j=1}^N \mu(A_j) \otimes \tau_{x_j}.$$

Then $f^* \in E^* \otimes \{\tau_x\}_{x \in X}$. Let $f \in \text{lip}(X, E)$ with $\|f\|_{\max} \leq 1$ be arbitrary. Set

$$g := \sum_{j=1}^N f(x_j) \chi_{A_j},$$

where χ_{A_j} is the characteristic function on A_j . For any $x \in \text{supp } \mu$, there exists a unique $i \in \{1, \dots, N\}$ such that $x \in A_i$. By noting that $A_i \subset B_i$, we have

$$\begin{aligned} \|g(x) - f(x)\|_E &= \|f(x_i) - f(x)\|_E \leq d(x_i, x) \\ &\leq d(x_i, x'_i) + d(x'_i, x) < \frac{\gamma}{2|\mu|(\text{supp } \mu)}. \end{aligned}$$

Therefore, we have

$$\sup_{x \in \text{supp } \mu} \|g(x) - f(x)\|_E \leq \frac{\gamma}{2|\mu|(\text{supp } \mu)}.$$

Since

$$\int_X g(x) d\mu(x) = \sum_{j=1}^N (\mu(A_j))(f(x_j)) = \sum_{j=1}^N (\mu(A_j) \otimes \tau_{x_j})(f) = f^* f,$$

we have

$$\begin{aligned} |f^* f - f_\mu^* f| &= \left| \int_X g(x) d\mu(x) - \int_X f d\mu(x) \right| = \left| \int_X (g - f)(x) d\mu(x) \right| \\ &\leq \int_X \|g(x) - f(x)\|_E d|\mu|(x) = \int_{\text{supp } \mu} \|g(x) - f(x)\|_E d|\mu|(x) \\ &\leq |\mu|(\text{supp } \mu) \left(\sup_{x \in \text{supp } \mu} \|g(x) - f(x)\|_E \right) \leq \frac{\gamma}{2}. \end{aligned}$$

Hence we have

$$\|f^* - f_\mu^*\|_{\text{lip}(X, E)^*} < \gamma.$$

Therefore, $E^* \otimes \{\tau_x\}_{x \in X}$ is dense in $\mathcal{N}$. □

Propositions 2.5 and 2.6 imply that $E^* \otimes \{\tau_x\}_{x \in X}$ is dense in $\text{lip}(X, E)^*$ if $\text{lip}(X)$ separates points uniformly. Therefore, $E^* \otimes \text{lip}(X)^*$ is dense in $\text{lip}(X, E)^*$ if $\text{lip}(X)$ separates points uniformly.

3. Proof of Theorem 1

In this section, we prove Theorem 1. To prove it, let us introduce some notation. For any Banach spaces A and B , $A \otimes_\varepsilon B$ and $A \otimes_\pi B$ denote the algebraic tensor product of A and B with injective cross norm ε and projective cross norm π , respectively. Let $\widehat{A \otimes_\varepsilon B}$ and $\widehat{A \otimes_\pi B}$ denote the completion of $A \otimes_\varepsilon B$ and $A \otimes_\pi B$ with respect to the cross norms ε and π , respectively.

Also, let us introduce several conditions as follows:

- (A1) $\text{lip}(X)$ separates the points uniformly;
- (A2) either $\text{lip}(X)^*$ or E^* has the approximation property.

By Proposition 2.2, the condition (A1) implies that $\text{lip}(X)^*$ has the Radon–Nikodým property.

By using [7, Theorem 5.33], we get the following lemma.

LEMMA 3.1. *Suppose (A1) and (A2) hold. Then $E^* \widehat{\otimes}_\pi \text{lip}(X)^*$ is isometrically isomorphic to $(E \widehat{\otimes}_\varepsilon \text{lip}(X))^*$. Moreover, if $T: E^* \widehat{\otimes}_\pi \text{lip}(X)^* \rightarrow (E \widehat{\otimes}_\varepsilon \text{lip}(X))^*$ is an isometric isomorphism, then T satisfies*

$$\left(T \left(\sum_{j=1}^N e_j^* \otimes h_j^* \right) \right) \left(\sum_{i=1}^M e_i \otimes h_i \right) = \sum_{j=1}^N \sum_{i=1}^M (e_j^* e_i) (h_j^* h_i) \quad (3)$$

for any $\sum_{j=1}^N e_j^* \otimes h_j^* \in E^* \otimes_\pi \text{lip}(X)^*$ and $\sum_{i=1}^M e_i \otimes h_i \in E \otimes_\varepsilon \text{lip}(X)$.

By Lemma 3.1, $(E \widehat{\otimes}_\varepsilon \text{lip}(X))^{**}$ is isometrically isomorphic to $(E^* \widehat{\otimes}_\pi \text{lip}(X)^*)^*$.

LEMMA 3.2. *The mapping $U_0: E \otimes_\varepsilon \text{lip}(X) \rightarrow \text{lip}(X, E)$ defined by*

$$(U_0 z)(x) := \sum_{i=1}^M h_i(x) e_i \quad \left(z = \sum_{i=1}^M e_i \otimes h_i \in E \otimes_\varepsilon \text{lip}(X), x \in X \right) \quad (4)$$

is a linear isometry.

Proof. Let $z = \sum_{i=1}^M e_i \otimes h_i \in E \otimes_\varepsilon \text{lip}(X)$ be arbitrary. Then we have

$$\begin{aligned} \|U_0 z\|_{C(X, E)} &= \sup_{x \in X} \left\| \sum_{i=1}^M h_i(x) e_i \right\|_E = \sup_{x \in X} \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \left| e^* \left(\sum_{i=1}^M h_i(x) e_i \right) \right| \\ &= \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \sup_{x \in X} \left| \sum_{i=1}^M (e^* e_i) h_i(x) \right| = \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \left\| \sum_{i=1}^M (e^* e_i) h_i \right\|_{C(X)} \end{aligned} \quad (5)$$

and

$$\begin{aligned} \mathcal{L}_{X, E}(U_0 z) &= \sup_{\substack{x, x' \in X \\ x \neq x'}} \frac{\left\| \sum_{i=1}^M h_i(x) e_i - \sum_{i=1}^M h_i(x') e_i \right\|_E}{d(x, x')} \\ &= \sup_{\substack{x, x' \in X \\ x \neq x'}} \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \frac{|e^*(\sum_{i=1}^M h_i(x) e_i - \sum_{i=1}^M h_i(x') e_i)|}{d(x, x')} \\ &= \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \sup_{\substack{x, x' \in X \\ x \neq x'}} \frac{|\sum_{i=1}^M (e^* e_i) h_i(x) - \sum_{i=1}^M (e^* e_i) h_i(x')|}{d(x, x')} \\ &= \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \mathcal{L}_{X, \mathbb{K}} \left(\sum_{i=1}^M (e^* e_i) h_i \right). \end{aligned} \quad (6)$$

By using equations (5) and (6), we have

$$\begin{aligned} \|U_0 z\|_{\max} &= \max \left\{ \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \left\| \sum_{i=1}^M (e^* e_i) h_i \right\|_{C(X)}, \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \mathcal{L}_{X, \mathbb{K}} \left(\sum_{i=1}^M (e^* e_i) h_i \right) \right\} \\ &= \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \max \left\{ \left\| \sum_{i=1}^M (e^* e_i) h_i \right\|_{C(X)}, \mathcal{L}_{X, \mathbb{K}} \left(\sum_{i=1}^M (e^* e_i) h_i \right) \right\} \\ &= \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \left\| \sum_{i=1}^M (e^* e_i) h_i \right\|_{\max} = \varepsilon(z). \end{aligned}$$

Therefore, U_0 is a linear isometry. □

By noting that $E \widehat{\otimes}_\varepsilon \text{lip}(X)$ is the completion of $E \otimes_\varepsilon \text{lip}(X)$, there exists a unique linear isometry $U: E \widehat{\otimes}_\varepsilon \text{lip}(X) \rightarrow \text{lip}(X, E)$ such that

$$Uz = U_0 z \quad (z \in E \otimes_\varepsilon \text{lip}(X)). \quad (7)$$

LEMMA 3.3. *Suppose (A1) and (A2) hold. Then U is an isometric isomorphism from $E \widehat{\otimes}_\varepsilon \text{lip}(X)$ to $\text{lip}(X, E)$.*

Proof. It is sufficient to show that U is surjective. Let $\text{ran}(U)$ denote the range of U . Clearly, $\text{ran}(U)$ is a closed subspace of $\text{lip}(X, E)$. Here, we show that $\text{ran}(U)$ is dense in $\text{lip}(X, E)$. Note that $E^* \otimes \text{lip}(X)^*$ is a subspace of $(E \widehat{\otimes}_\varepsilon \text{lip}(X))^*$ (see e.g. [7, Proposition 3.1(c)]), and that $\|\cdot\|_{\text{lip}(X, E)^*}$ is the cross norm on $E^* \otimes \text{lip}(X)^*$ by Proposition 2.3. It is easy to see that

$$\varepsilon^*(z^*) \leq \|z^*\|_{\text{lip}(X, E)^*} \leq \pi(z^*) \quad (z^* \in E^* \otimes \text{lip}(X)^*), \quad (8)$$

where ε^* is the dual of the injective cross norm ε . By Lemma 3.1, we have

$$\varepsilon^*(z^*) = \pi(z^*) \quad (z^* \in E^* \otimes \text{lip}(X)^*). \quad (9)$$

By combining (8) and (9) we have

$$\|z^*\|_{\text{lip}(X, E)^*} = \pi(z^*) \quad (z^* \in E^* \otimes \text{lip}(X)^*).$$

Therefore, the mapping

$$J_0 : (E^* \otimes \text{lip}(X)^*, \|\cdot\|_{\text{lip}(X, E)^*}) \ni z^* \mapsto z^* \in E^* \widehat{\otimes}_\pi \text{lip}(X)^*$$

is a linear isometry. By using Proposition 2.4, there exists a unique linear isometry $J : \text{lip}(X, E)^* \rightarrow E^* \widehat{\otimes}_\pi \text{lip}(X)^*$ such that

$$Jz^* = J_0z^* \quad (z^* \in E^* \otimes \text{lip}(X)^*).$$

Let $z^* = \sum_{j=1}^N e_j^* \otimes h_j^* \in E^* \otimes \text{lip}(X)^*$ and $z = \sum_{i=1}^M e_i \otimes h_i \in E \otimes_\varepsilon \text{lip}(X)$ be arbitrary. Then we have

$$\begin{aligned} (TJz^*)(z) &= (Tz^*)(z) = \sum_{j=1}^N \sum_{i=1}^M (e_j^* e_i)(h_j^* h_i) \\ &= \sum_{j=1}^N h_j^* \left(\sum_{i=1}^M (e_j^* e_i) h_i \right) = \sum_{j=1}^N h_j^* (e_j^* \circ (Uz)) \\ &= \sum_{j=1}^N (e_j^* \otimes h_j^*)(Uz) = z^*(Uz) = (U^* z^*)(z). \end{aligned}$$

By noting that $E \widehat{\otimes}_\varepsilon \text{lip}(X)$ is the completion of $E \otimes_\varepsilon \text{lip}(X)$, and that TJz^* and $U^* z^*$ are bounded on $E \widehat{\otimes}_\varepsilon \text{lip}(X)$, the equation $TJz^* = U^* z^*$ holds on $E \widehat{\otimes}_\varepsilon \text{lip}(X)$. By noting that $E^* \otimes \text{lip}(X)^*$ is dense in $\text{lip}(X, E)^*$ by Proposition 2.4, and that TJ and U^* are bounded on $\text{lip}(X, E)^*$, the equation $TJ = U^*$ holds on $\text{lip}(X, E)^*$. Since $T : E^* \widehat{\otimes}_\pi \text{lip}(X)^* \rightarrow (E \widehat{\otimes}_\varepsilon \text{lip}(X))^*$ and $J : \text{lip}(X, E)^* \rightarrow E^* \widehat{\otimes}_\pi \text{lip}(X)^*$ are isometries, respectively, $U^* : \text{lip}(X, E)^* \rightarrow (E \widehat{\otimes}_\varepsilon \text{lip}(X))^*$ is an isometry. Therefore, $\text{ran}(U)$ is dense in $\text{lip}(X, E)$. $\square$

By Lemma 3.3, $\text{lip}(X, E)^{**}$ is isometrically isomorphic to $(E \widehat{\otimes}_\varepsilon \text{lip}(X))^{**}$.

For any Banach spaces A and B , $\mathcal{B}(A, B)$ denotes the Banach space of all bounded operators from A into B with operator norm $\|\cdot\|_{\mathcal{B}(A, B)}$.

LEMMA 3.4. *Suppose (A1) holds. Then $(E^* \widehat{\otimes}_\pi \text{lip}(X)^*)^*$ is isometrically isomorphic to $\mathcal{B}(E^*, \text{Lip}(X))$.*

Proof. By using [5, Corollary 1.22] hold, $(E^* \widehat{\otimes}_\pi \text{lip}(X)^*)^*$ is isometrically isomorphic to $\mathcal{B}(E^*, \text{lip}(X)^{**})$. Suppose that $\Psi: (E^* \widehat{\otimes}_\pi \text{lip}(X)^*)^* \rightarrow \mathcal{B}(E^*, \text{lip}(X)^{**})$ is an isometric isomorphism. Then Ψ is of the form

$$((\Psi f^{**})(e^*))(h^*) = f^{**}(e^* \otimes h^*) \quad (10)$$

for any $f^{**} \in (E^* \widehat{\otimes}_\pi \text{lip}(X)^*)^*$, $e^* \in E^*$ and $h^* \in \text{lip}(X)^*$ (see e.g. [5, Theorem 1.21 and Corollary 1.22]).

By Theorem A, $\mathcal{B}(E^*, \text{lip}(X)^{**})$ is isometrically isomorphic to $\mathcal{B}(E^*, \text{Lip}(X))$, via the map $\Phi: \mathcal{B}(E^*, \text{lip}(X)^{**}) \rightarrow \mathcal{B}(E^*, \text{Lip}(X))$ defined by

$$((\Phi A)(e^*))(x) := (Ae^*)(\tau_x) \quad (11)$$

for any $A \in \mathcal{B}(E^*, \text{lip}(X)^{**})$, $e^* \in E^*$ and $x \in X$.

Set $S := \Phi\Psi$. Since Φ and Ψ are isometric isomorphisms, S is an isometric isomorphism from $(E^* \widehat{\otimes}_\pi \text{lip}(X)^*)^*$ to $\mathcal{B}(E^*, \text{Lip}(X))$. $\square$

LEMMA 3.5. *$\mathcal{B}(E^*, \text{Lip}(X))$ is isometrically isomorphic to $\text{Lip}(X, E^{**})$.*

Proof. The mapping $R: \mathcal{B}(E^*, \text{Lip}(X)) \rightarrow \text{Lip}(X, E^{**})$ is defined by

$$((RA)(x))(e^*) := (Ae^*)(x) \quad (A \in \mathcal{B}(E^*, \text{Lip}(X)), x \in X, e^* \in E^*). \quad (12)$$

Then,

$$\|RA\|_{C(X, E^{**})} = \sup_{x \in X} \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} |(Ae^*)(x)| = \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \|Ae^*\|_{C(X)}$$

and

$$\mathcal{L}_{X, E^{**}}(RA) = \sup_{\substack{x, x' \in X \\ x \neq x'}} \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \frac{|(Ae^*)(x) - (Ae^*)(x')|}{d_X(x, x')} = \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \mathcal{L}_{X, \mathbb{K}}(Ae^*).$$

Therefore, we have

$$\begin{aligned} \|RA\|_{\max} &= \max\{\|RA\|_{C(X, E^{**})}, \mathcal{L}_{X, E^{**}}(RA)\} \\ &= \max\left\{ \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \|Ae^*\|_{C(X)}, \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \mathcal{L}_{X, \mathbb{K}}(Ae^*) \right\} \\ &= \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \max\{\|Ae^*\|_{C(X)}, \mathcal{L}_{X, \mathbb{K}}(Ae^*)\} = \|A\|_{\mathcal{B}(E^*, \text{Lip}(X))}. \end{aligned}$$

Hence, R is a linear isometry. Let $g \in \text{Lip}(X, E^{**})$. The mapping $A: E^* \rightarrow \text{Lip}(X)$ is defined by

$$(Ae^*)(x) = (g(x))(e^*) \quad (e^* \in E^*, x \in X).$$

It is clear that A is a linear operator. Also, we have

$$\begin{aligned} \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \|Ae^*\|_{C(X)} &= \sup_{x \in X} \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} |(Ae^*)(x)| = \sup_{x \in X} \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} |(g(x))(e^*)| \\ &= \sup_{x \in X} \|g(x)\|_{E^{**}} = \|g\|_{C(X, E^{**})} \end{aligned} \quad (13)$$

and

$$\begin{aligned} \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \mathcal{L}_{X, \mathbb{K}}(Ae^*) &= \sup_{\substack{x, x' \in X \\ x \neq x'}} \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \frac{|(Ae^*)(x) - (Ae^*)(x')|}{d_X(x, x')} \\ &= \sup_{\substack{x, x' \in X \\ x \neq x'}} \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \frac{|(g(x))(e^*) - (g(x'))(e^*)|}{d_X(x, x')} \\ &= \sup_{\substack{x, x' \in X \\ x \neq x'}} \frac{\|(g(x)) - (g(x'))\|_{E^{**}}}{d_X(x, x')} = \mathcal{L}_{X, E^{**}}(g). \end{aligned} \quad (14)$$

Equations (13) and (14) show that

$$\begin{aligned} \|A\|_{\mathcal{B}(E^*, \text{Lip}(X))} &= \max \left\{ \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \|Ae^*\|_{C(X)}, \sup_{\substack{e^* \in E^* \\ \|e^*\|_{E^*} \leq 1}} \mathcal{L}_{X, \mathbb{K}}(Ae^*) \right\} \\ &= \max\{\|g\|_{C(X, E^{**})}, \mathcal{L}_{X, E^{**}}(g)\} = \|g\|_{\max}. \end{aligned}$$

Therefore, we have $A \in \mathcal{B}(E^*, \text{Lip}(X))$. Also, we have

$$((RA)(x))(e^*) = (Ae^*)(x) = (g(x))(e^*)$$

for any $x \in X$ and $e^* \in E^*$. Hence, R is an isometric isomorphism from $\mathcal{B}(E^*, \text{Lip}(X))$ to $\text{Lip}(X, E^{**})$. $\square$

Proof of Theorem 1. Put

$$Qf^{**} := RST^*(U^{-1})^{**} f^{**} \quad (f^{**} \in \text{lip}(X, E)^{**}),$$

where R, S, T and U are as in the above lemmas. Since R, S, T^* and $(U^{-1})^{**}$ are isometric isomorphisms, Q is an isometric isomorphism from $\text{lip}(X, E)^{**}$ to $\text{Lip}(X, E^{**})$. Therefore, $\text{lip}(X, E)^{**}$ is isometrically isomorphic to $\text{Lip}(X, E^{**})$.

Next, we compute Qf^{**} for any $f^{**} \in \text{lip}(X, E)^{**}$. By using equations (10), (11) and (12), we have

$$\begin{aligned} ((Qf^{**})(x))(e^*) &= ((RST^*(U^{-1})^{**} f^{**})(x))(e^*) = ((ST^*(U^{-1})^{**} f^{**})(e^*))(x) \\ &= (T^*(U^{-1})^{**} f^{**})(e^* \otimes \tau_x) = ((U^{-1})^{**} f^{**})(T(e^* \otimes \tau_x)) \\ &= f^{**}((U^{-1})^* T(e^* \otimes \tau_x)) \end{aligned} \quad (15)$$

for any $f^{**} \in \text{lip}(X, E)^{**}$, $x \in X$ and $e^* \in E^*$. Here, we compute $(U^{-1})^* T(e^* \otimes \tau_x)$. Let $f \in \text{lip}(X, E)$ be arbitrary. Then there exists a sequence $\{z_k\}_{k=1}^\infty$ in $E \otimes \text{lip}(X)$ such that

$$\|Uz_k - f\|_{\max} \rightarrow 0 \quad (k \rightarrow \infty). \quad (16)$$

For each $k = 1, 2, 3, \dots$, z_k is represented as follows:

$$z_k = \sum_{i=1}^{M_k} e_i^{(k)} \otimes h_i^{(k)}$$

with some $e_i^{(k)} \in E$ and $h_i^{(k)} \in \text{lip}(X)$. Since U is an isometric isomorphism, we have

$$\varepsilon(U^{-1}f - z_k) \rightarrow 0 \quad (k \rightarrow \infty). \quad (17)$$

By using equations (1), (3), (4) and (7), we get the following estimate:

$$\begin{aligned} & |((U^{-1})^*T(e^* \otimes \tau_x))(f) - (e^* \otimes \tau_x)(f)| \\ & \leq |(T(e^* \otimes \tau_x))(U^{-1}f) - (T(e^* \otimes \tau_x))(z_k)| + |(T(e^* \otimes \tau_x))(z_k) - (e^* \otimes \tau_x)(f)| \\ & = |(T(e^* \otimes \tau_x))(U^{-1}f - z_k)| + \left| \sum_{i=1}^{M_k} (e^* e_i^{(k)}) h_i^{(k)}(x) - e^*(f(x)) \right| \\ & = |(T(e^* \otimes \tau_x))(U^{-1}f - z_k)| + \left| e^* \left(\sum_{i=1}^{M_k} h_i^{(k)}(x) e_i^{(k)} - f(x) \right) \right| \\ & \leq \|T(e^* \otimes \tau_x)\|_{E^*} \varepsilon(U^{-1}f - z_k) + \|e^*\|_{E^*} \left\| \sum_{i=1}^{M_k} h_i^{(k)}(x) e_i^{(k)} - f(x) \right\|_E \\ & = \pi(e^* \otimes \tau_x) \varepsilon(U^{-1}f - z_k) + \|e^*\|_{E^*} \| (U z_k)(x) - f(x) \|_E \\ & \leq \|e^*\|_{E^*} \|\tau_x\|_{\text{lip}(X)^*} \varepsilon(U^{-1}f - z_k) + \|e^*\|_{E^*} \|U z_k - f\|_{\max}. \end{aligned}$$

By using (16) and (17), we have

$$((U^{-1})^*T(e^* \otimes \tau_x))(f) = (e^* \otimes \tau_x)(f).$$

Now f is arbitrary, so we have

$$(U^{-1})^*T(e^* \otimes \tau_x) = e^* \otimes \tau_x. \quad (18)$$

Therefore, by combining (15) and (18), we have

$$((Qf^{**})(x))(e^*) = f^{**}(e^* \otimes \tau_x). \quad \square$$

4. Examples

In this section, we give several examples using Theorem 1.

Example 4.1. Suppose that $\text{lip}(X)$ separates points uniformly. Let K be a compact Hausdorff space. Let $C(K)$ denote the Banach space of $\mathbb{K}$ -valued continuous functions on K . Then $C(K)^*$ has the approximation property (see e.g. [7, p. 74]). It follows from Theorem 1 that $\text{lip}(X, C(K))^{**}$ is isometrically isomorphic to $\text{Lip}(X, C(K)^{**})$.

Example 4.2. Suppose that $\text{lip}(X)$ separates points uniformly. Let $(\Omega, \mathfrak{M}, \mu)$ be a σ -finite measure space and let $p \in [1, \infty]$. Then $L^p(\Omega)^*$ has the approximation property (see e.g. [6, Theorem 6.16] and [7, pp. 73–74]). It follows from Theorem 1 that $\text{lip}(X, L^p(\Omega))^{**}$

is isometrically isomorphic to $\text{Lip}(X, L^p(\Omega)^{**})$. By noting that $L^p(\Omega)$ is reflexive for any $p \in (1, \infty)$, $\text{lip}(X, L^p(\Omega))^{**}$ is isometrically isomorphic to $\text{Lip}(X, L^p(\Omega))$ for any $p \in (1, \infty)$.

Example 4.3. Suppose that $\text{lip}(X)$ separates points uniformly. Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. Szankowski showed in [8] that $\mathcal{B}(\mathcal{H})$ does not have the approximation property. Therefore, by [7, Corollary 4.7], $\mathcal{B}(\mathcal{H})^*$ does not have the approximation property. On the other hand, if $\text{Lip}(X)$ has the approximation property, then so does $\text{lip}(X)^*$ by Theorem A and [7, Corollary 4.7]. It follows from Theorem 1 that $\text{lip}(X, \mathcal{B}(\mathcal{H}))^{**}$ is isometrically isomorphic to $\text{Lip}(X, \mathcal{B}(\mathcal{H})^{**})$ if $\text{Lip}(X)$ has the approximation property.

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REFERENCES

- [1] W. G. Bade, P. C. Curtis Jr and H. G. Dales. Amenability and weak amenability for Beurling and Lipschitz algebras. *Proc. London Math. Soc.* **55** (1987), 359–377.
- [2] P. Cembranos and J. Mendoza. *Banach Spaces of Vector-Valued Functions* (Lecture Notes in Mathematics, 1676). Springer, Berlin, 1997.
- [3] L. G. Hanin. Kantorovich-Rubinstein norm and its application in the theory of Lipschitz spaces. *Proc. Amer. Math. Soc.* **115** (1992), 345–352.
- [4] J. A. Johnson. Banach spaces of Lipschitz functions and vector-valued Lipschitz functions. *Trans. Amer. Math. Soc.* **148** (1970), 147–169.
- [5] W. A. Light and E. W. Cheney. *Approximation Theory in Tensor Product Spaces* (Lecture Notes in Mathematics, 1169). Springer, Berlin, 1985.
- [6] W. Rudin. *Real and Complex Analysis*, 3rd edn. McGraw-Hill, New York, 1987.
- [7] R. A. Ryan. *Introduction to Tensor Products of Banach Spaces* (Springer Monographs in Mathematics). Springer, London, 2002.
- [8] A. Szankowski. $\mathcal{B}(\mathcal{H})$ does not have the approximation property. *Acta Math.* **147** (1981), 89–108.
- [9] N. Weaver. Subalgebras of little Lipschitz algebras. *Pacific J. Math.* **173** (1996), 283–293.
- [10] N. Weaver. *Lipschitz Algebras*. World Scientific, River Edge, NJ, 1999.

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