COHOMOLOGY ON NEIGHBORHOODS OF NON-PLURIHARMONIC LOCI IN PSEUDOCONVEX KÄHLER MANIFOLDS

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Abstract. We study the cohomology groups of vector bundles on neighborhoods of non-pluriharmonic loci in q -complete Kähler manifolds and in compact Kähler manifolds. Applying our results, we show variants of the Lefschetz hyperplane theorem.

1. Introduction

We are interested in cohomology groups of holomorphic vector bundles on neighborhoods of non-pluriharmonic loci, especially an isomorphism theorem of natural restriction maps and vanishing of higher cohomologies of certain degree. A generalization of the Lefschetz hyperplane theorem is a typical example of our interests. In a paper Tiba [13], studied the cohomology of holomorphic vector bundles over a Stein manifold on open neighborhoods of the non-pluriharmonic locus $\text{supp } i\partial\bar{\partial}\varphi$ of an exhaustive plurisubharmonic function φ , and showed a cohomology vanishing. He also obtained similar results for projective manifolds by taking an ω_0 -psh function φ , namely $\omega_0 + i\partial\bar{\partial}\varphi \geq 0$, for a Kähler form ω_0 . Using these, a result analogous to the Lefschetz hyperplane theorem was obtained.

In this paper we extend Tiba's result above to a certain class of pseudoconvex Kähler manifolds, X , including the compact case, which is not necessarily projective. One example that does not belong to Tiba's situation is a quasi-torus. It is natural to expect that by considering q -completeness, we can prove the vanishing of cohomology up to the middle. Let ψ be a smooth exhaustive plurisubharmonic function on X (which has a certain decreasing approximation of smooth plurisubharmonic functions, see Definition 1.1). Let $f : X \rightarrow S$ be a holomorphic proper mapping to a Stein manifold S . As a generalization of Tiba's result, this paper shows the vanishing and isomorphism of the cohomology of holomorphic vector bundles on open neighborhoods of $\text{supp } i\partial\bar{\partial}\psi$, or $f^{-1}(\text{supp } i\partial\bar{\partial}\varphi)$ for an exhaustive plurisubharmonic function φ on S under the condition that X is a q -complete manifold. We first consider the special case of this result for $\text{supp } i\partial\bar{\partial}\psi$ in terms of saying that a non-pluriharmonic locus $\text{supp } i\partial\bar{\partial}\psi$ coincides with a sublevel set of ψ , and then extend this case to Kähler manifolds. Then we get a certain extension of the Serre duality considered on compact manifolds to weakly pseudoconvex Kähler manifolds when a holomorphic vector bundle has positivity. Finally, using the case where the holomorphic vector bundle of the

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main results is $\Lambda^p T_X^*$, we obtain a variant of the Lefschetz hyperplane theorem on q -complete manifolds.

We now explain a few notions to state our results more precisely. Let F be a holomorphic vector bundle over X and let $\mathcal{E}^{p,q}(F)$ be the sheaf of germs of $\mathcal{C}^\infty$ sections of $\Lambda^{p,q} T_X^* \otimes F$. Let $K \neq \emptyset$ be a closed set. For any open neighborhoods $V \subset U$ of K , the inclusion map induces $H^p(U, F) \rightarrow H^p(V, F)$. We define the direct limit of cohomology to a closed set K by $\lim_{\rightarrow K \subset U} H^p(U, F)$, where U runs through all open neighborhoods of K . The set of plurisubharmonic functions on X is denoted by $PSH(X)$.

DEFINITION 1.1. Let X be a complex manifold and φ be a plurisubharmonic function on it. Let K be a closed set such that $\text{supp } i\partial\bar{\partial}\varphi \subset K$. A sequence of functions $(\varphi_j)_{j \in \mathbb{N}}$ is said to be a decreasing approximation of smooth plurisubharmonic functions to φ and K on X , if φ_j satisfies the following conditions:

- (i) $\varphi_j \in PSH \cap \mathcal{E}(X)$,
- (ii) $\varphi_j > \varphi$ on K ,
- (iii) the family $(\varphi_j)_{j \in \mathbb{N}}$ is decreasing in j and $\lim_{j \rightarrow +\infty} \varphi_j = \varphi$ on X , and
- (iv) for any open neighborhood U of K , there exists an integer j_0 such that $\{z \in X \mid \varphi(z) < \varphi_{j_0}(z)\} \subset U$.

If $K = \text{supp } i\partial\bar{\partial}\varphi$, then $(\varphi_j)_{j \in \mathbb{N}}$ is simply said to be a decreasing approximation of smooth plurisubharmonic functions to φ .

Let q be an integer with $1 \leq q \leq n = \dim X$. A function $\varphi : X \rightarrow [-\infty, +\infty[$ is said to be *exhaustive* if all sublevel sets $X_c := \{z \in X \mid \varphi(z) < c\}$, $c < \sup_X \varphi$, are relatively compact (see [6, Chapter I]). A function $\psi : X \rightarrow \mathbb{R}$ is said to be *q -convex* if ψ is of class $\mathcal{C}^2$ and if its Levi form $i\partial\bar{\partial}\psi$ has at least $n - q + 1$ positive eigenvalues on the tangent space $T_{X,x}$ for each $x \in X$. A complex manifold is said to be *weakly pseudoconvex* if there exists a smooth exhaustive plurisubharmonic function, and is said to be *q -complete* if there exists a smooth exhaustive q -convex function (see [3, 10]). Andreotti and Grauert [3] and Ohsawa [10] proved a vanishing result for the higher-degree cohomology of q -complete manifolds, namely $H^p(X, F) = 0$ for any $p \geq q$.

The goal of this paper is to prove the following main results.

THEOREM 1.2. Let X be a weakly pseudoconvex Kähler manifold of dimension n ($n \geq 3$) such that there exists a smooth plurisubharmonic function ψ_q on X that is q -convex. Let F be a holomorphic vector bundle over X . Let $\varphi_1, \dots, \varphi_m$ be non-constant smooth exhaustive plurisubharmonic functions on X . We assume that there exists a decreasing approximation $(\varphi_{j,k})_{k \in \mathbb{N}}$ to each φ_j on X as in Definition 1.1. Then the natural map

$$H^p(X, F) \longrightarrow \varinjlim_{(\bigcap_{j=1}^m \text{supp } i\partial\bar{\partial}\varphi_j) \subset U} H^p(U, F)$$

is an isomorphism for $0 \leq p < n - q - m$ and is injective for $p = n - q - m$.

In particular, the following holds from the q -completeness of X :

$$\varinjlim_{(\bigcap_{j=1}^m \text{supp } i\partial\bar{\partial}\varphi_j) \subset U} H^p(U, F) = 0$$

for $q \leq p < n - q - m$.

As for the case where we do not assume the existence of decreasing approximations as in Definition 1.1, we obtain the following theorem by using Theorem 1.2.

THEOREM 1.3. *Let X be a Kähler manifold of dimension n ($n \geq 3$) and S be a Stein manifold. Let $f : X \rightarrow S$ be a holomorphic proper mapping and F be a holomorphic vector bundle over X . Let $\varphi_1, \dots, \varphi_m$ be non-constant exhaustive plurisubharmonic functions on S such that $f^{-1}(\text{supp } i\partial\bar{\partial}\varphi_j) \neq \emptyset$. We assume that there exists a smooth strictly plurisubharmonic function ψ on S such that $f^*\psi$ is q -convex. Then the natural map*

$$H^p(X, F) \longrightarrow \varinjlim_{f^{-1}(\bigcap_{j=1}^m \text{supp } i\partial\bar{\partial}\varphi_j) \subset U} H^p(U, F)$$

is an isomorphism for $0 \leq p < n - q - m$ and is injective for $p = n - q - m$.

In particular, the following holds from the q -completeness of X :

$$\varinjlim_{f^{-1}(\bigcap_{j=1}^m \text{supp } i\partial\bar{\partial}\varphi_j) \subset U} H^p(U, F) = 0$$

for $q \leq p < n - q - m$. We note that, if furthermore f is flat, then

$$f^{-1}\left(\bigcap_{j=1}^m \text{supp } i\partial\bar{\partial}\varphi_j\right) = \bigcap_{j=1}^m \text{supp } i\partial\bar{\partial}f^*\varphi_j.$$

When the base space S is a projective manifold instead of a Stein manifold, we consider ample line bundles and their continuous hermitian metrics instead of plurisubharmonic functions. For a projective manifold Y , we let $\mathcal{K}_{NS}(Y) \subset H^{1,1}(Y, \mathbb{R})$ be the open cone generated by classes of ample divisors, which is equal to the ample cone $\text{Amp}(Y)$ (see [5, Chapter 6]). Here the space $H^{1,1}(Y, \mathbb{R}) := H^{1,1}(Y, \mathbb{C}) \cap H^2(Y, \mathbb{R})$ consists of real $(1, 1)$ cohomology classes. Let T be a closed positive current of type $(1, 1)$ on Y such that the cohomology class $\{T\}$ belongs to $\mathcal{K}_{NS}(Y)$. There exist very ample line bundles $L_1, \dots, L_l$ and positive numbers $a_1, \dots, a_l$ such that $\{T\} = a_1 c_1(L_1) + \dots + a_l c_1(L_l)$, where $c_1(L_j)$ is the first Chern class of L_j . Here there exist Kähler forms ω_j such that $\omega_j \in c_1(L_j)$. Let $\omega := a_1 \omega_1 + \dots + a_l \omega_l$ be a Kähler form. Then we get $\omega \in \{T\}$.

THEOREM 1.4. *Let X be a compact Kähler manifold of dimension n ($n \geq 3$) and Y be a projective manifold. Let $f : X \rightarrow Y$ be a surjective holomorphic mapping and F be a holomorphic vector bundle over X . Let $T_1, \dots, T_m$ be closed positive currents of type $(1, 1)$ on Y whose cohomology classes belong to $\mathcal{K}_{NS}(Y)$. We assume that there exist Kähler forms ω_j such that $\omega_j \in \{T_j\}$ and that $f^*\omega_j$ has at least $n - q + 1$ positive eigenvalues. Then the natural map*

$$H^p(X, F) \longrightarrow \varinjlim_{f^{-1}(\bigcap_{j=1}^m \text{supp } T_j) \subset U} H^p(U, F)$$

is an isomorphism for $0 \leq p < n - q - m$ and is injective for $p = n - q - m$.

The organization of this paper is as follows. We first consider the $\bar{\partial}$ -problem involving support of $\bar{\partial}$ -closed F -valued forms using L^2 -estimates. Then we proceed to discuss the properties of decreasing approximations as in Definition 1.1 in Section 3. We will prove the $m = 1$ case of the main results and the main results (Theorems 1.2–1.4) in

Sections 4–6. We consider a quasi-tori as a concrete example of Theorem 1.2 in Section 7. We obtain a certain extension of the Serre duality considered on compact manifolds to weakly pseudoconvex Kähler manifolds when a holomorphic vector bundle has positivity in Section 8. Finally, applying our results, we show variants of the Lefschetz hyperplane theorem in Section 9.

Theorem 4.1 (the $m = 1$ case of Theorem 1.2) is proved by first proving the sublevel set case using the Donnelly–Fefferman–Berndtsson type L^2 -estimate in Section 2, and then extending this result to the whole X using the properties of decreasing approximations as in Definition 1.1 in Section 3. Theorem 5.1 (the $m = 1$ case of Theorem 1.3) is shown as an application of Theorem 4.1, using the properties of decreasing approximations as in Definition 1.1. Theorem 6.1 (the $m = 1$ case of Theorem 1.4) is shown using Theorem 5.1 because considering the inverse image by f of the set from Y minus the zeros of a holomorphic section of a very ample line bundle over Y is the same as the situation of this theorem. The main results are shown using the Mayer–Vietoris exact sequence by induction of m .

2. L^2 -estimate

A complex manifold is said to be (weakly) *hyperconvex* if there exists a smooth plurisubharmonic function that is bounded exhaustive. Tiba showed [13, Proposition 1] using the Donnelly–Fefferman–Berndtsson type L^2 -estimate considered on the bounded Stein manifold and L^2 -Serre duality. In this section we show the Donnelly–Fefferman–Berndtsson type L^2 -estimate for (n, p) -forms and Proposition 2.7 below corresponding to [13, Proposition 1] on (weakly) hyperconvex manifolds that have a smooth q -convex plurisubharmonic function.

Let (X, ω) be a Kähler manifold. Let F be a holomorphic vector bundle over X and h be a smooth Hermitian metric of F . We denote by $L^2_{p,q}(X, F, h, \omega)$ the Hilbert space of F -valued (p, q) -forms u that satisfy

$$\|u\|_{h,\omega}^2 = \int_X |u|_{h,\omega}^2 dV_\omega < +\infty.$$

Let $\star_F$ be the Hodge-star operator $L^2_{p,q}(X, F, h, \omega) \rightarrow L^2_{n-p,n-q}(X, F^*, h^*, \omega)$ as in [4, 6] and let $\bar{\partial}_F^*$ be the Hilbert space adjoint to $\bar{\partial}_F : L^2_{p,q}(X, F, h, \omega) \rightarrow L^2_{p,q+1}(X, F, h, \omega)$. Let $i\Theta_{F,h}$ be the Chern curvature tensor of (F, h) and Λ_ω be the adjoint of multiplication of ω .

THEOREM 2.1. (L^2 -existence theorem, cf. [2, 5, 6]) *If ω is complete on X and the curvature operator $[i\Theta_{F,h}, \Lambda_\omega]$ is positive definite on $\Lambda^{p,q}T_X^* \otimes F$ for some $q \geq 1$, then for any form $g \in L^2_{p,q}(X, F, h, \omega)$ satisfying $\bar{\partial}g = 0$ and $\int_X \langle [i\Theta_{F,h}, \Lambda_\omega]^{-1}g, g \rangle_{h,\omega} dV_\omega < +\infty$, there exists $f \in L^2_{p,q-1}(X, F, h, \omega)$ such that $\bar{\partial}f = g$, $f \in (\text{Ker } \bar{\partial})^\perp$ and*

$$\|f\|_{h,\omega}^2 \leq \int_X \langle [i\Theta_{F,h}, \Lambda_\omega]^{-1}g, g \rangle_{h,\omega} dV_\omega.$$

In particular, because of the ellipticity of Δ'' , if g is smooth then f is also smooth.

Let X be a Kähler manifold of dimension n and ω_X be a Kähler metric on X . Let D be a relatively compact subdomain in X . Assume that there exists $\psi_q \in \mathcal{E}(X)$

which is plurisubharmonic and q -convex and there exist $\varphi, \eta \in \mathcal{E}(\overline{D})$ which are negative and plurisubharmonic on D . We assume that $\max\{\varphi, \eta\} \rightarrow 0$ when $z \rightarrow \partial D$ and that $\inf_D \eta < -1$. Define $\phi := -\log(-\varphi)$ and $\rho := M_\varepsilon(-\log(-\eta), 0)$. Here $\varepsilon > 0$ is a small positive number and M_ε is a regularized max function (see [6, Chapter I, Section 5]). Let F be a holomorphic vector bundle over X and h be a smooth Hermitian metric of F . We define F^* and h^* to be the duals of F and h . Let $\delta > 0$ be a positive number and let $\tilde{\psi}_q := \psi_q/\delta + \rho/\delta + \phi \in \mathcal{E}(D)$ be an exhaustive plurisubharmonic function. Put $\omega := \omega_\varepsilon := \varepsilon\omega_X + i\partial\bar{\partial}\tilde{\psi}_q$. Then ω is a complete Kähler metric on D . For any smooth function $f \in \mathcal{E}(X)$, we denote $(\gamma_j : f : \omega)$ by the condition that $\gamma_1 \geq \dots \geq \gamma_n$ are eigenvalues of $i\partial\bar{\partial}f$ with respect to ω . Then for all positive numbers $c > 0$, we have that $(\gamma_j/c : f : c\omega)$, $(c\gamma_j : cf : \omega)$ and $(\gamma_j : cf : c\omega)$. Let μ_j, σ_j and α_j be the eigenvalues of $\tilde{\psi}_q$ and ψ_q that satisfy conditions $(\mu_j : \tilde{\psi}_q : \omega)$, $(\sigma_j : \tilde{\psi}_q : \omega_X)$ and $(\alpha_j : \psi_q : \omega_X)$. We get $\sigma_j \geq \alpha_j/\delta$ and $\min_{\overline{D}} \alpha_{n-q+1} > 0$. Let m be a positive number $\min_{\overline{D}} \alpha_{n-q+1}$ and $M := M_{\varepsilon\delta}$ be a positive number $m/(\varepsilon\delta + m) < 1$.

LEMMA 2.2. *Let μ_j and σ_j be as in the above setup. Then we have the following conditions:*

- (i) $\mu_j = \frac{\sigma_j}{\varepsilon + \sigma_j} = 1 - \frac{\varepsilon}{\varepsilon + \sigma_j} \leq 1$,
- (ii) $\mu_1 \geq \dots \geq \mu_{n-q+1} = \frac{\sigma_{n-q+1}}{\varepsilon + \sigma_{n-q+1}} \geq M_{\varepsilon\delta} = \frac{m}{\varepsilon\delta + m}$.

Proof. Fix a point x_0 in D . Then there exists an orthonormal basis $(\zeta_1, \dots, \zeta_n)$ of T_{X, x_0} with respect to ω_X such that

$$i\partial\bar{\partial}\tilde{\psi}_q = i \sum \sigma_j \zeta_j^* \wedge \bar{\zeta}_j^*, \quad \omega_X = i \sum \zeta_j^* \wedge \bar{\zeta}_j^*.$$

We obtain that $\omega = \varepsilon\omega_X + i\partial\bar{\partial}\tilde{\psi}_q = i \sum (\varepsilon + \sigma_j) \zeta_j^* \wedge \bar{\zeta}_j^*$. By putting $\chi_j = (1/\sqrt{\varepsilon + \sigma_j}) \zeta_j$, we have that

$$\omega = i \sum \chi_j^* \wedge \bar{\chi}_j^*, \quad i\partial\bar{\partial}\tilde{\psi}_q = i \sum \frac{\sigma_j}{\varepsilon + \sigma_j} \chi_j^* \wedge \bar{\chi}_j^*$$

and that $\sigma_1/(\varepsilon + \sigma_1) \geq \dots \geq \sigma_n/(\varepsilon + \sigma_n)$, where $(\chi_1, \dots, \chi_n)$ is an orthonormal basis of T_{X, x_0} with respect to ω . From the second condition and $\sigma_{n-q+1} \geq \alpha_{n-q+1}/\delta \geq m/\delta$ on D , we have the conditions (i) and (ii). $\square$

Here we prove that there exists a positive integer N such that the curvature operator $A_p := [i\Theta_{F^*, h^* e^{-N\psi_q}}, \Lambda_{\varepsilon\omega_X}]$ is positive definite on $\Lambda^{n, p} T_X^* \otimes F|_D$ for any $p \geq q$.

Now we have that $(\alpha_j/\varepsilon : \psi_q : \varepsilon\omega_X)$ and that $A_p = [i\Theta_{F^*, h^*}, \Lambda_{\varepsilon\omega_X}] + N[i\partial\bar{\partial}\tilde{\psi}_q \otimes \text{id}_{F^*}, \Lambda_{\varepsilon\omega_X}]$. For any (n, p) -form, we get the inequality

$$[i\partial\bar{\partial}\tilde{\psi}_q, \Lambda_{\varepsilon\omega_X}] \geq \frac{\alpha_n}{\varepsilon} + \dots + \frac{\alpha_{n-p+1}}{\varepsilon} \geq \frac{\alpha_{n-q+1}}{\varepsilon} \geq \frac{m}{\varepsilon}.$$

Therefore we have that $A_p \geq [i\Theta_{F^*, h^*}, \Lambda_{\varepsilon\omega_X}] + Nm/\varepsilon$ on X . Because of the boundedness of $[i\Theta_{F^*, h^*}, \Lambda_{\varepsilon\omega_X}]$ by the relative-compactness of $D \subset\subset X$, we obtain the above claim.

LEMMA 2.3. (cf. [6, Chapter VIII]) *Let γ, ω be Hermitian metrics on X such that $\gamma \geq \omega$. If the curvature operator $A_p := [i\Theta_{F, h}, \Lambda_\omega]$ is positive definite on $\Lambda^{n, p} T_X^* \otimes F$, then the curvature operator $A_{p, \gamma} := [i\Theta_{F, h}, \Lambda_\gamma]$ is also positive definite on $\Lambda^{n, p} T_X^* \otimes F$ for any integer p in $\{1, \dots, n\}$.*

Proof. Let $x_0 \in X$ be a given point and $(z_1, \dots, z_n)$ be a coordinate such that

$$\begin{aligned}\omega &= i \sum_{1 \leq j \leq n} dz_j \wedge d\bar{z}_j, \quad \gamma = i \sum_{1 \leq j \leq n} \gamma_j dz_j \wedge d\bar{z}_j, \\ i\Theta_{F,h} &= i \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu \quad \text{at } x_0,\end{aligned}$$

where $\gamma_1 \leq \dots \leq \gamma_n$ are the eigenvalues of γ with respect to ω . We have $|dz_j|_\gamma^2 = \gamma_j^{-1}$ and $|dz_K|_\gamma^2 = \gamma_K^{-1}$ for any multi-index K , with the notation $\gamma_K = \prod_{j \in K} \gamma_j$.

For every $u = \sum u_{K,\lambda} dz_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_n \otimes e_\lambda$, $|K| = p$, $1 \leq \lambda \leq \text{rank } F$, the computations yield

$$\begin{aligned}\Lambda_\gamma u &= \sum_{|I|=p-1} \sum_{j,\lambda} i(-1)^{n+j-1} \gamma_j^{-1} u_{jI,\lambda} dz_1 \wedge \dots \wedge \widehat{dz_j} \wedge \dots \wedge dz_n \wedge d\bar{z}_I \otimes e_\lambda, \\ A_{p,\gamma} u &= \sum_{|I|=p-1} \sum_{j,k,\lambda,\mu} \gamma_j^{-1} c_{jk\lambda\mu} u_{jI,\lambda} dz_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_{kI} \otimes e_\mu, \\ \langle A_{p,\gamma} u, u \rangle_{h,\gamma} &= (\gamma_1 \dots \gamma_n)^{-1} \sum_{|I|=p-1} \gamma_I^{-1} \sum_{j,k,\lambda,\mu} \gamma_j^{-1} \gamma_k^{-1} c_{jk\lambda\mu} u_{jI,\lambda} \bar{u}_{kI,\mu} \\ &\geq (\gamma_1 \dots \gamma_n)^{-1} \sum_{|I|=p-1} \gamma_I^{-2} \sum_{j,k,\lambda,\mu} \gamma_j^{-1} \gamma_k^{-1} c_{jk\lambda\mu} u_{jI,\lambda} \bar{u}_{kI,\mu} \\ &= \gamma_1 \dots \gamma_n \langle A_p S_\gamma u, S_\gamma u \rangle_{h,\omega},\end{aligned}$$

where S_γ is the operator defined by

$$S_\gamma u = \sum_K (\gamma_1 \dots \gamma_n \gamma_K)^{-1} u_{K,\lambda} dz_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_K \otimes e_\lambda.$$

Therefore we get $\langle A_{p,\gamma} u, u \rangle_{h,\gamma} > 0$, by the positivity of A_p . $\square$

LEMMA 2.4. *We have that the curvature operator $A_{p,\omega} := [i\Theta_{F^*,h^*e^{-N\psi_q}}, \Lambda_\omega]$ is positive definite on $\Lambda^{n,p} T_X^* \otimes F|_D$ for any $p \geq q$.*

In particular, if $d\varphi$ or $d\eta \neq 0$ ($x_0 \rightarrow \partial D$) then $A_{p,\omega} \searrow 0$ ($x_0 \rightarrow \partial D$).

Proof. There exists $N \in \mathbb{N}$ such that $A_p = [i\Theta_{F^*,h^*e^{-N\psi_q}}, \Lambda_{\varepsilon\omega_X}] > 0$ on D , for every (n, p) -form. Since $\omega = \varepsilon\omega_X + i\partial\bar{\partial}\tilde{\psi}_q \geq \varepsilon\omega_X$ and using Lemma 2.3, we have that $A_{p,\omega} > 0$.

From the condition $(\sigma_j/\varepsilon : \tilde{\psi}_q : \varepsilon\omega_X)$, for any point $x_0 \in D$, there exists an orthonormal basis $(\zeta_1, \dots, \zeta_n)$ of T_{X,x_0} such that

$$\varepsilon\omega_X = i \sum \zeta_j^* \wedge \bar{\zeta}_j^*, \quad i\partial\bar{\partial}\tilde{\psi}_q = i \sum \frac{\sigma_j}{\varepsilon} \zeta_j^* \wedge \bar{\zeta}_j^*.$$

We have that

$$\omega = i \sum \left(1 + \frac{\sigma_j}{\varepsilon}\right) \zeta_j^* \wedge \bar{\zeta}_j^*.$$

Since $d\varphi$ or $d\eta \neq 0$ ($x_0 \rightarrow \partial D$), at least one eigenvalue σ_j tends to $+\infty$; therefore at least one eigenvalue of ω tends to $+\infty$, as x_0 tends to ∂D . $\square$

Let $\kappa \in \mathcal{E}(\mathbb{R})$ such that $\kappa'(t) \geq 1$, $\kappa''(t) \geq 0$ for $t \geq 0$. Put $\xi := (N+1)\psi_q + \kappa \circ \rho - \delta\phi$. The proof of the following lemma is similar to [9, Lemma 3.1] and [13, Lemma 1] except for the existence of a smooth strictly plurisubharmonic function.

LEMMA 2.5. *Let $f \in L^2_{n,p}(D, F^*, h^*e^{-\xi}, \omega)$ such that $\bar{\partial}f = 0$. Assume that $\delta < 2M_{\varepsilon\delta}$ and $p \geq q+1$. Then there exists $u \in L^2_{n,p-1}(D, F^*, h^*e^{-\xi}, \omega)$ such that $\bar{\partial}u = f$ and that*

$$\int_D |u|_{h^*,\omega}^2 e^{-\xi} dV_\omega \leq C_{\delta,\varepsilon} \int_D |f|_{h^*,\omega}^2 e^{-\xi} dV_\omega,$$

where $C_{\delta,\varepsilon}$ depends only on δ and ε .

Proof. There exist relatively compact weakly pseudoconvex subdomains $D_1 \subset D_2 \subset \dots \subset D$ which exhaust D . Since $\kappa \circ \rho$ is a plurisubharmonic function and $e^{-\delta\phi}$ is bounded on D_k , we have that $[i\Theta_{F^*,h^*e^{-(N+1)\psi_q - \kappa \circ \rho}}, \Lambda_\omega] = [i\Theta_{F^*,h^*e^{-\xi - \delta\phi}}, \Lambda_\omega] > 0$ on $\Lambda^{n,p}T_X^* \otimes F|_D$ and $f \in L^2_{n,p}(D_k, F^*, h^*e^{-\xi - \delta\phi}, \omega)$. Because of the L^2 -estimate, there exists a minimal solution u_k in $L^2_{n,p-1}(D_k, F^*, h^*e^{-\xi - \delta\phi}, \omega)$ such that $\bar{\partial}f = u_k$ and that

$$\int_{D_k} |u_k|_{h^*,\omega}^2 e^{-\xi - \delta\phi} dV_\omega \leq \int_{D_k} \langle [i\Theta_{F^*,h^*e^{-\xi - \delta\phi}}, \Lambda_\omega]^{-1} f, f \rangle_{h^*,\omega} e^{-\xi - \delta\phi} dV_\omega.$$

Since ϕ and $\bar{\partial}\phi$ are bounded in D_k , we have that $u_k e^{\delta\phi}$ is the minimal solution of $\bar{\partial}(u_k e^{\delta\phi}) = (f + \delta\bar{\partial}\phi \wedge u_k) e^{\delta\phi}$ in $L^2_{n,p-1}(D_k, F^*, h^*e^{-\xi - 2\delta\phi})$. Then we get the inequality:

$$\begin{aligned} & \int_{D_k} |u_k e^{\delta\phi}|_{h^*,\omega}^2 e^{-\xi - 2\delta\phi} dV_\omega \\ &= \int_{D_k} |u_k|_{h^*,\omega}^2 e^{-\xi} dV_\omega \\ &\leq \int_{D_k} \langle [i\Theta_{F^*,h^*e^{-\xi - 2\delta\phi}}, \Lambda_\omega]^{-1} (f + \delta\bar{\partial}\phi \wedge u_k), f + \delta\bar{\partial}\phi \wedge u_k \rangle_{h^*,\omega} e^{-\xi} dV_\omega \\ &\leq \left(1 + \frac{1}{t}\right) \int_{D_k} \langle [i\Theta_{F^*,h^*e^{-\xi - 2\delta\phi}}, \Lambda_\omega]^{-1} f, f \rangle_{h^*,\omega} e^{-\xi} dV_\omega \\ &\quad + (1+t)\delta^2 \int_{D_k} \langle [i\Theta_{F^*,h^*e^{-\xi - 2\delta\phi}}, \Lambda_\omega]^{-1} \bar{\partial}\phi \wedge u_k, \bar{\partial}\phi \wedge u_k \rangle_{h^*,\omega} e^{-\xi} dV_\omega \end{aligned}$$

for every $t > 0$. We have that

$$\langle [i\Theta_{F^*,h^*e^{-\xi - 2\delta\phi}}, \Lambda_\omega]^{-1} v, v \rangle_{h^*,\omega} \leq \frac{1}{2M\delta} |v|_{h^*,\omega}^2$$

for any F -valued (n, p) -form v . In fact, by the condition $(\mu_j : \tilde{\psi}_q : \omega)$ and Lemma 2.2, for any (n, p) -forms we get $[i\partial\bar{\partial}\tilde{\psi}_q, \Lambda_\omega] \geq \mu_n + \dots + \mu_{n-p+1} \geq \mu_{n-q+1} + \mu_{n-q} \geq 2M$. Since $\psi_q + \rho + \delta\phi = \delta\tilde{\psi}_q$, we have that

$$\begin{aligned} [i\Theta_{F^*,h^*e^{-\xi - 2\delta\phi}}, \Lambda_\omega] &= [i\Theta_{F^*,h^*e^{-(N+1)\psi_q - \kappa \circ \rho - \delta\phi}}, \Lambda_\omega] > [i\partial\bar{\partial}(\psi_q + \kappa \circ \rho + \delta\phi) \otimes \text{id}_{F^*}, \Lambda_\omega] \\ &\geq [i\partial\bar{\partial}(\psi_q + \rho + \delta\phi) \otimes \text{id}_{F^*}, \Lambda_\omega] = \delta[i\partial\bar{\partial}\tilde{\psi}_q \otimes \text{id}_{F^*}, \Lambda_\omega] \geq 2\delta M. \end{aligned}$$

Moreover $0 < M := M_{\varepsilon\delta}$ tends to 1, as $\varepsilon\delta$ tends to 0.

We show that $|\bar{\partial}\phi|_\omega \leq 1$. By the definition $\phi := -\log(-\varphi)$, we obtain that $i\partial\bar{\partial}\phi \geq i\partial\phi \wedge \bar{\partial}\phi$. Here

$$\omega := \varepsilon\omega_X + i\partial\bar{\partial}\tilde{\psi}_q = \varepsilon\omega_X + i\partial\bar{\partial}\left(\frac{\psi_q}{\delta} + \frac{\rho}{\delta} + \phi\right) \geq i\partial\bar{\partial}\phi \geq i\bar{\partial}\phi \wedge \partial\phi.$$

Therefore we have that $1 \geq |i\bar{\partial}\phi \wedge \partial\phi|_\omega = |\bar{\partial}\phi|_\omega^2$.

From the above two inequalities, we obtain the following estimate:

$$\langle [i\Theta_{F^*, h^* e^{-\xi-2\delta\phi}}, \Lambda_\omega]^{-1} \bar{\partial}\phi \wedge u_k, \bar{\partial}\phi \wedge u_k \rangle_{h^*, \omega} \leq \frac{1}{2M\delta} |\bar{\partial}\phi \wedge u_k|_{h^*, \omega}^2 \leq \frac{1}{2M\delta} |u_k|_{h^*, \omega}^2.$$

Let

$$U := \int_{D_k} |u_k|_{h^*, \omega}^2 e^{-\xi} dV_\omega \quad \text{and} \quad F := \int_{D_k} |f|_{h^*, \omega}^2 e^{-\xi} dV_\omega.$$

Then from the last L^2 -estimate, we obtain that

$$U \leq \frac{1}{2M - (1+t)\delta} \frac{1+t}{t\delta} F.$$

In the case of $\delta < 2M$, by taking t sufficiently small, we have that

$$\int_{D_k} |u_k|_{h^*, \omega}^2 e^{-\xi} dV_\omega \leq C_{\delta, \varepsilon} \int_{D_k} |f|_{h^*, \omega}^2 e^{-\xi} dV_\omega \leq C_{\delta, \varepsilon} \int_D |f|_{h^*, \omega}^2 e^{-\xi} dV_\omega.$$

Here, for any positive number $2 > \delta > 0$, there exists a sufficiently small positive number $\varepsilon > 0$ such that $\delta < 2M = 2m/(\varepsilon\delta + m)$. Hence we may choose a subsequence of $(u_k)_{k \in \mathbb{N}}$ converging weakly in $L^2_{n, p-1}(D, F^*, h^* e^{-\xi}, \omega)$ to u . $\square$

LEMMA 2.6. *Let $1 \leq p \leq n - q - 1$ and $\alpha \in L^2_{0, p}(D, F, h e^\xi, \omega)$ such that $\bar{\partial}\alpha = 0$. Assume that $\delta < 2M_{\varepsilon\delta}$. Then there exists $\beta \in L^2_{0, p-1}(D, F, h e^\xi, \omega)$ such that $\bar{\partial}\beta = \alpha$ and that*

$$\int_D |\beta|_{h\omega}^2 e^\xi dV_\omega \leq C_{\delta, \varepsilon} \int_D |\alpha|_{h\omega}^2 e^\xi dV_\omega,$$

where $C_{\delta, \varepsilon}$ depends only on δ and ε .

Proof. Lemma 2.6 follows from [13, Lemma 2], except for the index p . In Lemma 2.5, the condition on the index p is $q + 1 \leq p$. By the adjointness of [13, Lemma 2], we get the index condition $q + 1 \leq n - p$, i.e. $p \leq n - q - 1$. $\square$

PROPOSITION 2.7. *Let $1 \leq p \leq n - q - 1$ and $\partial'D = \{z \in \partial D \mid \varphi(z) = 0\}$. Assume that φ is pluriharmonic on a neighborhood of $\partial'D$ and $d\varphi \neq 0$ on $\partial'D$. Let $\alpha \in \mathcal{E}^{0, p}(\bar{D}, F)$ such that $\bar{\partial}\alpha = 0$ in D and that $\text{supp } \alpha \cap D \subset \{z \in D \mid \rho = 0\}$. Then there exists $\beta \in \mathcal{E}^{0, p-1}(D, F)$ such that $\bar{\partial}\beta = \alpha$ and that $\text{supp } \beta \subset \{z \in D \mid \rho \leq 1\}$.*

Proof. Proposition 2.7 follows from [13, Proposition 1], except for the index δ and the boundedness of α on a small neighborhood of any points $a \in \{z \in \partial D \mid \eta(z) \neq 0\}$. There exists a small neighborhood $U \subset X$ of the point a such that φ is pluriharmonic and $i\partial\bar{\partial}\phi = i\partial\varphi \wedge \bar{\partial}\varphi/\varphi^2$ on $U \cap D$. Since $\eta(a) \neq 0$, we may assume that $i\partial\bar{\partial}\rho$ and $\kappa \circ \rho$ are bounded. Then we obtain that $e^\xi \leq C|\varphi|^\delta$ on $U \cap D$, where C is a bounded positive constant. By the Kählerness of $\varepsilon\omega_X + i\partial\bar{\partial}(\psi_q/\delta + \rho/\delta)$, there exist a smaller neighborhood $U' \subset U$ of

the point a and a bounded Kähler potential $\Omega \in sPSH \cap \mathcal{E}(U')$ such that $i\partial\bar{\partial}\Omega = \varepsilon\omega_X + i\partial\bar{\partial}(\psi_q/\delta + \rho/\delta)$ on U' . Hence by [11, Lemma 5], if $1 < \delta < 2$ then

$$\int_{U' \cap D} |\alpha|_{h\omega}^2 dV_\omega \leq \tilde{C} \int_{U' \cap D} |\alpha|_{hi\partial\bar{\partial}\Omega}^2 |\varphi|^{\delta-2} (i\partial\bar{\partial}\Omega)^n < +\infty.$$

In Lemma 2.6, the condition on the index δ is $\delta < 2M_{\varepsilon\delta} = 2m/(\varepsilon\delta + m) < 2$. Therefore it is necessary that the condition $1 < \delta < 2m/(\varepsilon\delta + m) < 2$ holds. Clearly we can choose positive numbers δ and ε such that the condition is satisfied. $\square$

3. Decreasing approximation of smooth plurisubharmonic functions

In this section, we explain the properties of decreasing approximations defined in Section 1.

EXAMPLE 3.1. Let φ be a plurisubharmonic function on $\mathbb{C}^n$ and $\rho : \mathbb{C}^n \rightarrow \mathbb{R}_{\geq 0}$ be a smooth function depending only on $|z|$ such that $\text{supp } \rho \subset \mathbb{B}^n$ and that $\int_{\mathbb{C}^n} \rho(z) dV = 1$, where $\mathbb{B}^n$ is the unit ball. This function ρ is called a positive mollifier. Define $\rho_\varepsilon(z) = (1/\varepsilon^{2n})h(z/\varepsilon)$ for $\varepsilon > 0$. Let $\varphi_\varepsilon := \varphi * \rho_\varepsilon$ be a smooth plurisubharmonic function. Then the function sequence $(\varphi_{1/j})_{j \in \mathbb{N}}$ is a decreasing approximation to φ as in Definition 1.1.

This example does not hold on general complex manifolds that are not flat, because the convolution of a plurisubharmonic function and a positive mollifier is not necessarily plurisubharmonic. Examples of decreasing approximations as in Definition 1.1 on general complex manifolds include that in the following proposition.

PROPOSITION 3.2. *Let X be a complex manifold. Let ψ be a smooth plurisubharmonic function which is n -convex. Therefore $\text{supp } i\partial\bar{\partial}\psi = X$. Then we have the following properties:*

- (I) *If φ is a pluriharmonic function, then the plurisubharmonic function $\max\{\varphi, \psi\}$ has a decreasing approximation as in Definition 1.1.*
- (II) *If φ is a plurisubharmonic function which is smooth on an open neighborhood of $\{z \in X \mid \varphi(z) = \psi(z)\}$ and which has a decreasing approximation $(\varphi_j)_{j \in \mathbb{N}}$ to φ as in Definition 1.1, then the plurisubharmonic function $\max\{\varphi, \psi\}$ has a decreasing approximation as in Definition 1.1.*

Proof. (I) Define $\Psi := \max\{\varphi, \psi\}$ and $\Psi_\varepsilon := \max\{\varphi, \psi + \varepsilon\}$ for any $\varepsilon > 0$. From $\text{supp } i\partial\bar{\partial}\psi = X$, we have that $\text{supp } i\partial\bar{\partial}\Psi_\varepsilon = \{z \in X \mid \varphi(z) \leq \psi(z) + \varepsilon\}$. For any open neighborhood U of $\text{supp } i\partial\bar{\partial}\Psi = \{z \in X \mid \varphi(z) \leq \psi(z)\}$, there exists a sufficiently small $\varepsilon > 0$ such that $\{z \in X \mid \Psi(z) < \Psi_\varepsilon(z)\} \subset U$, i.e. the condition (iv) of Definition 1.1 with respect to the family $(\Psi_\varepsilon)_{\varepsilon > 0}$. Obviously, the decreasing family of the plurisubharmonic function $(\Psi_\varepsilon)_{\varepsilon > 0}$ satisfies the conditions (ii), (iii) and (iv) of Definition 1.1. We consider the smooth plurisubharmonic function $M_\delta(\varphi(z), \psi(z)) := M_\delta(\varphi, \psi)(z)$ for any small positive number $\delta > 0$. Here M_δ is a regularized max function.

For completeness, we now give the definition and properties of the regularized max function M_δ . Let $\theta \in \mathcal{E}(\mathbb{R})$ be a non-negative function with support in $[-1, 1]$ such that $\int_{\mathbb{R}} \theta(h) dh = 1$ and that $\int_{\mathbb{R}} h\theta(h) dh = 0$. The regularized max function is defined by

$$M_\delta(x, y) := \int_{\mathbb{R}^2} \max\{x + \delta h_1, y + \delta h_2\} \theta(h_1) \theta(h_2) dh_1 dh_2.$$

The function possesses the following properties:

- (a) $M_\delta(x, y)$ is non-decreasing in all variables, smooth and convex on $\mathbb{R}^2$;
- (b) $\max\{x, y\} \leq M_\delta(x, y) \leq \max\{x, y\} + \delta$; and
- (c) $M_\delta(x, y) = \max\{x, y\}$ on $\{(x, y) \in \mathbb{R}^2 \mid |x - y| \geq 2\delta\}$.

We now return to the proof. We prove that if $2\delta < \varepsilon$ then $\max\{\varphi, \psi\} \leq M_\delta(\varphi, \psi) \leq \max\{\varphi, \psi + \varepsilon\}$. This does not require that the function φ be pluriharmonic, but it only needs to be plurisubharmonic. The first inequality sign follows from the condition (b). From the condition (c), the second inequality sign is correct on $\{z \in X \mid |\varphi(z) - \psi(z)| \geq 2\delta\}$. From this, we show the second inequality sign on the set $\{z \in X \mid |\varphi(z) - \psi(z)| \leq 2\delta\}$. For simplicity, put $(\varphi, \psi) = (x, y)$. Since $M_\delta(x, y) = \max\{x, y\}$ on $\{(x, y) \in \mathbb{R}^2 \mid |x - y| = 2\delta\}$, we have that $M_\delta(x, x - 2\delta) = x$ and that $M_\delta(y - 2\delta, y) = y$. Let $(\xi, \eta) \in \{(x, y) \in \mathbb{R}^2 \mid |x - y| \leq 2\delta\}$, then there exist $t \in [0, 1]$ and two points $(\xi_1, \eta_1) \in \{(x, y) \in \mathbb{R}^2 \mid y = x - 2\delta\}$ and $(\xi_2, \eta_2) \in \{(x, y) \in \mathbb{R}^2 \mid y = x + 2\delta\}$ such that $(\xi, \eta) = t(\xi_1, \eta_1) + (1 - t)(\xi_2, \eta_2)$. From the convexity of M_δ , we obtain an inequality:

$$\begin{aligned} M_\delta(\xi, \eta) &\leq tM_\delta(\xi_1, \eta_1) + (1 - t)M_\delta(\xi_2, \eta_2) \\ &= t\xi_1 + (1 - t)\eta_2 = t(\eta_1 + 2\delta) + (1 - t)\eta_2 = t2\delta + \eta \leq 2\delta + \eta. \end{aligned}$$

Hence, we have that $M_\delta(\varphi, \psi) \leq \psi + 2\delta < \psi + \varepsilon$ on $\{z \in X \mid |\varphi(z) - \psi(z)| \leq 2\delta\}$ if $0 < 2\delta < \varepsilon$.

Therefore take the number sequence $(\varepsilon_j)_{j \in \mathbb{N}}$ such that $\varepsilon_1 > \varepsilon_2 > \dots$ and that $\varepsilon_j \searrow 0$ ($j \rightarrow +\infty$) and let δ_j be positive numbers that satisfy $0 < 2\delta_j < \varepsilon_j - \varepsilon_{j+1}$ for $j \in \mathbb{N}$. Define the smooth plurisubharmonic functions $\Psi_j(z) := M_{\delta_j}(\varphi, \psi + \varepsilon_{j+1})(z)$. From what we have shown, we get that $\Psi_{\varepsilon_{j+1}} \leq \Psi_j \leq \max\{\varphi, \psi + \varepsilon_{j+1} + (\varepsilon_j - \varepsilon_{j+1})\} = \Psi_{\varepsilon_j}$ and that $\text{supp } i\partial\bar{\partial}\Psi_{\varepsilon_{j+1}} \subset \text{supp } i\partial\bar{\partial}\Psi_j \subset \text{supp } i\partial\bar{\partial}\Psi_{\varepsilon_j}$. Then the conditions (ii), (iii) and (iv) of Definition 1.1 are inherited from $(\Phi_{\varepsilon_j})_{j \in \mathbb{N}}$ and $(\Psi_j)_{j \in \mathbb{N}}$ is a decreasing approximation to $\max\{\varphi, \psi\}$ as in Definition 1.1.

(II) Here, φ is a plurisubharmonic function. Define the functions Ψ and Ψ_ε as well as (I) and define $\Psi_{j,\varepsilon} := \max\{\varphi_j, \psi + \varepsilon\}$. Because of the hypothesis of φ , for small enough numbers $\varepsilon > 0$ we have that $\text{supp } i\partial\bar{\partial}\Psi = \text{supp } i\partial\bar{\partial}\varphi \cup \{z \in X \mid \varphi(z) \leq \psi(z)\}$, $\text{supp } i\partial\bar{\partial}\Psi_\varepsilon = \text{supp } i\partial\bar{\partial}\varphi \cup \{z \in X \mid \varphi(z) \leq \psi(z) + \varepsilon\}$ and $\text{supp } i\partial\bar{\partial}\Psi_{j,\varepsilon} = \text{supp } i\partial\bar{\partial}\varphi_j \cup \{z \in X \mid \varphi_j(z) \leq \psi(z) + \varepsilon\}$. From taking the same number sequences $(\varepsilon_j)_{j \in \mathbb{N}}$ and $(\delta_j)_{j \in \mathbb{N}}$ as in (I) and putting the smooth plurisubharmonic functions $\Psi_j(z) := M_{\delta_j}(\varphi_j, \psi + \varepsilon_{j+1})(z)$, we have that $\Psi_{j,\varepsilon_{j+1}} \leq \Psi_j \leq \Psi_{j,\varepsilon_j}$ as in (I). By $\varphi_j \geq \varphi_{j+1}$, we get that $\Psi_j \geq \Psi_{j,\varepsilon_{j+1}} \geq \Psi_{j+1,\varepsilon_{j+1}} \geq \Psi_{j+1}$, i.e. the function sequence $(\Psi_j)_{j \in \mathbb{N}}$ is a decreasing family.

Therefore we have proved that the function sequence $(\Psi_j)_{j \in \mathbb{N}}$ is a decreasing approximation to Ψ as in Definition 1.1. The function sequence satisfies the conditions (i) and (iii) of Definition 1.1; then we just need to show that the conditions (ii) and (iv) are satisfied.

(ii) We need to prove that $\Psi_j > \Psi$ on $\text{supp } i\partial\bar{\partial}\Psi = \text{supp } i\partial\bar{\partial}\varphi \cup \{z \in X \mid \varphi(z) \leq \psi(z)\}$. We consider sets separately. In the case of $\text{supp } i\partial\bar{\partial}\varphi$, from the condition (ii) of Definition 1.1 with respect to $(\varphi_j)_{j \in \mathbb{N}}$, i.e. $\varphi_j > \varphi$, we get the inequality $\Psi_j \geq \Psi_{j,\varepsilon_{j+1}} > \Psi$. In the case of $\{z \in X \mid \varphi(z) \leq \psi(z)\}$, we get the inequality $\Psi_j \geq \Psi_{j,\varepsilon_{j+1}} \geq \psi + \varepsilon_{j+1} > \psi = \Psi$.

(iv) Because of the inequality $\Psi_j \leq \Psi_{j,\varepsilon_j}$, we have that $\{z \in X \mid \Psi(z) < \Psi_j(z)\} \subset \{z \in X \mid \Psi(z) < \Psi_{j,\varepsilon_j}(z)\}$. From the inequality $\Psi = \max\{\varphi, \psi\} \leq \Psi_{j,\varepsilon_j} = \max\{\varphi_j, \psi + \varepsilon_j\}$

$= \max\{\varphi, \psi + \varepsilon_j\}$ on $\{z \in X \mid \varphi(z) < \varphi_j(z)\}^c = \{z \in X \mid \varphi(z) = \varphi_j(z)\}$, we get the set inequality $\{z \in X \mid \Psi(z) < \Psi_{j,\varepsilon_j}(z)\} \subset \{z \in X \mid \varphi(z) < \psi(z) + \varepsilon_j\}$. Hence we have that

$$\begin{aligned} \text{supp } i\partial\bar{\partial}\Psi &= \text{supp } i\partial\bar{\partial}\varphi \cup \{z \in X \mid \varphi(z) \leq \psi(z)\} \\ &\subset \{z \in X \mid \Psi(z) < \Psi_j(z)\} \subset \{z \in X \mid \Psi(z) < \Psi_{j,\varepsilon_j}(z)\} \\ &\subset \{z \in X \mid \varphi(z) < \varphi_j(z)\} \cup \{z \in X \mid \varphi(z) < \psi(z) + \varepsilon_j\}. \end{aligned}$$

Since the set $\{z \in X \mid \varphi(z) < \varphi_j(z)\} \cup \{z \in X \mid \varphi(z) < \psi(z) + \varepsilon_j\}$ approaches $\text{supp } i\partial\bar{\partial}\Psi$ when j tends to $+\infty$, the set $\{z \in X \mid \Psi(z) < \Psi_j(z)\}$ also approaches $\text{supp } i\partial\bar{\partial}\Psi$. $\square$

LEMMA 3.3. *Let X be a complex manifold such that there exists a smooth plurisubharmonic function $\Psi \in \mathcal{E}(X)$ which is n -convex. Let φ be an exhaustive plurisubharmonic function. Let r, s and t be real numbers such that $r < s < t < \sup_X \varphi$. Then there exists a plurisubharmonic function Φ such that*

- (a) $\varphi = \Phi$ on $X_t^c := \{z \in X \mid \varphi(z) \geq t\}$, and
- (b) $\text{supp } i\partial\bar{\partial}\varphi \cup \overline{X_r} \subset \text{supp } i\partial\bar{\partial}\Phi \subset \text{supp } i\partial\bar{\partial}\varphi \cup X_t$.

Proof. Lemma 3.3 follows from [12, Lemma 8] and $\text{supp } i\partial\bar{\partial}\Psi = X$. In particular, the function Φ is defined by

$$\Phi(z) = \begin{cases} \varphi(z) & \text{if } z \in X_t^c, \\ \max\{\varphi(z), \tau(z)\} & \text{if } z \in X_t, \end{cases}$$

where $M := \sup_{\overline{X_t}} \Psi$ and $\tau(z) = r + \Psi(z)(s - r)/(M + 1)$. It is easy to see that Φ satisfies the conditions of the lemma. $\square$

From Proposition 3.2, in Lemma 3.3 if φ is smooth then the plurisubharmonic function Φ has a decreasing approximation as in Definition 1.1.

LEMMA 3.4. *Let X, Y be complex manifolds and let $f : X \rightarrow Y$ be a holomorphic mapping. Let φ be a plurisubharmonic function on Y . If φ has a decreasing approximation $(\varphi_j)_{j \in \mathbb{N}}$ as in Definition 1.1, then the function sequence $(f^*\varphi_j)_{j \in \mathbb{N}}$ is a decreasing approximation to the plurisubharmonic function $f^*\varphi$ and the closed set $f^{-1}(\text{supp } i\partial\bar{\partial}\varphi)$ on X as in Definition 1.1.*

Proof. Immediately, the function sequence $(f^*\varphi_j)_{j \in \mathbb{N}}$ satisfies the conditions (i), (ii) and (iii) of Definition 1.1. Because of the condition (iv) of Definition 1.1 with respect to $(\varphi_j)_{j \in \mathbb{N}}$, we get the equation $\overline{\{y \in Y \mid \varphi(y) < \varphi_j(y)\}}^c = (\text{supp } i\partial\bar{\partial}\varphi)^c$. Therefore from the following two conditions,

$$\begin{aligned} \bigcup f^{-1}(\overline{\{y \in Y \mid \varphi(y) < \varphi_j(y)\}})^c &= f^{-1}(\text{supp } i\partial\bar{\partial}\varphi)^c, \\ f^{-1}(\overline{\{y \in Y \mid \varphi(y) < \varphi_j(y)\}})^c &\subset f^{-1}(\overline{\{y \in Y \mid \varphi(y) < \varphi_{j+1}(y)\}})^c, \end{aligned}$$

we have that, for any open neighborhood U of $f^{-1}(\text{supp } i\partial\bar{\partial}\varphi)$, there exists an integer j_0 such that

$$\begin{aligned} \{x \in X \mid f^*\varphi(x) < f^*\varphi_{j_0}(x)\} &= f^{-1}(\{y \in Y \mid \varphi(y) < \varphi_{j_0}(y)\}) \\ &\subset f^{-1}(\overline{\{y \in Y \mid \varphi(y) < \varphi_{j_0}(y)\}}) \subset U. \end{aligned}$$

From the above, this proof is completed. $\square$

From Example 3.1 and Lemma 3.4, we get the following example.

EXAMPLE 3.5. Let X be a non-compact complex manifold. Let φ be a plurisubharmonic function on $\mathbb{C}^n$ and $f: X \rightarrow \mathbb{C}^n$ be a holomorphic map. Then the function sequence $(f^*\varphi_{1/j})_{j \in \mathbb{N}}$ is a decreasing approximation to a plurisubharmonic function $f^*\varphi$ and a closed set $f^{-1}(\text{supp } i\partial\bar{\partial}\varphi)$ on X as in Definition 1.1. Here it is clear that $\text{supp } i\partial\bar{\partial}f^*\varphi \subset f^{-1}(\text{supp } i\partial\bar{\partial}\varphi)$. Moreover f is flat, then $\text{supp } i\partial\bar{\partial}f^*\varphi = f^{-1}(\text{supp } i\partial\bar{\partial}\varphi)$.

4. Proof of Theorem 1.2 in the case of $m = 1$

In this section, we prove the $m = 1$ case of Theorem 1.2 and extend this to closed F -valued forms as in [13].

THEOREM 4.1. (The $m = 1$ case of Theorem 1.2) *Let X be a weakly pseudoconvex Kähler manifold of dimension n ($n \geq 3$) such that there exists a smooth plurisubharmonic function ψ_q which is q -convex. Let F be a holomorphic vector bundle over X . Let φ be a non-constant smooth exhaustive plurisubharmonic function on X . We assume that there exists a decreasing approximation $(\varphi_k)_{k \in \mathbb{N}}$ to φ as in Definition 1.1. Then the natural map*

$$H^p(X, F) \longrightarrow \varinjlim_{\text{supp } i\partial\bar{\partial}\varphi \subset U} H^p(U, F)$$

is an isomorphism for $0 \leq p < n - q - 1$ and is injective for $p = n - q - 1$.

In particular, the following holds from the q -completeness of X :

$$\varinjlim_{\text{supp } i\partial\bar{\partial}\varphi \subset U} H^p(U, F) = 0$$

for $q \leq p < n - q - 1$.

Let X , ψ_q and φ be as in Theorem 4.1 which satisfy the condition. We define $X_r := \{z \in X \mid \varphi(z) < r\}$. The following proposition does not require the smoothness of φ .

PROPOSITION 4.2. *Let $\beta \in \mathcal{E}^{0,p}(X, F)$ such that $\bar{\partial}\beta = 0$ on X and that $\beta = 0$ on a neighborhood of $\text{supp } i\partial\bar{\partial}\varphi$ ($1 \leq p \leq n - q - 1$). Let $r < \sup_X \varphi$ such that $d\varphi \neq 0$ on $\partial X_r \setminus \text{supp } i\partial\bar{\partial}\varphi$. Assume that there exists a decreasing approximation to φ as in Definition 1.1. Then there exists $\gamma \in \mathcal{E}^{0,p-1}(X_r, F)$ such that $\bar{\partial}\gamma = \beta$ and that $\gamma = 0$ on a neighborhood of $\text{supp } i\partial\bar{\partial}\varphi \cap X_r$.*

Proof. There exist a positive integer j_0 and a smooth plurisubharmonic function φ_{j_0} on a neighborhood of $\bar{X}_r$ such that $\varphi_{j_0} \geq \varphi$ on $\bar{X}_r$, $\varphi_{j_0} > \varphi$ on $\text{supp } i\partial\bar{\partial}\varphi \cap \bar{X}_r$ and $\varphi_{j_0} = \varphi$ on $\text{supp } \beta$. Let $\eta = \varphi_{j_0} - \varphi$. It follows that η is plurisubharmonic on $X_r \setminus \text{supp } i\partial\bar{\partial}\varphi$ and that $\eta = 0$ on $\text{supp } \beta$. Since η is lower semi-continuous on $\bar{X}_r$, there exists $c > 0$ such that $\eta > c$ on $\bar{X}_r \cap \text{supp } i\partial\bar{\partial}\varphi$. Let $X'_r = \{z \in D \mid \varphi(z) - r < 0, \eta(z) - c/2 < 0\}$. From $\varphi \in \mathcal{E}(X \setminus \text{supp } i\partial\bar{\partial}\varphi)$, we get $\eta \in \mathcal{E}(\bar{X}'_r)$. By Proposition 2.7, there exist $c' < c/2$ and $\gamma \in \mathcal{E}^{0,p-1}(X'_r, F)$ such that $\bar{\partial}\gamma = \beta$ and that $\text{supp } \gamma \subset \{z \in X'_r \mid \eta(z) \leq c'\}$. We extend γ by 0 on $X_r \setminus X'_r$, and γ can be considered an element of $\mathcal{E}^{0,p-1}(X_r, F)$. Therefore we obtain that $\bar{\partial}\gamma = \beta$ on X_r and that $\gamma = 0$ on a neighborhood of $\text{supp } i\partial\bar{\partial}\varphi \cap X_r$ from $X_r \setminus X'_r \subset X_r \setminus \text{supp } i\partial\bar{\partial}\varphi$. $\square$

From Proposition 3.2, Proposition 4.2 holds even if we replace φ by Φ in Lemma 3.3. Using Proposition 4.2 in these cases, we get the following lemma.

LEMMA 4.3. *Let X and φ be as in Theorem 4.1. Let $U \subset X$ be an open neighborhood of $\text{supp } i\partial\bar{\partial}\varphi$. Let $\alpha \in \mathcal{E}^{0,p}(X, F)$ such that $\bar{\partial}\alpha = 0$ and let $u \in \mathcal{E}^{0,p-1}(U, F)$ such that $\bar{\partial}u = \alpha$ ($1 \leq p \leq n - q - 1$). Let $s_1 < s_2 < \dots < \sup_X \varphi$ such that $\lim_{j \rightarrow +\infty} s_j = \sup_X \varphi$ and that $X_{s_1} \subset \subset X_{s_2} \subset \subset \dots \subset \subset X$. Then there exist $U_j \subset X$ and $v_j \in \mathcal{E}^{0,p-1}(U_j, F)$ ($j = 1, 2, \dots$) which satisfy the following:*

- (i) U_j is an open neighborhood of $\text{supp } i\partial\bar{\partial}\varphi \cup \overline{X_{s_j}}$,
- (ii) $\bar{\partial}v_j = \alpha$ on U_j ,
- (iii) $v_{j+1} = v_j$ on X_{s_j} , and
- (iv) $v_j = u$ on a neighborhood of $\text{supp } i\partial\bar{\partial}\varphi$.

Proof. Let $\chi_0 \in \mathcal{E}(X)$ such that $\text{supp } \chi_0 \subset U$, $0 \leq \chi_0 \leq 1$, and that $\chi_0|_{\text{supp } i\partial\bar{\partial}\varphi} \equiv 1$. We define $\beta_1 := \alpha - \bar{\partial}(\chi_0 u)$ on X which satisfies $\bar{\partial}\beta_1 = 0$ on X_{s_3} and $\beta_1 = 0$ on a neighborhood of $\text{supp } i\partial\bar{\partial}\varphi$. By Proposition 4.2, there exists $v \in \mathcal{E}^{0,p-1}(X_{s_3}, F)$ such that $\bar{\partial}v = \beta_1$ and that $v = 0$ on a neighborhood of $\text{supp } i\partial\bar{\partial}\varphi \cap X_{s_3}$. Therefore we can choose an open neighborhood V_1 of $\text{supp } i\partial\bar{\partial}\varphi$ such that $\text{supp } i\partial\bar{\partial}\varphi \subset V_1 \subset U$; then v on X_{s_2} and 0 on V_1 can be glued together to give the form $\tilde{v}_1$ on $U_1 = V_1 \cup X_{s_2}$, and we get $\bar{\partial}\tilde{v}_1 = \beta$ on U_1 . Then define $v_1 := \chi_0 u + \tilde{v}_1$ on U_1 which satisfy $\bar{\partial}v_1 = \alpha$ and $v_1 = u$ on a neighborhood of $\text{supp } i\partial\bar{\partial}\varphi$. By repeating this argument, we assume that there exist U_j and v_j which satisfy the conditions of the lemma for some $j \geq 1$. By Lemma 3.3, there exists a plurisubharmonic function Φ_j on X such that $\varphi = \Phi_j$ on $X_{s_{j+1}}^c$ and that $\text{supp } i\partial\bar{\partial}\varphi \cup \overline{X_{s_j}} \subset \text{supp } i\partial\bar{\partial}\Phi_j \subset \text{supp } i\partial\bar{\partial}\varphi \cup X_{s_{j+1}}$. We can choose an open neighborhood V_j of $\text{supp } i\partial\bar{\partial}\varphi$ such that $U_j = V_j \cup X_{s_{j+1}}$, $\text{supp } i\partial\bar{\partial}\Phi_j \subset U_j$ and $V_j \cap X_{s_{j+1}} \subset V_{j-1} \cap X_{s_{j+1}}$. Then v_j considered on U_j is defined on a neighborhood of $\text{supp } i\partial\bar{\partial}\Phi_j$. Let $\chi_j \in \mathcal{E}(X)$ such that $\text{supp } \chi_j \subset U_j$, $0 \leq \chi_j \leq 1$, and that $\chi_j|_{\text{supp } i\partial\bar{\partial}\Phi_j} \equiv 1$. We define $\beta_{j+1} := \alpha - \bar{\partial}(\chi_j v_j)$ on X such that $\bar{\partial}\beta_{j+1} = 0$ on X and that $\beta_{j+1} = 0$ on a neighborhood of $\text{supp } i\partial\bar{\partial}\Phi_j$. By Propositions 3.2 and 4.2, there exists $v \in \mathcal{E}^{0,p-1}(X_{s_{j+3}}, F)$ such that $\bar{\partial}v = \beta_{j+1}$ and that $v = 0$ on an open neighborhood of $X_{s_{j+3}} \cap \text{supp } i\partial\bar{\partial}\Phi_j$. Therefore we can choose an open neighborhood V_{j+1} of $\text{supp } i\partial\bar{\partial}\varphi$ such that $\text{supp } i\partial\bar{\partial}\varphi \subset V_{j+1} \subset V_j$. Let $U_{j+1} = V_{j+1} \cup X_{s_{j+2}}$. Then v on $X_{s_{j+2}}$ and 0 on V_{j+1} can be glued together to give the form $\tilde{v}_{j+1}$ on U_{j+1} such that $\bar{\partial}\tilde{v}_{j+1} = \beta_{j+1}$ on U_{j+1} . Then define $v_{j+1} := \chi_j v_j + \tilde{v}_{j+1}$ on U_{j+1} which satisfy the conditions of the lemma. Hence we obtain U_k and v_k ($k = 1, 2, \dots$) inductively. $\square$

Proof of Theorem 4.1 (the $m = 1$ case of Theorem 1.2). First we show that

$$H^p(X, F) \rightarrow \varinjlim_{\text{supp } i\partial\bar{\partial}\varphi \subset U} H^p(U, F)$$

is injective for $0 \leq p \leq n - q - 1$. The case of $p = 0$ is obvious. We consider the $1 \leq p \leq n - q - 1$ case. Let $\alpha \in \mathcal{E}^{0,p}(X, F)$ such that $\bar{\partial}\alpha = 0$. Suppose that there exists an open neighborhood $U \subset X$ of $\text{supp } i\partial\bar{\partial}\varphi$ and $u \in \mathcal{E}^{0,p-1}(U, F)$ such that $\alpha = \bar{\partial}u$ on U . Take $s_1 < s_2 < \dots < \sup_X \varphi$, open neighborhoods U_j of $\text{supp } i\partial\bar{\partial}\varphi \cup \overline{X_{s_j}}$ and $v_j \in \mathcal{E}^{0,p-1}(U_j, F)$ as in Lemma 4.3. Therefore we define $v(z) = v_j(z)$ when $z \in X_{s_j}$ for $j \geq 1$. Then $v \in \mathcal{E}^{0,p-1}(X, F)$ satisfies that $\bar{\partial}v = \alpha$ and that $v = u$ on a neighborhood of $\text{supp } i\partial\bar{\partial}\varphi$.

Next, we show that

$$H^p(X, F) \rightarrow \varinjlim_{\text{supp } i\partial\bar{\partial}\varphi \subset U} H^p(U, F)$$

is surjective for $0 \leq p \leq n - q - 2$. Let U be an open neighborhood of $\text{supp } i\partial\bar{\partial}\varphi$ in X and let $u \in \mathcal{E}^{0,p}(U, F)$ such that $\bar{\partial}u = 0$. Take $s_1 < s_2 < \cdots < \sup_X \varphi$, open neighborhoods U_j of $\text{supp } i\partial\bar{\partial}\varphi \cup \bar{X}_{s_j}$ and $v_j \in \mathcal{E}^{0,p}(U_j, F)$ as in Lemma 4.3 for the $\alpha = 0$ case. Therefore we define $v(z) = v_j(z)$ when $z \in X_{s_j}$ for $j \geq 1$. Then $v \in \mathcal{E}^{0,p}(X, F)$ satisfies that $\bar{\partial}v = 0$ and that $v = u$ on a neighborhood of $\text{supp } i\partial\bar{\partial}\varphi$. $\square$

Here, we obtain the following corollary from Proposition 3.2 and Theorem 4.1.

COROLLARY 4.4. *Let X be a weakly pseudoconvex Kähler manifold of dimension n ($n \geq 3$) such that there exists a smooth plurisubharmonic function ψ_q on X which is exhaustive and q -convex. Let F be a holomorphic vector bundle over X . Define $X_r := \{z \in X \mid \psi_q(z) < r\}$ for any number $r < \sup_X \psi_q$. Then the natural map*

$$H^p(X, F) \longrightarrow \varinjlim_{X_r^c \subset U} H^p(U, F)$$

is an isomorphism for $0 \leq p < n - q - 1$ and is injective for $p = n - q - 1$.

In particular, the following holds from the q -completeness of X :

$$\varinjlim_{X_r^c \subset U} H^p(U, F) = 0$$

for $q \leq p < n - q - m$, where X_r^c is the complement of X_r .

5. Proof of Theorem 1.3 in the case of $m = 1$

From Lemma 3.4, in the setting of Theorem 1.3 if a plurisubharmonic function φ has a decreasing approximation as in Definition 1.1 then a plurisubharmonic function $f^*\varphi$ has a decreasing approximation for a closed set $f^{-1}(\text{supp } i\partial\bar{\partial}\varphi)$ as in Definition 1.1. Therefore, the $m = 1$ case of Theorem 1.3 is expected to be proved in the same way as Theorem 4.1. In this section, we prove the following theorem from this approach.

THEOREM 5.1. (The $m = 1$ case of Theorem 1.3) *Let X be a Kähler manifold of dimension n ($n \geq 3$) and S be a Stein manifold. Let $f : X \rightarrow S$ be a holomorphic proper mapping and F be a holomorphic vector bundle over X . Let φ be a non-constant exhaustive plurisubharmonic function on S such that $f^{-1}(\text{supp } i\partial\bar{\partial}\varphi) \neq \emptyset$. We assume that there exists a smooth strictly plurisubharmonic function ψ on S such that $f^*\psi$ is q -convex.*

Then the natural map

$$H^p(X, F) \longrightarrow \varinjlim_{f^{-1}(\text{supp } i\partial\bar{\partial}\varphi) \subset U} H^p(U, F)$$

is an isomorphism for $0 \leq p < n - q - 1$ and is injective for $p = n - q - 1$.

In particular, the following holds from the q -completeness of X :

$$\varinjlim_{f^{-1}(\text{supp } i\partial\bar{\partial}\varphi) \subset U} H^p(U, F) = 0$$

for $q \leq p < n - q - 1$. We note that, if furthermore f is flat, then $f^{-1}(\text{supp } i\partial\bar{\partial}\varphi) = \text{supp } i\partial\bar{\partial}f^*\varphi$.

Because of the exhaustiveness of φ on S and the properness of f , $f^*\varphi$ is too an exhaustive plurisubharmonic function on X and X is a weakly pseudoconvex manifold. From the weak pseudoconvexity and the existence of a q -convex function, X is a q -complete manifold. Define $S_r := \{z \in S \mid \varphi(z) < r\}$ and $X_r := \{z \in X \mid f^*\varphi(z) < r\}$.

LEMMA 5.2. (cf. [13, Lemma 12]) *Let S be a Stein manifold and D be a relatively compact open subset. Let φ be a plurisubharmonic function on S . Then φ has a decreasing approximation on D as in Definition 1.1.*

Proof. We may assume that S is a submanifold of $\mathbb{C}^m$. By the theorem of Docquier and Grauert, there exists an open neighborhood $W \subset \mathbb{C}^m$ of S and a holomorphic retraction $\mu : W \rightarrow S$ (cf. [7, Chapter VIII] and [8, Chapter V]). Put the function $\rho_\varepsilon : \mathbb{C}^m \rightarrow \mathbb{R}_{\geq 0}$ in the same way as in the Example 3.1 for $\varepsilon > 0$. Let $V \subset W$ be an relatively compact open neighborhood of D . If $\varepsilon > 0$ is sufficiently small, we can define $(\varphi \circ \mu)_\varepsilon = (\varphi \circ \mu) * \rho_\varepsilon$ on V . Put $\varphi_\varepsilon = (\varphi \circ \mu)_\varepsilon$ on V . Then we have that $\varphi_\varepsilon \geq \varphi$ on D and $\varphi_\varepsilon > \varphi$ on $\text{supp } i\partial\bar{\partial}\varphi$.

Therefore we prove that the plurisubharmonic function family $(\varphi_\varepsilon)_{\varepsilon>0}$ satisfies the condition (iv) of Definition 1.1. Let $U \subset D$ be an open neighborhood of $\text{supp } i\partial\bar{\partial}\varphi \cap D$. Since $\mu^{-1}(\text{supp } i\partial\bar{\partial}\varphi) \cap (V \setminus \mu^{-1}(U)) = \emptyset$, there exists a small number $\varepsilon > 0$ such that the ball of radius ε centered at every point of $V \setminus \mu^{-1}(U)$ does not intersect $\mu^{-1}(\text{supp } i\partial\bar{\partial}\varphi)$. Then we get $(\varphi \circ \mu)_\varepsilon = \varphi \circ \mu$ on $V \setminus \mu^{-1}(U)$, i.e. $\{z \in V \mid \varphi \circ \mu(z) < (\varphi \circ \mu)_\varepsilon(z)\} \subset \mu^{-1}(U)$. Hence we have that $\{z \in D \mid \varphi(z) < \varphi_\varepsilon(z)\} \subset U$. $\square$

Using Lemma 5.2, we get the following proposition as well as Proposition 4.2.

PROPOSITION 5.3. *Let $\beta \in \mathcal{E}^{0,p}(U, F)$ ($1 \leq p \leq n - q - 1$) such that $\bar{\partial}\beta = 0$ and that $\beta = 0$ on a neighborhood of $f^{-1}(\text{supp } i\partial\bar{\partial}\varphi)$. Let $r < \sup \varphi$ such that $df^*\varphi \neq 0$ on $\partial X_r \setminus f^{-1}(\text{supp } i\partial\bar{\partial}\varphi)$. (Note that $f^*\varphi$ is smooth on $X \setminus f^{-1}(\text{supp } i\partial\bar{\partial}\varphi)$.) Assume that there exists a decreasing approximation to $f^*\varphi$ and $f^{-1}(\text{supp } i\partial\bar{\partial}\varphi)$ as in Definition 1.1. Then there exists $\gamma \in \mathcal{E}^{0,p-1}(X_r, F)$ such that $\bar{\partial}\gamma = \beta$ and that $\gamma = 0$ on a neighborhood of $f^{-1}(\text{supp } i\partial\bar{\partial}\varphi) \cap X_r$.*

LEMMA 5.4. *Let X, S, f and φ be as in Theorem 5.1. Let r, s, t and u be real numbers such that $r < s < t < u < \sup_X \varphi$. Then there exists a plurisubharmonic function Φ on S such that*

- (a) $f^*\varphi = f^*\Phi$ on X_t^c , and
- (b) $f^{-1}(\text{supp } i\partial\bar{\partial}\varphi) \cup \overline{X_r} \subset f^{-1}(\text{supp } i\partial\bar{\partial}\Phi) \subset f^{-1}(\text{supp } i\partial\bar{\partial}\varphi) \cup X_t$.

Furthermore, f^Φ has a decreasing approximation to a closed set $f^{-1}(\text{supp } i\partial\bar{\partial}\Phi)$ on X_u as in Definition 1.1.*

Proof. The existence of a function Φ satisfying the conditions (a) and (b) follows directly from Lemma 3.3.

Because of Lemma 5.2, Φ has a decreasing approximation on S_u as in Definition 1.1. Therefore the proof is complete from Lemma 3.4. Moreover, if $f^*\varphi$ is smooth, then this can be shown directly by using Proposition 3.2. $\square$

From Lemma 5.4, Proposition 5.3 holds even if we replace $f^*\varphi$ by $f^*\Phi$. Therefore we obtain the following lemma in the same way as Lemma 4.3.

LEMMA 5.5. *Let X , S , f and φ be as in Theorem 5.1. Let $U \subset X$ be an open neighborhood of $f^{-1}(\text{supp } i\partial\bar{\partial}\varphi)$. Let $\alpha \in \mathcal{E}^{0,p}(X, F)$ such that $\bar{\partial}\alpha = 0$ and let $u \in \mathcal{E}^{0,p-1}(U, F)$ such that $\bar{\partial}u = \alpha$ ($1 \leq p \leq n - q - 1$). Let $s_1 < s_2 < \cdots < \sup_X f^*\varphi$ such that $\lim_{j \rightarrow +\infty} s_j = \sup_X f^*\varphi$ and that $X_{s_1} \subset \subset X_{s_2} \subset \subset \cdots \subset \subset X$. Then there exist $U_j \subset X$ and $v_j \in \mathcal{E}^{0,p-1}(U_j, F)$ ($j = 1, 2, \dots$) which satisfy the following:*

- (i) U_j is an open neighborhood of $f^{-1}(\text{supp } i\partial\bar{\partial}\varphi) \cup \bar{X}_{s_j}$;
- (ii) $\bar{\partial}v_j = \alpha$ on U_j ;
- (iii) $v_{j+1} = v_j$ on X_{s_j} ; and
- (iv) $v_j = u$ on a neighborhood of $f^{-1}(\text{supp } i\partial\bar{\partial}\varphi)$.

Proof of Theorem 5.1 (the $m = 1$ case of Theorem 1.3). Theorem 5.1 follows from Lemma 5.5 in the same way that Theorem 4.1 follows from Lemma 4.3. $\square$

6. Proofs of Theorem 1.4 in the case of $m = 1$ and of the main results

In this section, we prove the $m = 1$ case of Theorem 1.4. Therefore this theorem is set up so that the set of compact manifolds minus the zeros of holomorphic sections is q -complete, and is shown using an extension of closed F -valued forms on q -complete Kähler manifolds considered in a similar way as Lemma 5.5.

THEOREM 6.1. (The $m = 1$ case of Theorem 1.4) *Let X be a compact Kähler manifold of dimension n ($n \geq 3$) and Y be a projective manifold. Let $f : X \rightarrow Y$ be a surjective holomorphic mapping and F be a holomorphic vector bundle over X . Let T be a closed positive current of type $(1, 1)$ on Y whose cohomology class belongs to $\mathcal{K}_{NS}(Y)$. We assume that there exists a Kähler form ω such that $\omega \in \{T\}$ and that $f^*\omega$ has at least $n - q + 1$ positive eigenvalues. Then the natural map*

$$H^p(X, F) \longrightarrow \varinjlim_{f^{-1}(\text{supp } T) \subset U} H^p(U, F)$$

is an isomorphism for $0 \leq p < n - q - 1$ and is injective for $p = n - q - 1$.

A holomorphic line bundle L over X is said to be q -positive if there exists a smooth Hermitian metric h whose Chern curvature $i\Theta_{L,h}$ has at least $n - q + 1$ positive eigenvalues at any point on X . If there exists a line bundle L such that $T \in c_1(L)$, the assumption of $f^*\omega$ is saying to take the line bundle f^*L to be q -positive. As we said in Section 1, there exist very ample line bundles $L_1, \dots, L_l$ and positive numbers $a_1, \dots, a_l$ such that $\{T\} = a_1c_1(L_1) + \cdots + a_lc_1(L_l)$. Here there exist Kähler forms ω_j such that $\omega_j \in c_1(L_j)$. Let $\omega := a_1\omega_1 + \cdots + a_l\omega_l$ be a Kähler form. Then we have that $\omega \in \{T\}$. Since T is a closed positive current, there exists an almost plurisubharmonic function φ on Y such that $T = \omega + i\partial\bar{\partial}\varphi$.

Take non-zero holomorphic sections $s_j \in H^0(Y, L_j)$ ($1 \leq j \leq l$). Define $s := s_1 \otimes \cdots \otimes s_l \in H^0(Y, L_1 \otimes \cdots \otimes L_l)$. Let $Z_s := \{z \in Y \mid s(z) = 0\}$ and let h_j be smooth Hermitian metrics of L_j whose Chern curvature is ω_j , i.e. $(i/(2\pi))\Theta_{L_j, h_j} = \omega_j$. Then $Y \setminus Z_s$ is a Stein manifold and $X \setminus f^{-1}(Z_s) = X \setminus \{z \in X \mid f^*s(z) = 0\}$ is a q -complete manifold from the assumption of $f^*\omega$. In fact, the smooth function $-(1/(2\pi)) \sum \log \|s_j\|_{h_j}^{2a_j}$ on Z_s is strictly plurisubharmonic and exhaustive. And the smooth function $-(1/(2\pi))f^*(\sum \log \|s_j\|_{h_j}^{2a_j})$

on $X \setminus f^{-1}(Z_s)$ is plurisubharmonic, q -convex and exhaustive. Therefore let $Y_r^s := \{z \in Y \mid -(1/(2\pi)) \sum \log \|s_j\|_{h_j}^{2a_j}(z) < r\}$ and $X_r^s := \{z \in X \mid -(1/(2\pi)) f^*(\sum \log \|s_j\|_{h_j}^{2a_j})(z) < r\}$ be sublevel sets from exhaustiveness.

First we prove the case where φ is bounded on Y .

LEMMA 6.2. *Let $U \subset X$ be an open neighborhood of $f^{-1}(\text{supp } T) = f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi))$. Here φ is a bounded almost plurisubharmonic function. Let $u \in \mathcal{E}^{0,p-1}(U, F)$ such that $\bar{\partial}u = 0$ ($1 \leq p \leq n - q - 1$). Let $K \subset X \setminus \{z \in X \mid f^*s(z) = 0\}$ be a compact subset. Then there exist an open neighborhood $U_1 \subset X$ of $K \cup f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi))$, a $\bar{\partial}$ -closed F -valued form $u_1 \in \mathcal{E}^{0,p-1}(U_1, F)$ and a bounded almost plurisubharmonic function φ_1 on Y which satisfy the following conditions:*

- (a) $u = u_1$ on a neighborhood of $f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi))$;
- (b) $\omega + i\partial\bar{\partial}\varphi_1 \geq 0$; and
- (c) $K \cup f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi)) \subset f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi_1)) \subset U_1$.

Proof. Let $\tilde{\varphi} := -(1/(2\pi)) \sum \log \|s_j\|_{h_j}^{2a_j} + \varphi$ on $Y \setminus Z_s$. Then we have that $\tilde{\varphi}$ is an exhaustive plurisubharmonic function and satisfies the condition $\text{supp } i\partial\bar{\partial}\tilde{\varphi} = (Y \setminus Z_s) \cap \text{supp}(\omega + i\partial\bar{\partial}\varphi)$ and similarly that $f^*\tilde{\varphi}$ is an exhaustive plurisubharmonic function on $X \setminus f^{-1}(Z_s)$ and satisfies the condition $f^{-1}(\text{supp } i\partial\bar{\partial}\tilde{\varphi}) = f^{-1}(Y \setminus Z_s) \cap f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi)) = (X \setminus f^{-1}(Z_s)) \cap f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi))$. Let $Y_r^{\tilde{\varphi}} := \{z \in Y \setminus Z_s \mid \tilde{\varphi}(z) < r\}$ and $X_r^{\tilde{\varphi}} := f^{-1}(Y_r^{\tilde{\varphi}}) = \{z \in X \setminus f^{-1}(Z_s) \mid f^*\tilde{\varphi}(z) < r\}$ and let r be a large enough positive number such that $K \subset X_{r-1}^{\tilde{\varphi}}$.

We take $\chi \in \mathcal{E}(X \setminus f^{-1}(Z_s))$ such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ on a neighborhood of $f^{-1}(\text{supp } i\partial\bar{\partial}\tilde{\varphi})$ and that $\text{supp } \chi \subset U$. Then $\beta := \bar{\partial}(\chi u) \in \mathcal{E}^{0,p-1}(X \setminus f^{-1}(Z_s), F)$ is $\bar{\partial}$ -closed and $\beta = 0$ on a neighborhood of $f^{-1}(\text{supp } i\partial\bar{\partial}\tilde{\varphi})$. Because of Proposition 5.3 about a q -complete manifold $X \setminus f^{-1}(Z_s)$ and an exhaustive plurisubharmonic function $f^*\tilde{\varphi}$, there exists $\gamma \in \mathcal{E}^{0,p-1}(X_r^{\tilde{\varphi}}, F)$ such that $\bar{\partial}\gamma = \beta$ and that $\gamma = 0$ on a neighborhood of $X_r^{\tilde{\varphi}} \cap f^{-1}(\text{supp } i\partial\bar{\partial}\tilde{\varphi}) = X_r^{\tilde{\varphi}} \cap f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi))$. Then $v := \chi u - \gamma \in \mathcal{E}^{0,p-1}(X_r^{\tilde{\varphi}}, F)$ satisfies that $\bar{\partial}v = 0$ and that $u = v$ on a neighborhood of $X_r^{\tilde{\varphi}} \cap f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi))$.

As in the proof of Lemma 5.5, we can glue u and v together, and we get an open neighborhood $U_1 \subset X$ of $K \cup f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi))$ such that $X_r^{\tilde{\varphi}} \subset U_1$ and get $u_1 \in \mathcal{E}^{0,p-1}(U_1, F)$ which satisfies (a). As in Lemma 3.3, there exists a plurisubharmonic function $\tilde{\varphi}_1 \in C(Y \setminus Z_s)$ such that $\tilde{\varphi} = \tilde{\varphi}_1$ on $Y \setminus (Z_s \cup Y_r^{\tilde{\varphi}})$ and that $Y_{r-1}^{\tilde{\varphi}} \cup \text{supp } i\partial\bar{\partial}\tilde{\varphi} \subset \text{supp } i\partial\bar{\partial}\tilde{\varphi}_1 \subset Y_r^{\tilde{\varphi}} \cup \text{supp } i\partial\bar{\partial}\tilde{\varphi}$. Therefore we have that $f^*\tilde{\varphi} = f^*\tilde{\varphi}_1$ on $f^{-1}(Y \setminus (Z_s \cup Y_r^{\tilde{\varphi}})) = X \setminus (f^{-1}(Z_s) \cup X_r^{\tilde{\varphi}})$ and that $K \cup f^{-1}(\text{supp } i\partial\bar{\partial}\tilde{\varphi}) \subset X_{r-1}^{\tilde{\varphi}} \cup f^{-1}(\text{supp } i\partial\bar{\partial}\tilde{\varphi}) \subset f^{-1}(\text{supp } i\partial\bar{\partial}\tilde{\varphi}_1) \subset X_r^{\tilde{\varphi}} \cup f^{-1}(\text{supp } i\partial\bar{\partial}\tilde{\varphi}) \subset U_1$.

Put $\varphi_1 := \tilde{\varphi}_1 + (1/(2\pi)) \sum \log \|s_j\|_{h_j}^{2a_j}$. Since $\varphi_1 = \varphi$ on a neighborhood of Z_s , we can consider φ_1 as a bounded almost plurisubharmonic function on Y . Then φ_1 is the function we are looking for. In fact, from $i\partial\bar{\partial}\varphi_1 = i\partial\bar{\partial}\tilde{\varphi}_1 + (i/(2\pi))\partial\bar{\partial} \sum \log \|s_j\|_{h_j}^{2a_j} - \omega = i\partial\bar{\partial}\tilde{\varphi}_1 - \omega$ on $Y \setminus Z_s$, we have this lemma condition (b) and the set condition $K \cup f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi))|_{X \setminus f^{-1}(Z_s)} = K \cup f^{-1}(\text{supp } i\partial\bar{\partial}\tilde{\varphi}) \subset f^{-1}(\text{supp } i\partial\bar{\partial}\tilde{\varphi}_1) = f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi_1))|_{X \setminus f^{-1}(Z_s)} \subset U_1$, i.e. condition (c). $\square$

From the line bundle $L := L_1 \otimes \cdots \otimes L_l \rightarrow Y$ is very ample, there exist holomorphic sections $t_1, \dots, t_m \in H^0(Y, L)$ such that $\bigcap_{j=1}^m Z_{t_j} = \emptyset$, where $Z_{t_j} = \{z \in Y \mid t_j(z) = 0\}$.

Here, as above, $Y \setminus Z_{t_j}$ are Stein manifolds by the strictly plurisubharmonic functions $-\log \|t_j\|_{h_L}^2$ which are smooth and exhaustive. Then there exist sufficiently large positive numbers r_j ($1 \leq j \leq m$) such that $\bigcup_{j=1}^m Y_{r_j}^{t_j} = Y$; in particular $\bigcup_{j=1}^m X_{r_j}^{t_j} = X$. Clearly we have that $\overline{X_{r_j}^{t_j}}$ is compact, $\overline{X_{r_j}^{t_j}} \cap f^{-1}(Z_{t_j}) = \emptyset$ and that $\bigcup_{j=1}^m \overline{X_{r_j}^{t_j}} = X$.

Proof of Theorem 6.1 where φ is bounded. We first show that

$$H^p(X, F) \rightarrow \varinjlim_{f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi)) \subset U} H^p(U, F)$$

is surjective for $0 \leq p \leq n - q - 2$. Let $u \in \mathcal{E}^{0,p}(U, F)$ be a $\bar{\partial}$ -closed F -valued form where U is an open neighborhood of $f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi))$. Take $K \subset X$ as in Lemma 6.2. Then we obtain U_1, u_1 and φ_1 as in Lemma 6.2. We can take compact sets $K_j = \overline{X_{r_j}^{t_j}}$ ($1 \leq j \leq m$) as above. If we replace K, u, U and φ by K_1, u_1, U_1 and φ_1 respectively, we obtain u_2, U_2 and φ_2 which satisfy suitable conditions. By repeating the same process, there exist u_j, U_j ($1 \leq j \leq m+1$) and φ_j ($1 \leq j \leq m$) such that $u_j \in \mathcal{E}^{0,p}(U_j, F)$, $\bigcup_{k=1}^{j-1} K_k \cup f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi)) \subset f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi_j)) \subset U_j$ and that $u_j = u_{j-1}$ on a neighborhood of $f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi_{j-1}))$. Hence $u_{m+1} \in \mathcal{E}^{0,p}(X, F)$ is a $\bar{\partial}$ -closed F -valued form such that $u_{m+1} = u$ on a neighborhood of $f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi))$.

Next, we show that

$$H^p(X, F) \rightarrow \varinjlim_{f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi)) \subset U} H^p(U, F)$$

is injective for $0 \leq p \leq n - q - 1$. Let $\alpha \in \mathcal{E}^{0,p}(X, F)$ such that $\bar{\partial}\alpha = 0$. Suppose that there exists an open neighborhood $U \subset X$ of $f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi))$ and $v \in \mathcal{E}^{0,p-1}(U, F)$ such that $\alpha = \bar{\partial}v$ on U . Let $\chi \in \mathcal{E}(X)$ be a function such that $0 \leq \chi \leq 1$, $\chi = 1$ on a neighborhood of $f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi))$ and that $\text{supp } \chi \subset U$. Then $\alpha - \bar{\partial}(\chi v)$ is a $\bar{\partial}$ -closed F -valued form which vanishes on a neighborhood of $f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi))$. Take $K, K_1, \dots, K_m$ as in the proof of the surjectivity. By Proposition 5.3, there exists an F -valued $(0, p)$ -form v'_1 which is defined in an open neighborhood of K such that $\bar{\partial}v'_1 = \alpha - \bar{\partial}(\chi v)$ and that $v'_1 = 0$ on a neighborhood of $f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi))$. By a trivial extension, we may assume that v'_1 is defined on a neighborhood U_1 of $K \cup f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi))$. Define $v_1 = \chi v + v'_1$ on U_1 . We have that $\bar{\partial}v_1 = \alpha$ and that $v_1 = v$ on a neighborhood of $f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi))$. As in Lemma 6.2, there exists a function $\varphi_1 \in C(Y)$ such that $\omega + i\partial\bar{\partial}\varphi_1 \geq 0$ and that $K \cup f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi)) \subset f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi_1)) \subset U_1$. If we replace K, v, U and φ by K_1, v_1, U_1 and φ_1 respectively, we obtain v_2, U_2 and φ_2 which satisfy suitable conditions. By repeating this process, we obtain $v_{m+1} \in \mathcal{E}^{0,p-1}(X, F)$ such that $\bar{\partial}v_{m+1} = \alpha$ and that $v_{m+1} = v$ on a neighborhood of $f^{-1}(\text{supp}(\omega + i\partial\bar{\partial}\varphi))$. $\square$

Proof of Theorem 6.1. This can be shown using the case where φ is bounded, just similar to Tiba's proof [13, Theorem 2]. $\square$

Proof of the main results. Theorems 1.2–1.4 can be shown using the Mayer–Vietoris exact sequence by induction of m as well as Tiba's proof [13, Section 6]. $\square$

7. Quasi-torus case

In this section, we consider quasi-tori as a concrete example of Theorem 1.2.

Let Λ be a discrete subgroup of $\mathbb{C}^n$. The quotient group $X := \mathbb{C}^n / \Lambda$ is called a *quasi-torus* if $H^0(X, \mathcal{O}) = \mathbb{C}$. Here if $\text{rank}_{\mathbb{R}} \Lambda = 2n$, X is a complex torus. A basis of a discrete subgroup $\Lambda \subset \mathbb{C}^n$ is an ordered set $P = (\lambda_1, \dots, \lambda_r)$ of $\mathbb{R}$ -independent $\mathbb{Z}$ -generators of Λ . Let $\mathbb{R}_{\Lambda} := \{x_1\lambda_1 + \dots + x_r\lambda_r \mid x_1, \dots, x_r \in \mathbb{R}\}$ be the $\mathbb{R}$ -span of Λ . If the complex rank of $\Lambda \subset \mathbb{C}^n$ is n and the real rank is $n + q$, then after a change of coordinates we can get $e_1, \dots, e_n \in \Lambda$ and $\Lambda = \mathbb{Z}^n \oplus \Gamma$ with a discrete subgroup $\Gamma \subset \mathbb{C}^n$ of the real rank q . Let $M\mathbb{C}_{\Lambda} := \mathbb{R}_{\Lambda} \cap i\mathbb{R}_{\Lambda}$ be the maximal $\mathbb{C}$ -linear subspace of $\mathbb{R}_{\Lambda}$. A quasi-torus $X = \mathbb{C}^n / \Lambda$ is of *type* q if $M\mathbb{C}_{\Lambda}$ has the complex dimension q . The concept of toroidal coordinates was introduced by Kopfermann, Gherardelli and Andreotti (cf. [1]). Toroidal coordinates have the following properties:

- (1) $M\mathbb{C}_{\Lambda} = \{z \in \mathbb{C}^n \mid z_{q+1} = \dots = z_n = 0\}$ is the subspace of the first q coordinate;
- (2) $\mathbb{R}_{\Lambda} = \{z \in \mathbb{C}^n \mid \text{Im } z_{q+1} = \dots = \text{Im } z_n = 0\} = M\mathbb{C}_{\Lambda} \oplus V$, where V is the real subspace generated by the $n - q$ units $e_{q+1}, \dots, e_n \in \Lambda$; and
- (3) $\mathbb{C}^n = M\mathbb{C}_{\Lambda} \oplus V \oplus iV = \mathbb{R}_{\Lambda} \oplus iV$.

PROPOSITION 7.1. (cf. [1]) *Every Abelian Lie group $X = \mathbb{C}^n / \Lambda$ of type q is $(q + 1)$ -complete. Moreover there exists a smooth exhaustive plurisubharmonic function which is $(q + 1)$ -convex.*

Proof. In toroidal coordinates the $\mathbb{R}$ -span of Λ is $\mathbb{R}_{\Lambda} = \{z = x + iy \in \mathbb{C}^n \mid y_{q+1} = \dots = y_n = 0\}$. The Λ -periodic smooth function $\tilde{\psi}(z) := \sum_{k=q+1}^n y_k^2$ on $\mathbb{C}^n$ which is exhaustive and plurisubharmonic induces a smooth exhaustive plurisubharmonic function ψ on X . Moreover ψ is $(q + 1)$ -convex, because the Levi form

$$i\partial\bar{\partial}\tilde{\psi} = i \sum_{j,k=1}^n \frac{\partial^2 \tilde{\psi}}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k = \frac{i}{4} \sum_{j,k=1}^n \frac{\partial^2 \tilde{\psi}}{\partial y_j \partial y_k} dz_j \wedge d\bar{z}_k = \frac{i}{4} \sum_{k=q+1}^n dz_k \wedge d\bar{z}_k$$

is positive semi-definite with $n - q$ positive eigenvalues. $\square$

If f is a Λ -periodic function, then the convolution of f and a mollifier ρ is also a Λ -periodic function. From this, we obtain the following theorem as a quasi-torus case of Theorem 1.2.

THEOREM 7.2. *Let $X := \mathbb{C}^n / \Lambda$ be a quasi-torus of type q and F be a holomorphic vector bundle over X . Let $\varphi_1, \dots, \varphi_m$ be exhaustive plurisubharmonic functions on X . Then the natural map*

$$H^p(X, F) \longrightarrow \varinjlim_{(\bigcap_{j=1}^m \text{supp } i\partial\bar{\partial}\varphi_j) \subset U} H^p(U, F)$$

is an isomorphism for $0 \leq p < n - q - m - 1$ and is injective for $p = n - q - m - 1$.

In particular, the following holds from the $(q + 1)$ -completeness of X .

$$\varinjlim_{(\bigcap_{j=1}^m \text{supp } i\partial\bar{\partial}\varphi_j) \subset U} H^p(U, F) = 0$$

for $q + 1 \leq p < n - q - m - 1$.

Proof. Let $\pi : \mathbb{C}^n \rightarrow X = \mathbb{C}^n / \Lambda$ be a natural projection. Fix an integer j such that $j \in \{1, \dots, m\}$. Then $\pi^* \varphi_j$ is a Λ -periodic plurisubharmonic function on $\mathbb{C}^n$ and induces φ_j . From Example 3.1, $\pi^* \varphi_j$ has a decreasing approximation $(\pi^* \varphi_j * \rho_{1/k})_{k \in \mathbb{N}}$ as in Definition 1.1. Because the Λ -periodicity of the function $\pi^* \varphi_j$ is inherited, the convolution $\pi^* \varphi_j * \rho_{1/k}$ is also a Λ -periodic function. In fact, for any element $a \in \Lambda$, we have that

$$\begin{aligned} \pi^* \varphi_j * \rho_{1/k}(z + a) &= \int \pi^* \varphi_j(z + a - w) \rho_{1/k}(w) dw \\ &= \int \pi^* \varphi_j(z - w) \rho_{1/k}(w) dw = \pi^* \varphi_j * \rho_{1/k}(z). \end{aligned}$$

Therefore the function $\pi^* \varphi_j * \rho_{1/k}$ on $\mathbb{C}^n$ induces a smooth plurisubharmonic function $\varphi_{j,k}$ on X . Since $(\pi^* \varphi_j * \rho_{1/k})_{k \in \mathbb{N}}$ is a decreasing approximation to $\pi^* \varphi_j$ on $\mathbb{C}^n$ as in Definition 1.1, we obtain a decreasing approximation $(\varphi_{j,k})_{k \in \mathbb{N}}$ to φ_j on X as in Definition 1.1. From Theorem 1.2 and Proposition 7.1, this proof is complete. $\square$

8. Extension to the Kähler case

Corollary 4.4 is a special case of Theorem 1.2 in terms of saying that the support of a semi-positive current is a sublevel set of an exhaustive function, and since this theorem requires the existence of a smooth plurisubharmonic function which is q -convex, X must be a q -complete manifold. Also, if X is a weak pseudoconvex Kähler manifold and a holomorphic vector bundle F has some positivity, a similar argument as in Corollary 4.4 would hold, but the existence of a smooth plurisubharmonic function ψ which satisfies $\text{supp } i\partial\bar{\partial}\psi = X$ must be assumed.

Then, we consider the case in which the conditions of q -completeness and the existence of a smooth plurisubharmonic function satisfying $\text{supp } i\partial\bar{\partial}\psi = X$ are eliminated.

LEMMA 8.1. *Let (X, ω) be a Kähler manifold. Let (F, h) be a holomorphic Hermitian vector bundle over X . Then for any (n, p) -form with values in F , we have that*

$$\star_F[i\Theta_{F,h}, \Lambda_\omega] = [i\Theta_{F^*,h^*}, \Lambda_\omega]\star_F,$$

$$[i\Theta_{F,h}, \Lambda_\omega] > 0 \text{ on } \Lambda^{n,p} T_X^* \otimes F \iff [i\Theta_{F^*,h^*}, \Lambda_\omega] > 0 \text{ on } \Lambda^{0,n-p} T_X^* \otimes F^*.$$

Furthermore, for any Kähler metric γ such that $\gamma \geq \omega$, if $[i\Theta_{F,h}, \Lambda_\omega] > 0$ on $\Lambda^{0,p} T_X^* \otimes F$ then $[i\Theta_{F,h}, \Lambda_\gamma] > 0$ on $\Lambda^{0,p} T_X^* \otimes F$ for any $p \geq 1$.

Proof. Let $x_0 \in X$ and let $(z_1, \dots, z_n)$ be local coordinates such that $(\partial/\partial z_1, \dots, \partial/\partial z_n)$ is an orthonormal basis of (T_X, ω) at x_0 . Let $(e_1, \dots, e_r)$ be an orthonormal basis of F_{x_0} . We can write

$$\begin{aligned} \omega_{x_0} &= i \sum_{1 \leq j \leq n} dz_j \wedge d\bar{z}_j, \\ i\Theta_{F,h,x_0} &= i \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu, \end{aligned}$$

$$\begin{aligned}
i\Theta_{F^*, h^*, x_0} &= -i\Theta_{F, h, x_0}^\dagger = -i \sum_{j, k, \lambda, \mu} c_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes (e_\lambda^*)^* \otimes e_\mu^* \\
&= -i \sum_{j, k, \lambda, \mu} c_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes (e_\mu^*)^* \otimes e_\lambda^*.
\end{aligned}$$

Let $dz_N = dz_1 \wedge \cdots \wedge dz_n$ and let

$$\begin{aligned}
u &= \sum_{|K|=p, \lambda} u_{K, \lambda} dz_N \wedge d\bar{z}_K \otimes e_\lambda \in \Lambda^{n, p} T_{X, x_0}^* \otimes F_{x_0}, \\
v &= \sum_{|J|=p, |K|=q, \lambda} v_{K, \lambda} dz_J \wedge d\bar{z}_K \otimes e_\lambda \in \Lambda^{p, q} T_{X, x_0}^* \otimes F_{x_0}.
\end{aligned}$$

As is well known, we have that

$$\begin{aligned}
\star_F u &= \sum_{|K|=p, \lambda} \text{sgn}(K, \widehat{K}) \bar{u}_{K, \lambda} d\bar{z}_{\widehat{K}} \otimes e_\lambda^* \in \Lambda^{0, n-p} T_{X, x_0}^* \otimes F_{x_0}^*, \\
\Lambda_\omega v &= i(-1)^p \sum_{J, K, \lambda, s} u_{J, K, \lambda} \left(\frac{\partial}{\partial z_s} \lrcorner dz_J \right) \wedge \left(\frac{\partial}{\partial \bar{z}_s} \lrcorner d\bar{z}_K \right) \otimes e_\lambda,
\end{aligned}$$

where $\xi \lrcorner$ denotes an interior product ι_ξ for $\xi \in T_X$ and $\widehat{K}$ is a multi-index such that $\{K, \widehat{K}\} = \{1, \dots, n\}$. Define the number $\varepsilon(s, K) \in \{-1, 1\}$ such that $\partial/\partial \bar{z}_s \lrcorner d\bar{z}_K = \varepsilon(s, K) d\bar{z}_{K \setminus s}$ for any number $s \in K$. Then, since $\bar{c}_{jk\lambda\mu} = c_{kj\mu\lambda}$, a simple computation gives

$$\begin{aligned}
\star_F [i\Theta_{F, h}, \Lambda_\omega] u &= \star_F (i\Theta_{F, h} \wedge \Lambda_\omega u) \\
&= \star_F \left\{ i\Theta_{F, h} \wedge i(-1)^n \sum u_{K, \lambda} \left(\frac{\partial}{\partial z_s} \lrcorner dz_N \right) \wedge \left(\frac{\partial}{\partial \bar{z}_s} \lrcorner d\bar{z}_K \right) \otimes e_\lambda \right\} \\
&= \star_F \left\{ \sum c_{jk\lambda\mu} u_{K, \lambda} dz_j \wedge \left(\frac{\partial}{\partial z_s} \lrcorner dz_N \right) \wedge d\bar{z}_k \wedge \left(\frac{\partial}{\partial \bar{z}_s} \lrcorner d\bar{z}_K \right) \otimes e_\mu \right\} \\
&= \star_F \left\{ \sum_{j \in K} \sum c_{jk\lambda\mu} u_{K, \lambda} dz_N \wedge d\bar{z}_k \wedge \left(\frac{\partial}{\partial \bar{z}_j} \lrcorner d\bar{z}_K \right) \otimes e_\mu \right\} \\
&= \star_F \left\{ \sum_{j \in K} \sum \varepsilon(j, K) c_{jk\lambda\mu} u_{K, \lambda} dz_N \wedge d\bar{z}_k \wedge d\bar{z}_{K \setminus j} \otimes e_\mu \right\} \\
&= \star_F \left\{ \sum_{j \in K} \sum \varepsilon(j, K) c_{jk\lambda\mu} u_{K, \lambda} dz_N \wedge d\bar{z}_{k, K \setminus j} \otimes e_\mu \right\} \\
&= \sum_{j \in K} \sum \text{sgn}(k, K \setminus j, \widehat{k, K \setminus j}) \varepsilon(j, K) \bar{c}_{jk\lambda\mu} \bar{u}_{K, \lambda} d\bar{z}_{\widehat{k, K \setminus j}} \otimes e_\mu^* \\
&= \sum_{k \in K} \sum \text{sgn}(j, K \setminus k, \widehat{j, K \setminus k}) \varepsilon(k, K) c_{jk\lambda\mu} \bar{u}_{K, \mu} d\bar{z}_{\widehat{j, K \setminus k}} \otimes e_\lambda^*,
\end{aligned}$$

$$\begin{aligned}
[i\Theta_{F^*,h^*}, \Lambda_\omega] \star_F u &= -\Lambda_\omega \wedge i\Theta_{F^*,h^*} \wedge \star_F u \\
&= -\Lambda_\omega \wedge i\Theta_{F^*,h^*} \wedge \sum \operatorname{sgn}(K, \widehat{K}) \bar{u}_{K,\lambda} d\bar{z}_{\widehat{K}} \otimes e_\lambda^* \\
&= -\Lambda_\omega \wedge i(-1) \sum_{k \in K} \sum \operatorname{sgn}(K, \widehat{K}) c_{jk\lambda\mu} \bar{u}_{K,\mu} dz_j \wedge d\bar{z}_k \wedge d\bar{z}_{\widehat{K}} \otimes e_\lambda^* \\
&= \sum_{k \in K} \sum \operatorname{sgn}(K, \widehat{K}) c_{jk\lambda\mu} \bar{u}_{K,\mu} \left(\frac{\partial}{\partial z_s} \lrcorner dz_j \right) \wedge \left(\frac{\partial}{\partial \bar{z}_s} \lrcorner d\bar{z}_k \wedge d\bar{z}_{\widehat{K}} \right) \otimes e_\lambda^* \\
&= \sum_{k \in K} \left\{ \sum_{k=j} \operatorname{sgn}(K, \widehat{K}) c_{jj\lambda\mu} \bar{u}_{K,\mu} d\bar{z}_{\widehat{K}} \right. \\
&\quad \left. - \sum_{k \neq j} \operatorname{sgn}(K, \widehat{K}) c_{jk\lambda\mu} \bar{u}_{K,\mu} d\bar{z}_k \wedge \left(\frac{\partial}{\partial \bar{z}_j} \lrcorner d\bar{z}_{\widehat{K}} \right) \right\} \otimes e_\lambda^* \\
&= \sum_{k \in K} \left\{ \sum_{k=j} \operatorname{sgn}(K, \widehat{K}) c_{jj\lambda\mu} \bar{u}_{K,\mu} d\bar{z}_{\widehat{K}} \right. \\
&\quad \left. - \sum_{k \neq j} \operatorname{sgn}(K, \widehat{K}) \varepsilon(j, \widehat{K}) c_{jk\lambda\mu} \bar{u}_{K,\mu} d\bar{z}_{k, \widehat{K} \setminus j} \right\} \otimes e_\lambda^*.
\end{aligned}$$

Therefore $\star_F[i\Theta_{F,h}, \Lambda_\omega] = [i\Theta_{F^*,h^*}, \Lambda_\omega] \star_F$ if and only if

$$\begin{aligned}
&\operatorname{sgn}(K, \widehat{K}) d\bar{z}_{\widehat{K}} = \operatorname{sgn}(j, K \setminus j, \widehat{j, K \setminus j}) \varepsilon(j, K) d\bar{z}_{\widehat{j, K \setminus j}}, \\
&-\operatorname{sgn}(K, \widehat{K}) \varepsilon(j, \widehat{K}) d\bar{z}_{k, \widehat{K} \setminus j} = \operatorname{sgn}(j, K \setminus k, \widehat{j, K \setminus k}) \varepsilon(k, K) d\bar{z}_{\widehat{j, K \setminus k}}.
\end{aligned}$$

From the definition, we have that $\varepsilon(s, K) d\bar{z}_s \wedge d\bar{z}_{K \setminus s} = d\bar{z}_K$ and that $\operatorname{sgn}(K, \widehat{K}) d\bar{z}_{\widehat{K}}$ is characterized by $\operatorname{sgn}(K, \widehat{K}) d\bar{z}_K \wedge d\bar{z}_{\widehat{K}} = d\bar{z}_N$. Then the above two equations are proven from the following two equations:

$$\begin{aligned}
&\operatorname{sgn}(K, \widehat{K}) \varepsilon(j, K) d\bar{z}_{j, K \setminus j} \wedge d\bar{z}_{\widehat{K}} \\
&= \operatorname{sgn}(K, \widehat{K}) \varepsilon(j, K) d\bar{z}_j \wedge d\bar{z}_{K \setminus j} \wedge d\bar{z}_{\widehat{K}} = d\bar{z}_N, \\
&-\operatorname{sgn}(K, \widehat{K}) \varepsilon(j, \widehat{K}) \varepsilon(k, K) d\bar{z}_{j, K \setminus k} \wedge d\bar{z}_{k, \widehat{K} \setminus j} \\
&= -\operatorname{sgn}(K, \widehat{K}) \varepsilon(j, \widehat{K}) \varepsilon(k, K) d\bar{z}_j \wedge d\bar{z}_{K \setminus k} \wedge d\bar{z}_k \wedge d\bar{z}_{\widehat{K} \setminus j} \\
&= \operatorname{sgn}(K, \widehat{K}) \varepsilon(j, \widehat{K}) \varepsilon(k, K) d\bar{z}_k \wedge d\bar{z}_{K \setminus k} \wedge d\bar{z}_j \wedge d\bar{z}_{\widehat{K} \setminus j} = d\bar{z}_N.
\end{aligned}$$

For any $u \in \Lambda^{n,p} T_X^* \otimes F$, we have that $\langle [i\Theta_{F,h}, \Lambda_\omega] u, u \rangle_{h,\omega} = \langle [i\Theta_{F^*,h^*}, \Lambda_\omega] \star_F u, \star_F u \rangle_{h^*,\omega}$. Then the second argument follows from the isomorphism of $\star_F$. In fact

$$\begin{aligned}
\langle [i\Theta_{F,h}, \Lambda_\omega] u, u \rangle_{h,\omega} dV_\omega &= \overline{\langle u, [i\Theta_{F,h}, \Lambda_\omega] u \rangle_{h,\omega}} dV_\omega = \overline{u \wedge \star_F [i\Theta_{F,h}, \Lambda_\omega] u} \\
&= \overline{u \wedge [i\Theta_{F^*,h^*}, \Lambda_\omega] \star_F u} \\
&= (-1)^{(n+p)(n-p)} [i\Theta_{F^*,h^*}, \Lambda_\omega] \star_F u \wedge u \\
&= (-1)^{(n+p)(n-p)} [i\Theta_{F^*,h^*}, \Lambda_\omega] \star_F u \wedge \star_{F^*}^{-1} \star_{F^*}^{-1} \star_F u \\
&= (-1)^{(n+p)(n-p)} (-1)^{n-p} [i\Theta_{F^*,h^*}, \Lambda_\omega] \star_F u \wedge \star_{F^*} \star_F u \\
&= \langle [i\Theta_{F^*,h^*}, \Lambda_\omega] \star_F u, \star_F u \rangle_{h^*,\omega} dV_\omega.
\end{aligned}$$

Because of Lemma 2.3, for any Hermitian metric γ such that $\gamma \geq \omega$, we have that

$$\begin{aligned} [i\Theta_{F,h}, \Lambda_\omega] > 0 \quad \text{on } \Lambda^{0,p}T_X^* \otimes F &\iff [i\Theta_{F^*,h^*}, \Lambda_\omega] > 0 \quad \text{on } \Lambda^{n,n-p}T_X^* \otimes F^* \\ \implies [i\Theta_{F^*,h^*}, \Lambda_\gamma] > 0 \quad \text{on } \Lambda^{n,n-p}T_X^* \otimes F^* \\ \iff [i\Theta_{F,h}, \Lambda_\gamma] > 0 \quad \text{on } \Lambda^{0,p}T_X^* \otimes F. \end{aligned}$$

From the above, this proof is completed. $\square$

If the assumption of the following proposition is $[i\Theta_{F^*,h^*}, \Lambda_{\omega_X}] > 0$ on $\Lambda^{0,p}T_X^* \otimes F^*$ ($n - q \leq p$) instead of $[i\Theta_{F,h}, \Lambda_{\omega_X}] > 0$ on $\Lambda^{0,p}T_X^* \otimes F$ ($1 \leq p \leq q$), and if this positivity of F^* is created by a plurisubharmonic and q -convex function, then the proposition can be shown by the methods of Lemmas 2.5 and 2.6 and Proposition 2.7 as the $\varphi = \eta$ case, when using the Donnelly–Fefferman–Berndtsson type L^2 -estimate and L^2 -Serre duality. However, these conditions for the positivity of F and F^* are equivalent by Lemma 8.1 and therefore the following proposition holds.

PROPOSITION 8.2. *Let (X, ω_X) be a weakly pseudconvex Kähler manifold. Let (F, h) be a holomorphic Hermitian vector bundle over X . Assume that $[i\Theta_{F,h}, \Lambda_{\omega_X}] > 0$ on $\Lambda^{0,p}T_X^* \otimes F$ ($1 \leq p \leq q$). Then for any $\alpha \in \mathcal{E}^{0,p}(X, F)$ satisfying $\bar{\partial}\alpha = 0$ and $\text{supp } \alpha \subset\subset X$, there exists $\beta \in \mathcal{E}^{0,p-1}(X, F)$ such that $\bar{\partial}\beta = \alpha$ and that $\text{supp } \beta \subset\subset X$.*

From Lemma 8.1, Proposition 8.2 can be shown directly from the L^2 -existence theorem without using the Donnelly–Fefferman–Berndtsson type L^2 -estimate, L^2 -Serre duality, and the existence of q -convex plurisubharmonic functions as in the following proof.

Proof. Let $\psi \in \mathcal{E}(X, \mathbb{R})$ be an exhaustive plurisubharmonic function on X . There exists $r < \sup_X \psi$ such that $\text{supp } \alpha \subset X_r := \{z \in X \mid \psi(z) < r\}$. Let $0 < \varepsilon < \sup_X \psi - r$ be a positive number that satisfies $X_r \subset\subset X_{r+\varepsilon} \subset\subset X$. Define the smooth plurisubharmonic function $\phi = -\log\{-(\psi - (r + \varepsilon))\} + \log \varepsilon = -\log(r + \varepsilon - \psi) + \log \varepsilon$ on $X_{r+\varepsilon}$ then; ϕ is exhaustive and satisfies $X_r = \{z \in X_{r+\varepsilon} \mid \phi(z) < 0\}$. Therefore let $\omega := \omega_X + i\partial\bar{\partial}\phi$ be a complete Kähler metric on $X_{r+\varepsilon}$. Because of Lemma 8.1, $\omega \geq \omega_X$ and Lemma 2.3, we have that $[i\Theta_{F^*,h^*e^{-\phi}}, \Lambda_{\omega_X}] \geq [i\Theta_{F^*,h^*}, \Lambda_{\omega_X}] > 0$ on $\Lambda^{n,n-p}T_X^* \otimes F^*$ and that $[i\Theta_{F^*,h^*e^{-\phi}}, \Lambda_\omega] > 0$ on $\Lambda^{n,n-p}T_{X_{r+\varepsilon}}^* \otimes F^*|_{X_{r+\varepsilon}}$. Thus we get $[i\Theta_{F,he^\phi}, \Lambda_\omega] > 0$ on $\Lambda^{0,p}T_{X_{r+\varepsilon}}^* \otimes F|_{X_{r+\varepsilon}}$. From the condition $\text{supp } \alpha \subset X_r \subset\subset X_{r+\varepsilon}$, we have that

$$\begin{aligned} \|\alpha\|_{he^\phi, \omega}^2 &= \int_{X_{r+\varepsilon}} |\alpha|_{h, \omega}^2 e^\phi dV_\omega \leq \int_{X_{r+\varepsilon}} |\alpha|_{h, \omega}^2 dV_\omega < +\infty, \\ \int_{X_{r+\varepsilon}} \langle [i\Theta_{F,he^\phi}, \Lambda_\omega]^{-1} \alpha, \alpha \rangle_{h, \omega} e^\phi dV_\omega &\leq \int_{X_{r+\varepsilon}} \langle [i\Theta_{F,he^\phi}, \Lambda_\omega]^{-1} \alpha, \alpha \rangle_{h, \omega} dV_\omega < +\infty, \end{aligned}$$

i.e. $\alpha \in L_{0,p}^2(X_{r+\varepsilon}, F, he^\phi, \omega)$. Hence by Theorem 2.1 (L^2 -existence theorem), there exists $\beta \in L_{0,p-1}^2(X_{r+\varepsilon}, F, he^\phi, \omega)$ such that $\bar{\partial}\beta = \alpha$ and that

$$\|\beta\|_{he^\phi, \omega}^2 = \int_{X_{r+\varepsilon}} |\beta|_{h, \omega}^2 e^\phi dV_\omega \leq \int_{X_{r+\varepsilon}} \langle [i\Theta_{F,he^\phi}, \Lambda_\omega]^{-1} \alpha, \alpha \rangle_{h, \omega} e^\phi dV_\omega.$$

Here we get the condition that $\text{supp } \beta \subset X_{r+\varepsilon} \subset\subset X$ from the weight function e^ϕ tends to $+\infty$ as x tends to $\partial X_{r+\varepsilon}$. In addition, from the smoothness of α this β is smooth. $\square$

THEOREM 8.3. *Let (X, ω_X) be a Kähler manifold of dimension n ($n \geq 2$). Let (F, h) be a holomorphic Hermitian vector bundle over X . Assume that an open subset D is a weakly pseudoconvex submanifold and that $[i\Theta_{F,h}, \Lambda_{\omega_X}] > 0$ on $\Lambda^{0,p}T_D^* \otimes F|_D$ for any $1 \leq p \leq q$. Then the natural map*

$$H^p(X, F) \longrightarrow \varinjlim_{D^c \subset U} H^p(U, F)$$

is an isomorphism for $p < q$ and is injective for $p = q$.

Proof. Let $\psi \in \mathcal{E}(D, \mathbb{R})$ be an exhaustive plurisubharmonic function on D and let $D_r = \{z \in X \mid \psi(z) < r\}$ be a sublevel set for any $r < \sup_D \psi$.

We first show that $H^p(X, F) \rightarrow \varinjlim_{D^c \subset U} H^p(U, F)$ is injective for $0 \leq p \leq q$. The case of $p = 0$ is obvious. We consider the $1 \leq p \leq q$ case. Let $\alpha \in \mathcal{E}^{0,p}(X, F)$ such that $\bar{\partial}\alpha = 0$. Suppose that there exists an open neighborhood U of D^c and $u \in \mathcal{E}^{0,p-1}(U, F)$ such that $\bar{\partial}u = \alpha$ on U . Here there exists a number r such that $\bar{D}_r^c \subset U$. Let $\chi \in \mathcal{E}(X, \mathbb{R})$ be a function such that $\text{supp } \chi \subset U$ and that $\chi = 1$ on $\bar{D}_r^c$. Then $\beta = \bar{\partial}(\chi u) - \alpha$ is a $\bar{\partial}$ -closed F -valued $(0, p)$ -form which satisfies $\text{supp } \beta \subset D_r \subset \subset D$. By Proposition 8.2, there exists $\gamma \in \mathcal{E}^{0,p-1}(D, F)$ such that $\bar{\partial}\gamma = \beta$ and that $\text{supp } \gamma \subset \subset D$. By extending γ to 0 on D^c , it becomes a differential form on X . Then $v = \chi u - \gamma \in \mathcal{E}^{0,p-1}(X, F)$ satisfies that $\bar{\partial}v = \alpha$ and that $v = u$ on a neighborhood of D^c .

Next, we show that $H^p(X, F) \rightarrow \varinjlim_{D^c \subset U} H^p(U, F)$ is surjective for $0 \leq p \leq q - 1$. Let U be an open neighborhood of D^c and let $u \in \mathcal{E}^{0,p}(U, F)$ such that $\bar{\partial}u = 0$. Take the same function $\chi \in \mathcal{E}(X, \mathbb{R})$ as above. Then $\beta = \bar{\partial}(\chi u)$ is a $\bar{\partial}$ -closed F -valued $(0, p+1)$ -form which satisfies $\text{supp } \beta \subset D_r \subset \subset D$. By Proposition 8.2, there exists $\gamma \in \mathcal{E}^{0,p}(D, F)$ such that $\bar{\partial}\gamma = \beta$ and that $\text{supp } \gamma \subset \subset D$. Then $v = \chi u - \gamma \in \mathcal{E}^{0,p}(X, F)$ satisfies that $\bar{\partial}v = 0$ and that $v = u$ on a neighborhood of D^c . $\square$

This result is similar to the well-known result in [10], but the place where positivity is assumed is different. As in the case of $D = X_r$ or X , we will find the following corollaries.

COROLLARY 8.4. (Extension of Corollary 4.4) *Let X be a Kähler manifold of dimension n ($n \geq 2$). Let F be a holomorphic vector bundle over X . Assume that an open subset D is a weakly pseudoconvex submanifold such that there exists a smooth plurisubharmonic function on D which is q -convex. Then the natural map*

$$H^p(X, F) \longrightarrow \varinjlim_{D^c \subset U} H^p(U, F)$$

is an isomorphism for $0 \leq p < n - q$ and is injective for $p = n - q$.

COROLLARY 8.5. *Let (X, ω_X) be a weakly pseudoconvex Kähler manifold of dimension n ($n \geq 2$) with a smooth exhaustive plurisubharmonic function ψ on X . Let (F, h) be a holomorphic Hermitian vector bundle over X and $X_r = \{z \in X \mid \psi(z) < r\}$ be a sublevel set for any $r < \sup_X \psi$. Assume that $[i\Theta_{F,h}, \Lambda_{\omega_X}] > 0$ on $\Lambda^{0,p}T_X^* \otimes F$ for any $p \leq q$.*

Then the natural map

$$H^p(X, F) \longrightarrow \varinjlim_{X_r^c \subset U} H^p(U, F)$$

is an isomorphism for $0 \leq p < q$ and is injective for $p = q$.

Furthermore, from Lemma 8.1, as is well known, it follows that

$$H^p(X, K_X \otimes F^*) = 0 \quad \text{for } n - q \leq p.$$

Consider the case when X is compact. Then any sublevel set becomes X , i.e. $X_r^c = \emptyset$. When the target closed set is the empty set, if we consider the direct limit to the empty set as $\lim_{\rightarrow \emptyset \subset U} H^p(U, F) = 0$, then Corollary 8.5 expresses that $H^p(X, F) = 0$ for $p \leq q$. Here we always have the condition $H^p(X, K_X \otimes F^*) = 0$ for $n - q \leq p$ without compactness.

Hence Corollary 8.5 can be thought of as a certain extension of the Serre duality considered on compact manifolds to weakly pseudoconvex Kähler manifolds.

9. Variants of the Lefschetz hyperplane theorem

Let X be a complex manifold and let F be a holomorphic vector bundle over X . Let $K \neq \emptyset$ be a closed subset and let U be an open subset of X . We denote by $\mathcal{E}^{p,q}(X, U, F)$ (respectively $\mathcal{E}^k(X, U, F)$) the subspace of elements α in $\mathcal{E}^{p,q}(X, F)$ (respectively $\mathcal{E}^k(X, F)$) such that the restriction of α to U is zero. There exists a canonical map $\mathcal{E}^{p,q}(X, U, F) \longrightarrow \mathcal{E}^{p,q}(X, U', F)$ for $U' \subset U$, and the direct limit

$$\varinjlim_{K \subset U} \mathcal{E}^{p,q}(X, U, F)$$

is defined where U runs through all open neighborhoods of K . Then there exists a short exact sequence

$$0 \longrightarrow \varinjlim_{K \subset U} \mathcal{E}^{p,q}(X, U, F) \longrightarrow \mathcal{E}^{p,q}(X, F) \longrightarrow \varinjlim_{K \subset U} \mathcal{E}^{p,q}(U, F) \longrightarrow 0.$$

We denote $H^p(X, U, F)$ by $H^p(\mathcal{E}^{0,*}(X, U, F))$ and similarly $H^p(X, U, \mathbb{C})$ by $H^p(\mathcal{E}^*(X, U, \mathbb{C}))$.

We get the following corollaries in the same way as Tiba proved.

COROLLARY 9.1. *Let $X, \psi_q, \varphi_1, \dots, \varphi_m$ be as in Theorem 1.2. Then the natural map*

$$H^p(X, \mathbb{C}) \longrightarrow \varinjlim_{(\bigcap_{j=1}^m \text{supp } i\partial\bar{\partial}\varphi_j) \subset U} H^p(U, \mathbb{C})$$

is an isomorphism for $p < n - q - m$ and is injective for $p = n - q - m$.

COROLLARY 9.2. *Let $X = \mathbb{C}^n/\Lambda, \varphi_1, \dots, \varphi_m$ be as in Theorem 7.2. Then the natural map*

$$H^p(X, \mathbb{C}) \longrightarrow \varinjlim_{(\bigcap_{j=1}^m \text{supp } i\partial\bar{\partial}\varphi_j) \subset U} H^p(U, \mathbb{C})$$

is an isomorphism for $p < n - q - m - 1$ and is injective for $p = n - q - m - 1$.

COROLLARY 9.3. *Let $X, S, f, \psi, \varphi_1, \dots, \varphi_m$ be as in Theorem 1.3. Then the natural map*

$$H^p(X, \mathbb{C}) \longrightarrow \varinjlim_{f^{-1}(\bigcap_{j=1}^m \text{supp } i\partial\bar{\partial}\varphi_j) \subset U} H^p(U, \mathbb{C})$$

is an isomorphism for $p < n - q - m$ and is injective for $p = n - q - m$.

COROLLARY 9.4. Let $X, Y, f, T_1, \dots, T_m$ be as in Theorem 1.4. Then the natural map

$$H^p(X, \mathbb{C}) \longrightarrow \varinjlim_{f^{-1}(\bigcap_{j=1}^m \operatorname{supp} T_j) \subset U} H^p(U, \mathbb{C})$$

is an isomorphism for $p < n - q - m$ and is injective for $p = n - q - m$.

LEMMA 9.5. (cf. [12, Lemma 1] and [8, Lemma 2.7.9]) Let m be a positive integer and K be a closed subset of X . If

$$\varinjlim_{K \subset U} H^q(X, U, \Omega_X^p) = 0$$

for $0 \leq p \leq n$ and $0 \leq q \leq m$, then

$$\varinjlim_{K \subset U} H^k(X, U, \mathbb{C}) = 0$$

holds for $0 \leq k \leq m$.

Proof. Fix an integer $k \in \{0, \dots, m\}$. Let $\alpha \in \varinjlim_{K \subset U} \mathcal{E}^k(X, U)$ such that $d\alpha = 0$. There exists an integer $l \in \{0, \dots, k\}$ and we can write α in the form

$$\alpha = \sum_{j=0}^l \alpha^{k-j, j},$$

where $\alpha^{p, q} \in \varinjlim_{K \subset U} \mathcal{E}^{p, q}(X, U)$. Then $\bar{\partial}\alpha^{k-l, l} = 0$. We will show this lemma by induction on r . This statement is obvious if $r = 0$ by the identity theorem. Assume that the statement has already been proved in the case $0 \leq r \leq l - 1$. We consider the case of $r = l$, i.e. $\alpha = \alpha^{k, 0} + \alpha^{k-1, 1} + \dots + \alpha^{k-l, l}$. By the assumptions

$$\varinjlim_{K \subset U} H^l(X, U, \Omega_X^{k-l}) = 0$$

and $\bar{\partial}\alpha^{k-l, l} = 0$, there exists

$$\beta^{k-l, l-1} \in \varinjlim_{K \subset U} \mathcal{E}^{k-l, l-1}(X, U)$$

such that $\bar{\partial}\beta^{k-l, l-1} = \alpha^{k-l, l}$. Then we have the decomposition

$$\alpha - d\beta^{k-l, l-1} = \sum_{j=0}^{l-1} \tilde{\alpha}^{k-j, j}.$$

Because of the inductive hypothesis, there exists $\gamma \in \varinjlim_{K \subset U} \mathcal{E}^{k-1}(X, U)$ such that $\bar{\partial}\gamma = \alpha - d\beta^{k-l, l-1}$. Then $\gamma - \beta^{k-l, l-1} \in \varinjlim_{K \subset U} \mathcal{E}^{k-1}(X, U)$ is the form we are looking for. $\square$

Proof of Corollaries 9.1–9.4. The proofs of Corollaries 9.2–9.4 are similar to the proof of Corollary 9.1. Hence we only prove Corollary 9.1. It follows from Theorem 1.2 and the natural long exact sequence that

$$\varinjlim_{(\bigcap_{j=1}^m \operatorname{supp} i\bar{\partial}\bar{\partial}\varphi_j) \subset U} H^p(X, U, F) = 0$$

for $0 \leq p \leq n - q - m$. By applying the case of $F = \Omega_X^p$ ($0 \leq p \leq n$), we get the assumptions of Lemma 9.5. Then we have that

$$\lim_{(\bigcap_{j=1}^m \operatorname{supp} i\partial\bar{\partial}\varphi_j) \subset U} H^p(X, U, \mathbb{C}) = 0$$

for $0 \leq p \leq n - q - m$, and Corollary 9.1 follows from the natural long exact sequence. $\square$

In addition, the following corollary follows from Corollary 8.4 (the extension of Corollary 4.4) and the above proof.

COROLLARY 9.6. *Let X be a Kähler manifold of dimension n ($n \geq 2$). Assume that an open subset D is a weakly pseudoconvex manifold such that there exists a smooth plurisubharmonic function on D which is q -convex. Then the natural map*

$$H^p(X, \mathbb{C}) \longrightarrow \varinjlim_{D^c \subset U} H^p(U, \mathbb{C})$$

is an isomorphism for $0 \leq p < n - q$ and is injective for $p = n - q$.

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