EXPLICIT RELATIONS BETWEEN
KANEKO–YAMAMOTO TYPE MULTIPLE
ZETA VALUES AND RELATED VARIANTS

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Abstract. In this paper we first establish several integral identities involving the multiple polylogarithm functions and the Kaneko–Tsumura A-function, which can be thought as a single-variable multiple polylogarithm function of level two. We find that these integrals can be expressed in terms of multiple zeta (star) values, their related variants (multiple t-values, multiple T-values, multiple S-values, etc.), and multiple harmonic (star) sums and their related variants (multiple T-harmonic sums, multiple S-harmonic sums, etc.), which are closely related to some special types of Schur multiple zeta values and their generalizations. Using these integral identities, we prove many explicit evaluations of Kaneko–Yamamoto multiple zeta values and their related variants. Further, we derive some relations involving multiple zeta (star) values and their related variants.

1. Introduction and notation

1.1. Multiple zeta values (MZVs) and Schur MZVs

We begin with some basic notation. A finite sequence \( k \equiv k_r := (k_1, \ldots, k_r) \) of positive integers is called a composition. We put

\[
|k| := k_1 + \cdots + k_r, \quad \text{dep}(k) := r,
\]

and call them the weight and the depth of \( k \), respectively.

For \( 0 \leq j \leq i \), we adopt the following notation:

\[
k_i^j := (k_{i+1-j}, k_{i+2-j}, \ldots, k_i)
\]

and

\[
k_i \equiv k_i^0 := (k_1, k_2, \ldots, k_i),
\]

where \( k_i^0 := \emptyset \) (\( i \geq 0 \)). If \( i < j \), then \( k_i^j := \emptyset \).

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For a composition $k_r = (k_1, \ldots, k_r)$ and positive integer $n$, the multiple harmonic sums (MHSs) and multiple harmonic star sums (MHSSs) are defined by

$$
\zeta_n(k_1, \ldots, k_r) := \sum_{0 < m_1 < \cdots < m_r \leq n} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}},
$$

and

$$
\zeta^*_n(k_1, \ldots, k_r) := \sum_{0 < m_1 \leq \cdots \leq m_r \leq n} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}},
$$

respectively. If $n < r$ then $\zeta_n(k_r) := 0$ and $\zeta^*_n(\emptyset) := 1$. The multiple zeta values (MZVs) and the multiple zeta-star values (MZSVs) are defined by

$$
\zeta(k) := \lim_{n \to \infty} \zeta_n(k) \quad \text{and} \quad \zeta^*(k) := \lim_{n \to \infty} \zeta^*_n(k),
$$

respectively. These series converge if and only if $k_r \geq 2$, so we call a composition $k_r$ admissible if this is the case.

The systematic study of multiple zeta values began in the early 1990s with the works of Hoffman [4] and Zagier [16]. Due to their surprising and sometimes mysterious appearance in the study of many branches of mathematics and theoretical physics, these special values have attracted a lot of attention and interest in the past three decades (for example, see the book by the second author [18]). A common generalization of the MZVs and MZSVs is given by the Schur multiple zeta values [10, 11], which are defined using the skew Young tableaux. For example, for integers $a, b, d, e, f \geq 1, c, g \geq 2$, the following sum is an example of Schur multiple zeta values:

$$
\zeta\left(\begin{array}{ccc}
a & b & c \\
d & e & f \\
f & g & \\
\end{array}\right) = \sum_{\substack{m_a \leq m_b \leq m_c \\
\leq m_d \leq \cdots \leq m_g \leq}} \frac{1}{m_a^a m_b^b m_c^c m_d^d m_e^e m_f^f m_g^g}. \quad (1)
$$

In this paper, we shall study some families of variations of MZVs. First, consider the following special form of the Schur multiple zeta values.

**Definition 1.1.** (cf. [8]) For any two compositions of positive integers $k = (k_1, \ldots, k_r)$ and $l = (l_1, \ldots, l_s)$, define

$$
\zeta(k \oplus l) := \sum_{0 < m_1 \leq \cdots \leq m_r \geq n_1 \geq \cdots \geq n_s > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} n_1^{l_1} \cdots n_s^{l_s}}
$$

$$
= \sum_{n=1}^{\infty} \zeta_{n-1}(k_1, \ldots, k_r-1) \zeta^*_n(l_1, \ldots, l_{s-1}) \frac{1}{n^{k_r+l_s}}. \quad (2)
$$

We call them Kaneko–Yamamoto multiple zeta values (K–Y MZVs for short).
The K–Y MZVs defined by (2) are special cases of the Schur multiple zeta values \( \zeta_\lambda(s) \) given by the following skew Young tableaux of anti-hook type:

\[
\begin{array}{cccc}
& & k_1 & \\
& & : & \\
& & k_{r-1} & \\
l_1 & \cdots & l_{r-1} & x
\end{array}
\]

where \( x = k_r + l_x \) and \( \lambda \) is simply the Young diagram underlying the above tableaux.

1.2. Variations of MZVs with even/odd summation indices

One may modify the definition of MZVs by restricting the summation indices to even/odd numbers. These values are apparently not in the class of Schur multiple zeta values. For instance, in recent papers [6, 7], Kaneko and Tsumura introduced a new kind of multiple zeta value of level two, called multiple T-values (MTVs), defined for admissible compositions \( k = (k_1, \ldots, k_r) \) by

\[
T(k) := 2^r \sum_{0 < m_1 < \cdots < m_r \atop m_i \equiv i \mod 2} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}} \sum_{0 < n_1 < \cdots < n_r} \frac{1}{(2n_1 - 1)^{k_1} (2n_2 - 2)^{k_2} \cdots (2n_r - r)^{k_r}}.
\]

This is in contrast to Hoffman’s multiple t-values (MtVs) defined in [5] as follows: for admissible compositions \( k = (k_1, \ldots, k_r) \),

\[
t(k) := \sum_{0 < m_1 < \cdots < m_r \atop \forall m_j: \text{odd}} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}} = \sum_{0 < n_1 < \cdots < n_r} \frac{1}{(2n_1 - 1)^{k_1} (2n_2 - 1)^{k_2} \cdots (2n_r - 1)^{k_r}}.
\]

Moreover, in [5] Hoffman also defined its star version, called multiple t-star value (MtSVs), as follows:

\[
t^*(k) := \sum_{0 < m_1 < \cdots < m_r \atop \forall m_j: \text{odd}} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}} = \sum_{0 < n_1 < \cdots < n_r} \frac{1}{(2n_1 - 1)^{k_1} (2n_2 - 1)^{k_2} \cdots (2n_r - 1)^{k_r}}.
\]

Very recently, the present authors have defined another variant of multiple zeta values in [13], called multiple mixed values or multiple M-values (MMVs for short). For \( \epsilon = (\epsilon_1, \ldots, \epsilon_r) \in \{\pm 1\}^r \) and admissible compositions \( k = (k_1, \ldots, k_r) \),

\[
M(k; \epsilon) := \sum_{0 < m_1 < \cdots < m_r} \frac{(1 + \epsilon_1(-1)^{m_1}) \cdots (1 + \epsilon_r(-1)^{m_r})}{m_1^{k_1} \cdots m_r^{k_r}}
\]

\[
= \sum_{0 < m_1 < \cdots < m_r} \frac{2^r}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}}.
\]
For brevity, we put a check on top of the component $k_j$ if $\varepsilon_j = -1$. For example,

$$
M(1, 2, \bar{3}) = \sum_{0 < m_1 < m_2 < m_3} \frac{(1 + (-1)^{m_1})(1 + (-1)^{m_2})(1 - (-1)^{m_3})}{m_1 m_2^2 m_3^3}
= \sum_{0 < l < m < n} \frac{8}{(2\ell)(2m)^2(2n - 1)^3}.
$$

It is obvious that MiVs satisfy the series shuffle relation; however, it is non-trivial to see that MTVs can be expressed using the iterated integral and satisfy both the duality relations (see [7, Theorem 3.1]) and the integral shuffle relations (see [7, Theorem 2.1]). Similar to MZVs, MMVs satisfy both series shuffle relations and the integral shuffle relations. Moreover, in [13], we also introduced and studied a class of MMVs that is opposite to MTVs, called multiple S-values (MSVs). For an admissible composition $k = (k_1, \ldots, k_r)$,

$$
S(k) := 2^r \sum_{0 < m_1 < \cdots < m_r, m_i \equiv -1 \mod 2} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}}.
$$

It is clear that every MMV can be written as a linear combination of alternating MZVs (also referred to as Euler sums or colored multiple zeta values) defined as follows. For $k \in \mathbb{N}^r$ and $\varepsilon \in \{\pm 1\}^r$, if $(k_r, \varepsilon_r) \neq (1, 1)$ (called admissible case) then

$$
\zeta(k; \varepsilon) := \sum_{0 < m_1 < \cdots < m_r} \frac{\varepsilon_1^{m_1} \cdots \varepsilon_r^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}}.
$$

We may compactly indicate the presence of an alternating sign as follows. Whenever $\varepsilon_j = -1$, we place a bar over the corresponding integer exponent $k_j$. For example,

$$
\zeta(\bar{2}, 3, 1, 4) = \zeta(2, 3, 1, 4; -1, 1, -1, 1).
$$

Similarly, a star version of alternating MZVs (called \emph{alternating multiple zeta-star values}) is defined by

$$
\zeta^*(k; \varepsilon) := \sum_{0 < m_1 \leq \cdots \leq m_r} \frac{\varepsilon_1^{m_1} \cdots \varepsilon_r^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}}.
$$

Deligne showed that the rational space generated by alternating MZVs of weight $w$ is bounded by the Fibonacci number $F_w$, where $F_0 = F_1 = 1$. In [13, Theorem 7.1] we showed that the rational space generated by MMVs of weight $w$ is bounded by $F_w - 1$. The missing piece is the one-dimensional space generated by $\ln^w 2$.

### 1.3. Variations of Kaneko–Yamamoto MZVs with even/odd summation indices

Now, we introduce the $T$-variant of Kaneko–Yamamoto MZVs. For positive integers $m$ and $n$ such that $n \geq m$, we define

$$
D_{n,m} := \begin{cases} 
\{(n_1, n_2, \ldots, n_m) \in \mathbb{N}^m \mid 0 < n_1 \leq n_2 < n_3 \leq \cdots \leq n_{m-1} < n_m \leq n\}, & \text{if } 2 \nmid m, \\
\{(n_1, n_2, \ldots, n_m) \in \mathbb{N}^m \mid 0 < n_1 \leq n_2 < n_3 \leq \cdots \leq n_{m-1} \leq n_m \leq n\}, & \text{if } 2 \mid m,
\end{cases}
$$

$$
E_{n,m} := \begin{cases} 
\{(n_1, n_2, \ldots, n_m) \in \mathbb{N}^m \mid 1 \leq n_1 < n_2 \leq n_3 < \cdots < n_{m-1} \leq n_m < n\}, & \text{if } 2 \nmid m, \\
\{(n_1, n_2, \ldots, n_m) \in \mathbb{N}^m \mid 1 \leq n_1 < n_2 \leq n_3 < \cdots < n_{m-1} < n_m \leq n\}, & \text{if } 2 \mid m.
\end{cases}
$$
Definition 1.2. [13, Definition 1.1] For a positive integer $m$, define

\[
T_n(k_{2m-1}) := \sum_{n \in D_{n,2m-1}} \frac{2^{2m-1}}{(\prod_{j=1}^{m-1} (2n_{2j-1} - 1)^{k_{2j-1}} (2n_{2j})^{k_{2j}})(2n_{2m-1} - 1)^{k_{2m-1}}},
\]

\[
T_n(k_{2m}) := \sum_{n \in D_{n,2m}} \frac{2^{2m}}{\prod_{j=1}^{m} (2n_{2j-1} - 1)^{k_{2j-1}}(2n_{2j})^{k_{2j}}},
\]

\[
S_n(k_{2m-1}) := \sum_{n \in E_{n,2m-1}} \frac{2^{2m-1}}{(\prod_{j=1}^{m-1} (2n_{2j-1} - 1)^{k_{2j-1}}(2n_{2j})^{k_{2j}})(2n_{2m-1})^{k_{2m-1}}},
\]

\[
S_n(k_{2m}) := \sum_{n \in E_{n,2m}} \frac{2^{2m}}{\prod_{j=1}^{m} (2n_{2j-1} - 1)^{k_{2j-1}}(2n_{2j})^{k_{2j}}}.
\]

where $T_n(k_{2m-1}) := 0$ if $n < m$, and $T_n(k_{2m}) = S_n(k_{2m-1}) = S_n(k_{2m}) := 0$ if $n \leq m$. Moreover, for convenience, we set $T_n(\emptyset) = S_n(\emptyset) := 1$. We call (6) and (7) multiple $T$-harmonic sums (MTHSs for short), and call (8) and (9) multiple $S$-harmonic sums (MSHSs for short).

In [13], we used the MTHSs and MSHSs to define the convoluted $T$-values $T(k \otimes I)$, which can be regarded as $S$- or $T$-variants of K–Y MZVs.

Definition 1.3. [13, Definition 1.2] For positive integers $m$ and $p$, the convoluted $T$-values are defined by

\[
T(k_{2m} \otimes l_{2p}) = 2 \sum_{n=1}^{\infty} \frac{T_n(k_{2m-1})T_n(l_{2p-1})}{(2n)^{k_{2m}+l_{2p}}},
\]

\[
T(k_{2m-1} \otimes l_{2p-1}) = 2 \sum_{n=1}^{\infty} \frac{T_n(k_{2m-2})T_n(l_{2p-2})}{(2n - 1)^{k_{2m-1}+l_{2p-1}}},
\]

\[
T(k_{2m} \otimes l_{2p-1}) = 2 \sum_{n=1}^{\infty} \frac{T_n(k_{2m-1})S_n(l_{2p-2})}{(2n)^{k_{2m}+l_{2p-1}}},
\]

\[
T(k_{2m-1} \otimes l_{2p}) = 2 \sum_{n=1}^{\infty} \frac{T_n(k_{2m-2})S_n(l_{2p-1})}{(2n - 1)^{k_{2m-1}+l_{2p}}},
\]

We may further define the convoluted $S$-values by

\[
S(k_{2m} \otimes l_{2p}) = 2 \sum_{n=1}^{\infty} \frac{S_n(k_{2m-1})S_n(l_{2p-1})}{(2n - 1)^{k_{2m}+l_{2p}}},
\]

\[
S(k_{2m-1} \otimes l_{2p-1}) = 2 \sum_{n=1}^{\infty} \frac{S_n(k_{2m-2})S_n(l_{2p-2})}{(2n)^{k_{2m-1}+l_{2p-1}}}. 
\]

In view of the interpretation of K–Y MZVs as special Schur MZVs, one may wonder if Schur MZVs can be generalized so that the convoluted $S$- and $T$-values are special cases.
1.4. Schur MZVs modulo $N$

We now generalize the concept of Schur multiple zeta functions (respectively values) to Schur multiple zeta functions (respectively values) modulo any positive integer $N$, the case $N = 2$ of which contains all the MMVs as special cases.

It turns out that when $N = 2$ the only differences between these values and the Schur MZVs is that each box in the Young diagram is decorated by either ‘0’ or ‘1’ at the upper left corner so that the running index appearing in that box must be either even or odd. For example, a variation of the example in (1) can be given as follows:

$$
\zeta \left( \begin{array}{ccc}
0_a & 0_b & 1_c \\
1_d & 1_e \\
0_f & 1_g \\
\end{array} \right) := \sum_{m_a \leq m_b \leq m_c \atop \wedge \atop m_d \leq m_e \atop \wedge \atop m_f \leq m_g} \frac{2^7}{m_a m_b m_c m_d m_e m_f m_g}.
$$

We now briefly describe this idea in general. For a skew Young diagram $\lambda$ with $n$ boxes (denoted by $\sharp(\lambda) = n$), let $T(\lambda, X)$ be the set of all Young tableaux of shape $\lambda$ over a set $X$. Let $D(\lambda) = \{(i, j) : 1 \leq i \leq r, \alpha_i \leq j \leq \beta_i\} = \{(i, j) : 1 \leq j \leq s, \alpha_j \leq i \leq b_j\}$ be the skew Young diagram of $\lambda$ so that $(i, j)$ refers to the box on the $i$th row and $j$th column of $\lambda$. Here, for a fixed row number $i$ (respectively column number $j$), $\alpha_i$ and $\beta_i$ (respectively $a_j$ and $b_j$) provide the starting and ending positions of the $i$th row (respectively $j$th column) entries. Fix any positive integer $N$; we may decorate $D(\lambda)$ by putting a residue class $\pi_{ij}$ mod $N$ at the upper left corner of $(i, j)$th box. We call such a decorated diagram a Young diagram modulo $N$, denoted by $\lambda^\pi$. Further, we define the set of semi-standard skew Young tableaux of shape $\lambda^\pi$ by

$$\text{SSYT}(\lambda^\pi) := \left\{ (m_{i,j}) \in T(\lambda, N) : \begin{array}{c}
m_{i,a_i} \leq m_{i,a_i+1} \leq \cdots \leq m_{i,\beta_i}, \\
\text{for}\ i \leq r, \alpha_i \leq j \leq \beta_i. \\
m_{a_i,j} < m_{a_i+1,j} < \cdots < m_{b_j,j}, \\
\text{for}\ i \leq j \leq b_j. \\
m_{i,j} \equiv \pi_{i,j} \pmod{N} \ \forall \ 1 \leq i \leq r, \alpha_i \leq j \leq \beta_i. \\
\end{array} \right\}.$$

For $s = (s_{i,j}) \in T(\lambda, \mathbb{C})$, the Schur multiple zeta function modulo $N$ associated with $\lambda^\pi$ is defined by the series

$$\zeta_{\lambda^\pi}(s) := \sum_{M \in \text{SSYT}(\lambda^\pi)} \frac{2^{|\lambda|}}{M^s},$$

where $M^s = (m_{i,j})^s := \prod_{(i, j) \in D(\lambda)} s_{i,j}^{m_{i,j}}$. Similar to [11, Lemma 2.1], it is not too hard to prove that the above series converges absolutely whenever $s \in W_\lambda$, where

$$W_\lambda := \left\{ s = (s_{i,j}) \in T(\lambda, \mathbb{C}) : \begin{array}{c}
\Re(s_{i,j}) \geq 1 \ \forall \ (i, j) \in D(\lambda) \setminus C(\lambda), \\
\Re(s_{i,j}) > 1 \ \forall \ (i, j) \in C(\lambda). \\
\end{array} \right\},$$

where $C(\lambda)$ is the set of all corners of $\lambda$. We remark that this domain of convergence is not the largest possible but a detailed discussion of this issue would take us too far afield.

Similar to K–Y MZVs, the above convoluted $S$- and $T$-values are all special cases of Schur MZVs modulo 2 corresponding to anti-hook type Young diagrams. The six convoluted
S- or T-values in (10)–(15) are all given by mod 2 Schur MZVs $\zeta_{k^j}$ $(1 \leq j \leq 6)$ below, respectively:

\[
\begin{align*}
\lambda_1^{\pi_1} &= \begin{array}{c}
\underbrace{1} \\
\vdots \\
\underbrace{1}
\end{array} \\
&= \begin{array}{c}
\underbrace{1} \\
\vdots \\
\underbrace{1}
\end{array}, \\
\lambda_2^{\pi_2} &= \begin{array}{c}
\underbrace{1} \\
\underbrace{0} \\
\vdots \\
\underbrace{0}
\end{array} \\
&= \begin{array}{c}
\underbrace{1} \\
\underbrace{0} \\
\vdots \\
\underbrace{0}
\end{array}, \\
\lambda_3^{\pi_3} &= \begin{array}{c}
\underbrace{1} \\
\underbrace{0} \\
\vdots \\
\underbrace{0}
\end{array} \\
&= \begin{array}{c}
\underbrace{1} \\
\underbrace{0} \\
\vdots \\
\underbrace{0}
\end{array}, \\
\lambda_4^{\pi_4} &= \begin{array}{c}
\underbrace{1} \\
\underbrace{0} \\
\vdots \\
\underbrace{0}
\end{array} \\
&= \begin{array}{c}
\underbrace{1} \\
\underbrace{0} \\
\vdots \\
\underbrace{0}
\end{array},
\end{align*}
\]

where $m' = 2m - 1$, $m'' = 2m - 2$, $p' = 2p - 1$, $p'' = 2p - 2$, $x_1 = k_{2m} + l_{2p}$, $x_2 = k_{2m-1} + l_{2p-1}$, $x_3 = k_{2m} + l_{2p-1}$, and $x_4 = k_{2m-1} + l_{2p}$.

The primary goals of this paper are to study the explicit relations of K–Y MZVs $\zeta(k \oplus l^*)$ and their related variants, such as $T$-variants $T(k \oplus l)$. Then, using these explicit relations, we establish some explicit formulas of multiple zeta (star) values and their related variants.

The remainder of this paper is organized as follows. In Section 2, we first establish the explicit evaluations of integrals $\int_0^1 x^{n-1} \text{Li}_k(x) \, dx$ and $\int_0^1 x^{2n+b} \text{A}(k; x) \, dx$ for all positive integers $n$ and $b \in \{-1, -2\}$, where $\text{Li}_k(x)$ is the single-variable multiple polylogarithm (see (17)) and $\text{A}(k; x)$ is the Kaneko–Tsumura A-function (see (24)). Then, for all compositions $k$ and $l$, using these explicit formulas obtained and by considering the two kind of integrals

\[
I_L(k; l) := \int_0^1 \frac{\text{Li}_k(x) \text{Li}_l(x)}{x} \, dx \quad \text{and} \quad I_A(k; l) := \int_0^1 \frac{\text{A}(k; x) \text{A}(l; x)}{x} \, dx,
\]

we establish some explicit relations of $\zeta(k \oplus l^*)$ and $T(k \oplus l)$. Further, we express the integrals $I_L(k; l)$ and $I_A(k; l)$ in terms of multiple integrals associated with 2-labeled posets following the idea of Yamamoto [14].
In Section 3, we first define a variation of the classical multiple polylogarithm function with \( r \)-variable \( \lambda_k(x_1, x_2, \ldots, x_r) \) (see (39)), and give the explicit evaluation of the integral

\[
\int_0^1 x^{n-1} \lambda_k(x_1, x_2, \ldots, x_r) \, dx, \quad \sigma_j \in \{\pm 1\}.
\]

Then we will consider the integral

\[
I_k((k; \sigma), (l; \epsilon)) := \int_0^1 \frac{\lambda_{k_r}(x_1, \ldots, x_r) \lambda_{l_s}(\epsilon_1 x, \ldots, \epsilon_s x)}{x} \, dx
\]

to find some explicit relations of alternating Kaneko–Yamamoto MZVs \( \zeta((k; \sigma) \otimes (l; \epsilon)^*) \).

Further, we will find some relations involving alternating MZVs. Finally, we express the integrals \( I_k((k; \sigma), (l; \epsilon)) \) in terms of multiple integrals associated with 3-labeled posets.

In Section 4, we define the multiple harmonic (star) sums and the function \( t(k; x) \) related to multiple \( t \)-values. Further, we establish some relations involving multiple \( t \)-star values.

2. Formulas of Kaneko–Yamamoto MZVs and \( T \)-variants

In this section we will prove several explicit formulas of Kaneko–Yamamoto MZVs and \( T \)-variants, and find some explicit relations among MZ(S)Vs and MTVs.

2.1. Some relations of Kaneko–Yamamoto MZVs

**Theorem 2.1.** Let \( r, n \in \mathbb{N} \) and \( k_r := (k_1, \ldots, k_r) \in \mathbb{N}^r \). Then

\[
\int_0^1 x^{n-1} \text{Li}_{k_r}(x) \, dx = \sum_{j=1}^{k_r-1} \frac{(-1)^{j-1}}{n^j} \zeta(k_{r-1}, k_r + 1 - j) + \frac{(-1)^{k_r-r}}{n^{k_r}} \zeta_n(1, k_{r-1})
\]

\[
+ \sum_{l=1}^{r-1} (-1)^{k_{l-1}} \sum_{j=1}^{k_{l-1}-1} \frac{(-1)^{j-1}}{n^{k_{l-1}}} \zeta_n(j, k_{l-1}) \zeta(k_{r-l-1}, k_{r-l} + 1 - j),
\]

where \( \text{Li}_{k_1,\ldots,k_r}(z) \) is the single-variable multiple polylogarithm function defined by

\[
\text{Li}_{k_1,\ldots,k_r}(z) := \sum_{0 < n_1 < \cdots < n_r} \frac{z^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}}, \quad z \in [-1, 1).
\]

**Proof.** It is well known that multiple polylogarithms can be expressed by the iterated integral

\[
\text{Li}_{k_1,\ldots,k_r}(x) = \int_0^x \frac{dt}{1-t} \left( \frac{dt}{t} \right)^{k_1-1} \cdots \frac{dt}{1-t} \left( \frac{dt}{t} \right)^{k_r-1},
\]

where for 1-forms \( \alpha_1(t) = f_1(t) \, dt, \ldots, \alpha_{\ell}(t) = f_{\ell}(t) \, dt \), we define iteratively

\[
\int_a^b \alpha_1(t) \cdots \alpha_{\ell}(t) = \int_a^b \left( \int_a^y \alpha_1(t) \cdots \alpha_{\ell-1}(t) \right) f_\ell(y) \, dy.
\]
Using integration by parts, we deduce the recurrence relation
\[
\int_0^1 x^{n-1} \operatorname{Li}_{k_r}(x) \, dx = \sum_{j=1}^{k_r-1} \frac{(-1)^{j-1}}{n^j} \zeta(k_{r-1}, k_r + 1 - j) + \frac{(-1)^{k_r-1}}{n^{k_r}} \sum_{j=1}^{n} \int_0^1 x^{j-1} \operatorname{Li}_{k_{r-1}}(x) \, dx.
\]
Thus, we arrive at the desired formula by a direct calculation.

For any string \(\{s_1, \ldots, s_d\}\) and \(r \in \mathbb{N}\), we denote by \(\{s_1, \ldots, s_d\}_r\) the concatenated string obtained by repeating \(\{s_1, \ldots, s_d\}\) exactly \(r\) times.

**Corollary 2.2.** (cf. [12]) For positive integers \(n\) and \(r\),
\[
\int_0^1 x^{n-1} \log'(1-x) \, dx = (-1)^r r! \zeta_n^*(\{1\}_r).
\]

For any non-trivial compositions \(k\) and \(l\), we consider the integral
\[
I_L(k; l) := \int_0^1 \frac{\operatorname{Li}_k(x) \operatorname{Li}_l(x)}{x} \, dx
\]
and use (16) to find some explicit relations of K–Y MZVs. We prove the following theorem.

**Theorem 2.3.** For compositions \(k_r = (k_1, \ldots, k_r) \in \mathbb{N}^r\) and \(l_s = (l_1, l_2, \ldots, l_s) \in \mathbb{N}^s\),
\[
\sum_{j=1}^{k_r-1} (-1)^{j-1} \zeta(k_{r-1}, k_r + 1 - j) \zeta(l_{s-1}, l_s + j) + (-1)^{|k_r|-r} \zeta(l_s \oslash (1, k_r)^*)
\]
\[
+ \sum_{i=1}^{r-1} (-1)^{|k_r|-i} \sum_{j=1}^{k_{r-i}-1} (-1)^{j-1} \zeta(k_{r-i-1}, k_{r-i} + 1 - j) \zeta(l_s \oslash (j, k_i)^*)
\]
\[
= \sum_{j=1}^{l_s-1} (-1)^{j-1} \zeta(l_{s-1}, l_s + 1 - j) \zeta(k_{r-1}, k_r + j) + (-1)^{|l_s|-s} \zeta(k_r \oslash (1, l_s)^*)
\]
\[
+ \sum_{i=1}^{s-1} (-1)^{|l_s|-i} \sum_{j=1}^{l_{s-i}-1} (-1)^{j-1} \zeta(l_{s-i-1}, l_{s-i} + 1 - j) \zeta(k_r \oslash (j, l_i)^*). \tag{18}
\]

**Proof.** According to the definition of multiple polylogarithm, we have
\[
\int_0^1 \frac{\operatorname{Li}_{k_r}(x) \operatorname{Li}_{l_s}(x)}{x} \, dx = \sum_{n=1}^\infty \frac{\zeta_{n-1}(l_{s-1})}{n^{l_s}} \int_0^1 x^{n-1} \operatorname{Li}_{k_r}(x) \, dx
\]
\[
= \sum_{n=1}^\infty \frac{\zeta_{n-1}(k_{r-1})}{n^{k_r}} \int_0^1 x^{n-1} \operatorname{Li}_{l_s}(x) \, dx
\]
Then using (16) with a direct calculation, we may deduce the desired evaluation.

The formula in Theorem 2.3 seems to be related to the harmonic product of Schur MZVs of anti-hook type in [10, Theorem 3.2] and the general harmonic product formula in [3, Lemma 2.2]. However, it does not seem to follow from them easily.
As a special case, setting \( r = 2, s = 1 \) in (18) and noting the fact that
\[
\zeta(l_1 \odot (1, k_1, k_2)^*) = \zeta^*(1, k_1, k_2 + l_1)
\]
and
\[
\zeta(l_1 \odot (j, k_2)^*) = \zeta^*(j, l_1 + k_2),
\]
we find that
\[
\sum_{j=1}^{k_2-1} (-1)^{j-1} \zeta(k_1, k_2 + 1 - j) \zeta(l_1 + j) + (-1)^{k_1+k_2} \zeta^*(1, k_1, k_2 + l_1) \\
+ (-1)^{k_2-1} \sum_{j=1}^{k_2-1} (-1)^{j-1} \zeta(k_1 + 1 - j) \zeta^*(j, l_1 + k_2)
= \sum_{j=1}^{l_1-1} (-1)^{j-1} \zeta(l_1 + 1 - j) \zeta(k_1, k_2 + j) + (-1)^{l_1-1} \zeta^*((k_1, k_2) \odot (1, l_1)^*). \tag{19}
\]

On the other hand, from the definition of K–Y MZVs, it is easy to find that
\[
\zeta((k_1, k_2) \odot (1, l_1)^*) = \zeta^*(k_1, 1, k_2 + l_1) + \zeta^*(1, k_1, k_2 + l_1) - \zeta^*(k_1 + 1, k_2 + l_1)
\]
\[
- \zeta^*(1, k_1 + k_2 + l_1).
\]
Hence, we can get the following corollary.

**Corollary 2.4.** For positive integers \( k_1, k_2 \) and \( l_1 \),
\[
((-1)^{l_1-1} + (-1)^{k_1+k_2-1}) \zeta^*(1, k_1, k_2 + l_1) + (-1)^{l_1-1} \zeta^*(k_1, 1, k_2 + l_1)
\]
\[
= \sum_{j=1}^{k_2-1} (-1)^{j-1} \zeta(k_1, k_2 + 1 - j) \zeta(l_1 + j) - (-1)^{k_2}
\]
\[
\times \sum_{j=1}^{k_2-1} (-1)^{j-1} \zeta(k_1 + 1 - j) \zeta^*(j, l_1 + k_2)
\]
\[
- \sum_{j=1}^{l_1-1} (-1)^{j-1} \zeta(l_1 + 1 - j) \zeta(k_1, k_2 + j) + (-1)^{l_1-1} \zeta^*(k_1 + 1, k_2 + l_1)
\]
\[
+ (-1)^{l_1-1} \zeta^*(1, k_1 + k_2 + l_1). \tag{20}
\]

Next, we establish an identity involving the *Arakawa–Kaneko zeta function* (see [1]), which is defined by
\[
\xi(k_1, \ldots, k_r; s) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} \log^r(1-e^{-t})}{e^t-1} \Li_{k_1,\ldots,k_r}(1-e^{-t}) \, dt \quad (\Re(s) > 0). \tag{21}
\]
Setting variables \( 1 - e^{-t} = x \) and \( s = p + 1 \in \mathbb{N} \), we deduce
\[
\xi(k_1, \ldots, k_r; p + 1) = \frac{(-1)^p}{p!} \int_0^1 \frac{\log^p(1-x)\Li_{k_1,k_2,\ldots,k_r}(x)}{x} \, dx
\]
\[
= \sum_{n=1}^\infty \frac{\zeta_{n-1}(k_1, \ldots, k_r) \zeta_n^*(\{1\}_p)}{n^{k_r+1}} = \zeta(k \odot (\{1\}_{p+1})^*).
\]
where we have used Corollary 2.2. Clearly, the Arakawa–Kaneko zeta value is a special case of integral $I_L(k; l)$. Further, setting $l_1 = l_2 = \cdots = l_s = 1$ in Theorem 2.3 yields

$$\xi(k_1, \ldots , k_r; s + 1) = \xi(k \odot \{(1)_{s+1}\}^*)$$

$$= \sum_{j=1}^{k_r-1} (-1)^{j-1} \xi(k_{r-1}, k_r + 1 - j) \xi((1)_{s-1}, 1 + j) + (-1)^{|k_{r-1}|} \xi((1)_{s} \odot (1, k)^*)$$

$$+ \sum_{i=1}^{r-1} (-1)^{|k_i| - 1} \sum_{j=1}^{k_{r-i} - 1} (-1)^{j-1} \xi(k_{r-i-1}, k_{r-i} + 1 - j) \xi((1)_{s} \odot (j, k_i^*)^*) .$$

We end this section with the following theorem and corollary.

**THEOREM 2.5.** For any positive integer $m$ and composition $k = (k_1, \ldots , k_r)$,

$$2 \sum_{j=0}^{m-1} \tilde{\xi}(2m - 1 - 2j) T_n((1)_{2j+1}) + \sum_{n=1}^{\infty} \frac{\xi_n - 1((k_{r-1}) S_n((1)_{2m})}{n^{k_r+1}}$$

$$= \sum_{j=1}^{k_r-1} (-1)^{j-1} \tilde{\xi}(k_{r-1}, k_r + 1 - j) T((1)_{2m-1}, j + 1) + (-1)^{|k_{r-1}|}$$

$$\times \sum_{n=1}^{\infty} \frac{T_n((1)_{2m-1}) \xi_n^* (1, k_{r-1})}{n^{k_r+1}} + \sum_{i=1}^{r-1} (-1)^{|k_i| - 1}$$

$$\times \sum_{j=1}^{k_{r-i} - 1} (-1)^{j-1} \xi(k_{r-i-1}, k_{r-i} + 1 - j) \sum_{n=1}^{\infty} \frac{T_n((1)_{2m-1}) \xi_n^*(j, k_{r-i}^*)}{n^{k_r+1}} ,$$

(22)

where $\tilde{\xi}(m) = -\xi(\tilde{m})$.

**Proof.** On the one hand, in [13, Theorem 3.6], we proved that

$$\int_0^1 \frac{1}{x} \cdot \text{Li}_k(x^2) \log^{2n} \left( \frac{1-x}{1+x} \right) dx = \frac{(2m)!}{2} \times \text{[left-hand side of (22)].}$$

On the other hand, we note that

$$\int_0^1 \frac{1}{x} \cdot \text{Li}_k(x^2) \log^{2m} \left( \frac{1-x}{1+x} \right) dx = (2m)! \sum_{n=1}^{\infty} \frac{T_n((1)_{2m-1})}{n} \int_0^1 x^{2n-1} \text{Li}_k(x) dx$$

$$= (2m)! \sum_{n=1}^{\infty} \frac{T_n((1)_{2m-1})}{2n} \int_0^1 x^{n-1} \text{Li}_k(x) dx .$$
Then using (16) with an elementary calculation, we have

$$\int_0^1 \frac{1}{x} \cdot \text{Li}_k(x^2) \log^2 x \left( \frac{1-x}{1+x} \right) \, dx = \frac{(2m)!}{2} \times \text{[right-hand side of (22)]}. $$

Thus, formula (22) holds. \qed

In particular, setting $k = (1, r-1, k)$ we obtain [13, Theorem 3.9]. Setting $k = (2, r-1, k)$ we get the following corollary.

**Corollary 2.6.** For any positive integers $k$, $m$ and $r$,

$$2 \sum_{j=0}^{m-1} \bar{\zeta}(2m - 1 - 2j) \sum_{n=1}^{\infty} \frac{\zeta_{n-1}(2r) T_n((1)_{2j+1})}{n^{k+1}} + \sum_{n=1}^{\infty} \frac{\zeta_{n-1}(2r) S_n((1)_{2m})}{n^{k+1}}$$

$$= \sum_{j=1}^{k-1} (-1)^{j-1} 2^j \zeta((2)_{r-1}, k + 1 - j) T_0((1)_{2m-1}, j + 1)$$

$$+ \sum_{l=1}^{r} (-1)^{l+k} \zeta((2)_{r-1}) \sum_{n=1}^{\infty} \frac{T_n((1)_{2m-1}) \zeta^*_n(1, (2)_{l-1})}{n^{k+1}}.$$  

(23)

### 2.2. Some relations of $T$-variant of Kaneko–Yamamoto MZVs

Recall that the Kaneko–Tsumura A-function $A(k_1, \ldots, k_r; z)$ (see [6]) is defined by

$$A(k_1, \ldots, k_r; z) := 2^r \sum_{1 \leq n_1 < \cdots < n_r \atop n_i \equiv i \mod 2} \frac{z^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}}, \quad z \in [-1, 1).$$ 

(24)

In this subsection, we present a series of results concerning this function.

**Theorem 2.7.** For positive integers $m$ and $n$,

$$\int_0^1 x^{2n-2} A(k_{2m}; x) \, dx$$

$$= \sum_{j=1}^{k_{2m-1}} (\frac{1}{2n-1})^{j-1} T(k_{2m-1}, k_{2m} + 1 - j) + \frac{(-1)^{k_{2m}}}{(2n-1)^{k_{2m}}} T_n(1, k_{2m-1})$$

$$+ \frac{1}{(2n-1)^{k_{2m}}} \sum_{i=1}^{m-1} (-1)^{|k_{2m}|} \sum_{j=1}^{k_{2m-2i-1}} (\frac{1}{2n-1})^{j-1} T(k_{2m-2i-1}, k_{2m-2i} + 1 - j) T_n(j, k_{2m-2i-1})$$

$$- \frac{1}{(2n-1)^{k_{2m}}} \sum_{i=0}^{m-1} (-1)^{|k_{2m}|} \sum_{j=1}^{k_{2m-2i-1}} (\frac{1}{2n-1})^{j-1} T(k_{2m-2i-2}, k_{2m-2i-1} + 1 - j) S_n(j, k_{2m-2i-1})$$

$$- \frac{1}{(2n-1)^{k_{2m}}} \sum_{i=0}^{m-1} (-1)^{|k_{2m}|} \left( \int_0^1 A(k_{2m-2i-1}, 1; x) \, dx \right) T_n(k_{2m-2i-1}),$$

(25)
\[ \int_0^1 x^{2n-1} A(k_{2m+1}; x) \, dx \]

\[ = \sum_{j=1}^{k_{2m+1} - 1} \frac{(-1)^{j-1}}{(2n)^j} T(k_{2m}, k_{2m+1} + 1 - j) - \frac{(-1)^{k_{2m+1}}}{(2n)^{k_{2m+1}}} T_n(1, k_{2m}) \]

\[ - \frac{1}{(2n)^{k_{2m+1}}} \sum_{i=0}^{m-1} (-1)^{i} k_{2m+1}^i \sum_{j=1}^{k_{2m+1} - 1} (-1)^{j-1} T(k_{2m-2i-1}, k_{2m-2i} + 1 - j) T_n(j, k_{2m+1}^{2i}) \]

\[ + \frac{1}{(2n)^{k_{2m+1}}} \sum_{i=0}^{m-1} (-1)^{i} k_{2m+1}^i \sum_{j=1}^{k_{2m+1} - 1} (-1)^{j-1} T(k_{2m-2i-2}, k_{2m-2i} + 1 - j) S_n(j, k_{2m+1}^{2i}) \]

\[ + \frac{1}{(2n)^{k_{2m+1}}} \sum_{i=0}^{m-1} (-1)^{i} k_{2m+1}^i \left( \int_0^1 A(k_{2m-2i-1}, 1; x) \, dx \right) T_n(k_{2m+1}^{2i+1}), \quad (26) \]

\[ \int_0^1 x^{2n-2} A(k_{2m+1}; x) \, dx \]

\[ = \sum_{j=1}^{k_{2m+1} - 1} \frac{(-1)^{j-1}}{(2n - 1)^j} T(k_{2m}, k_{2m+1} + 1 - j) - \frac{(-1)^{k_{2m+1}}}{(2n - 1)^{k_{2m+1}}} S_n(1, k_{2m}) \]

\[ + \frac{1}{(2n - 1)^{k_{2m+1}}} \sum_{i=1}^{m} (-1)^{i} k_{2m+1}^i \sum_{j=1}^{k_{2m+1} - 1} (-1)^{j-1} T(k_{2m-2i}, k_{2m+1-2i} + 1 - j) T_n(j, k_{2m+1}^{2i-1}) \]

\[ - \frac{1}{(2n - 1)^{k_{2m+1}}} \sum_{i=1}^{m} (-1)^{i} k_{2m+1}^i \sum_{j=1}^{k_{2m+1} - 1} (-1)^{j-1} T(k_{2m-2i-1}, k_{2m-2i} + 1 - j) S_n(j, k_{2m+1}^{2i}) \]

\[ - \frac{1}{(2n - 1)^{k_{2m+1}}} \sum_{i=0}^{m} (-1)^{i} k_{2m+1}^i \left( \int_0^1 A(k_{2m-2i}, 1; x) \, dx \right) T_n(k_{2m+1}^{2i}), \quad (27) \]

\[ \int_0^1 x^{2n-1} A(k_{2m}; x) \, dx \]

\[ = \sum_{j=1}^{k_{2m} - 1} \frac{(-1)^{j-1}}{(2n)^j} T(k_{2m-1}, k_{2m} + 1 - j) + \frac{(-1)^{k_{2m}}}{(2n)^{k_{2m}}} S_n(1, k_{2m-1}) \]

\[ - \frac{1}{(2n)^{k_{2m}}} \sum_{i=1}^{m} (-1)^{i} k_{2m}^i \sum_{j=1}^{k_{2m} - 1} (-1)^{j-1} T(k_{2m-2i}, k_{2m+1-2i} + 1 - j) T_n(j, k_{2m-1}^{2i-1}) \]

\[ + \frac{1}{(2n)^{k_{2m}}} \sum_{i=1}^{m} (-1)^{i} k_{2m}^i \sum_{j=1}^{k_{2m} - 1} (-1)^{j-1} T(k_{2m-2i-1}, k_{2m-2i} + 1 - j) S_n(j, k_{2m-1}^{2i}) \]

\[ + \frac{1}{(2n)^{k_{2m}}} \sum_{i=0}^{m} (-1)^{i} k_{2m}^i \left( \int_0^1 A(k_{2m-2i}, 1; x) \, dx \right) T_n(k_{2m-1}^{2i-1}), \quad (28) \]

where we allow \( m = 0 \) in (26) and (27).
Proof. It is easy to see that the A-function can be expressed by an iterated integral:

\[
A(k_1, \ldots, k_r; x) = \int_0^x \frac{2 \, dt}{1 - t^2} \left( \frac{dt}{t} \right)^{k_1-1} \cdots \frac{2 \, dt}{1 - t^2} \left( \frac{dt}{t} \right)^{k_r-1}.
\]

Using integration by parts, we deduce the recurrence relation

\[
\int_0^1 x^{2n-2} A(k_r; x) \, dx = \sum_{j=1}^{k_r-1} \frac{(-1)^{j-1}}{(2n-1)^j} T(k_r-1, k_r + 1 - j) + \frac{(-1)^{k_r-1}}{(2n-1)^{k_r}} \int_0^1 A(k_r-1, 1; x) \, dx
\]

and

\[
\int_0^1 x^{2n-1} A(k_r; x) \, dx = \sum_{j=0}^{k_r-2} \frac{(-1)^j}{(2n)^{j+1}} T(k_r-1, k_r - j) + \frac{(-1)^{k_r-1}}{(2n)^{k_r}} \sum_{k=1}^n \int_0^1 x^{2k-1} A(k_r-1; x) \, dx.
\]

Hence, using the recurrence formulas above, we may deduce the four desired evaluations after an elementary but rather tedious computation, which we leave to the interested reader.

**Lemma 2.8.** For any positive integer \( r \) we have

\[
\int_0^1 A(\{1\}; x) \, dx = -2^{1-r} \zeta(\bar{r}) = \begin{cases} 
\log 2, & \text{if } r = 1, \\
2^{1-r}(1 - 2^{1-r})\zeta(r), & \text{if } r \geq 2.
\end{cases}
\]

Proof. Consider the generating function

\[
G(u) := 1 + \sum_{r=1}^{\infty} \left( \int_0^1 A(\{1\}; x) \, dx \right)(-2u)^r.
\]

By definition

\[
G(u) = 1 + \sum_{r=1}^{\infty} (-2u)^r \int_0^1 \left( \int_0^x \frac{dt}{1 - t^2} \right)^r \, dx
\]

\[
= 1 + \sum_{r=1}^{\infty} \frac{(-2u)^r}{r!} \int_0^1 \left( \int_0^x \frac{dt}{1 - t^2} \right)^r \, dx
\]

\[
= 1 + \int_0^1 \left( \sum_{r=1}^{\infty} \frac{1}{r!} \left( -u \log \left( \frac{1+x}{1-x} \right) \right)^r \right) \, dx
\]

\[
= \int_0^1 \left( \frac{1 - x}{1 + x} \right)^u \, dx.
\]
Taking \( a = u, b = 1, c = u + 2 \) and \( t = -1 \) in the formula
\[
\binom{a, b}{c}^t = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 v^{b-1}(1 - v)^{c-b-1}(1 - vt)^{-a} dv,
\]
we obtain
\[
G(u) = \frac{1}{u + 1} \sum_{k \geq 0} \frac{u(u + 1)}{(u + k)(u + k + 1)}(-1)^k
= u \sum_{k \geq 0} (-1)^k \left( \frac{1}{u + k} - \frac{1}{u + k + 1} \right)
= 1 + \sum_{k \geq 1} 2(-1)^k \frac{u}{u + k}
= 1 - \sum_{r \geq 0} 2(-1)^k \sum_{r \geq 0} \left( \frac{-u}{k} \right)^{r+1}
= 1 - 2 \sum_{r \geq 1} \xi(-r)(-u)^r.
\]

The lemma follows immediately. \(\square\)

**Theorem 2.9.** For a composition \( k = (k_1, \ldots, k_r) \), the integral
\[
\int_0^1 A(k_1, \ldots, k_r, 1; x) \, dx
\]
can be expressed as a \( \mathbb{Q} \)-linear combination of alternating MZVs.

**Proof.** It suffices to prove that the integral can be expressed in terms of \( \log 2 \) and MMVs since these values generate the same \( \mathbb{Q} \)-vector space as that by alternating MZVs as shown in [13]. Suppose \( k_r > 1 \). Then
\[
\int_0^1 A(k_1, \ldots, k_r, 1; x) \, dx
= 2^{r+1} \sum_{\substack{0 < n_1 < \cdots < n_r < n_{r+1} \\ n_i \equiv i \pmod{2}}} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \left( \frac{1}{n_{r+1}} - \frac{1}{n_r + 1} \right)
= \begin{cases} M_s(\tilde{k}_1, k_2, \tilde{k}_3, \ldots, \tilde{k}_r, 1) - M_s(\tilde{k}_1, k_2, \ldots, \tilde{k}_r, 1), & \text{if } 2 \nmid r, \\ M_s(\tilde{k}_1, k_2, \tilde{k}_3, \ldots, k_r, \tilde{1}) - M_s(\tilde{k}_1, k_2, \ldots, k_r, 1), & \text{if } 2 \mid r, \\ M(\tilde{k}_1, k_2, \tilde{k}_3, \ldots, \tilde{k}_r)(M_s(1) - M_s(\tilde{1})) \pmod{\text{MMV}}, & \text{if } 2 \nmid r, \\ M(\tilde{k}_1, k_2, \tilde{k}_3, \ldots, k_r)(M_s(\tilde{1}) - M_s(1)) \pmod{\text{MMV}}, & \text{if } 2 \mid r, \\ -2M(\tilde{k}_1, k_2, \tilde{k}_3, \ldots, \tilde{k}_r) \log 2 \pmod{\text{MMV}}, & \text{if } 2 \nmid r, \\ 2M(\tilde{k}_1, k_2, \tilde{k}_3, \ldots, k_r) \log 2 \pmod{\text{MMV}}, & \text{if } 2 \mid r,
\end{cases}
\]
which can be expressed as a \( \mathbb{Q} \)-linear combination of MMVs by [13, Theorem 7.1].
In general, we may assume $k_r > 1$ and consider $\int_0^1 A(k_1, \ldots, k_r, \{1\}_{\ell}; x) \, dx$. By induction on $\ell$, we see that

$$
\int_0^1 A(k_1, \ldots, k_r, \{1\}_{\ell}; x) \, dx
$$

$$
= \left\{
\begin{array}{ll}
M(\tilde{k}_1, k_2, \tilde{k}_3, \ldots, \tilde{k}_r)(M_{u}(u, 1) - M_{u}(u, \tilde{1})) & \text{(mod MMV)}, \quad \text{if } 2 \nmid r, 2 \nmid \ell, \\
M(k_1, k_2, \tilde{k}_3, \ldots, \tilde{k}_r)(M_{u}(u', 1, \tilde{1}) - M_{u}(u', 1, 1)) & \text{(mod MMV)}, \quad \text{if } 2 \nmid r, 2 \nmid \ell, \\
M(k_1, k_2, \tilde{k}_3, \ldots, \tilde{k}_r)(M_{v}(v, \tilde{1}) - M_{v}(v, 1)) & \text{(mod MMV)}, \quad \text{if } 2 | r, 2 | \ell, \\
M(\tilde{k}_1, k_2, \tilde{k}_3, \ldots, k_r)(M_{v'}(v', \tilde{1}, 1) - M_{v'}(v', \tilde{1}, \tilde{1})) & \text{(mod MMV)}, \quad \text{if } 2 | r, 2 | \ell,
\end{array}
\right.
$$

where $u = \{1, \tilde{1}\}_{(\ell-1)/2}$, $u' = \{1, \tilde{1}\}_{(\ell-2)/2}$, $v = \{\tilde{1}, 1\}_{(\ell-1)/2}$, and $v' = \{\tilde{1}, 1\}_{(\ell-2)/2}$. By Lemma 2.8 we see that $M_{u}(\ldots, 1) - M_{u}(\ldots, \tilde{1}) = \mp 2\zeta(\ell)$. This finishes the proof of the theorem. \hfill \Box

Example 2.10. Applying the idea in the proof of Theorem 2.9, we can find that for any positive integer $k$,

$$
\int_0^1 A(k, 1; x) \, dx = M_{u}(\tilde{k}, 1) - M_{u}(\tilde{k}, \tilde{1})
$$

$$
= M(\tilde{k})(M_{u}(1) - M_{u}(\tilde{1})) + M(\tilde{1}, \tilde{k}) - M(1, \tilde{k}) + 2M((k+1)^\circ).
$$

Observing that $M_{u}(\tilde{1}) - M_{u}(1) = 2 \log(2)$, $M(\tilde{k}) = T(k)$, $M(\tilde{1}, \tilde{k}) = 4t(1, k)$ and $M(1, \tilde{k}) = S(1, k)$, we obtain

$$
\int_0^1 A(k, 1; x) \, dx = -2(2 \log(2)T(k) + 2T(k+1) + 4t(1, k) - S(1, k).
$$

From Lemma 2.8 we can get the following corollary, which was proved in [13].

**Corollary 2.11.** [13, Theorem 3.1] For positive integers $m$ and $n$, the following identities hold:

$$
\int_0^1 t^{2n-2} \log^2m \left(1 - \frac{t}{1 + t}\right) \, dt = \frac{2(2m)!}{2n - 1} \sum_{j=0}^{m} \zeta(2j)T_n([1]_{2m-2j}),
$$

(29)

$$
\int_0^1 t^{2n-2} \log^2m_{-1} \left(1 - \frac{t}{1 + t}\right) \, dt = -\frac{(2m - 1)!}{2n - 1}
$$

$$
\times \left(2 \sum_{j=1}^{m} \zeta(2j - 1)T_n([1]_{2m-2j}) + S_n([1]_{2m-1})\right),
$$

(30)

$$
\int_0^1 t^{2n-1} \log^2m \left(1 - \frac{t}{1 + t}\right) \, dt = \frac{(2m)!}{n} \left(\sum_{j=1}^{m} \zeta(2j - 1)T_n([1]_{2m-2j+1}) + S_n([1]_{2m})\right),
$$

(31)

$$
\int_0^1 t^{2n-1} \log^2m_{-1} \left(1 - \frac{t}{1 + t}\right) \, dt = -\frac{(2m - 1)!}{n} \sum_{j=0}^{m-1} \zeta(2j - 2)T_n([1]_{2m-2j-1}),
$$

(32)

where $\zeta(0)$ should be interpreted as $1/2$ wherever it occurs.
We now derive some explicit relations about the $T$-variant of K–Y MZV $T(k \oplus l)$ by considering the integral

$$I_A(k; l) := \int_0^1 \frac{A(k; x)A(l; x)}{x} \, dx.$$ 

**Theorem 2.12.** For positive integers $k$ and $l$, we have

$$((-1)^l - (-1)^k)S(1, k + l) = \sum_{j=1}^l (-1)^{j-1}T(l + 1 - j)T(k + j)$$

$$+ \sum_{j=1}^k (-1)^jT(k + 1 - j)T(l + j),$$

where $T(1) := 2 \log(2)$.

**Proof.** One may deduce the formula by a straightforward calculation of the integral

$$\int_0^1 \frac{A(k; x)A(l; x)}{x} \, dx.$$ 

We leave the details to the interested reader. $\Box$

For example, setting $k = 1$ and $l = 2p$ ($p \in \mathbb{N}$) in Theorem 2.12 yields

$$S(1, 2p + 1) = \sum_{j=0}^{p-1} (-1)^{j-1}T(2p + 1 - j)T(j + 1) - \frac{(-1)^p}{2}T^2(p + 1).$$

**Theorem 2.13.** For positive integers $k_1$, $k_2$ and $l$,

$$(-1)^l - (-1)^{k_1 + k_2 - 1}T(1, k_1, k_2 + l)$$

$$= \sum_{j=1}^{k_2-1} (-1)^{j-1}T(k_1, k_2 + 1 - j)T(l + j) - \sum_{j=1}^{l-1} (-1)^{j-1}T(l + 1 - j)T(k_1, k_2 + j)$$

$$- (-1)^{k_2} \sum_{j=1}^{k_1-1} (-1)^{j-1}T(k_1 + 1 - j)S(j, k_2 + l) - (-1)^{k_2}T(k_2 + l)$$

$$\times \int_0^1 A(k_1, 1; x) \, dx,$$

where $\int_0^1 A(k, 1; x) \, dx$ is given by Example 2.10.
Proof. From (25) and (26), we deduce

$$\int_0^1 x^{2n-1} A(k; x) \, dx = \sum_{j=1}^{k-1} \frac{(-1)^{j-1}}{(2n)^j} T(k + 1 - j) + \frac{(-1)^{k-1}}{(2n)^k} T_n(1)$$

and

$$\int_0^1 x^{2n-2} A(k_1, k_2; x) \, dx$$

$$= \sum_{j=1}^{k_2-1} \frac{(-1)^{j-1}}{(2n-1)^j} T(k_1, k_2 + 1 - j) + \frac{(-1)^{k_1+k_2}}{(2n-1)k_2} T_n(1, k_1)$$

$$+ \frac{(-1)^{k_2-1}}{(2n-1)^k_2} \sum_{j=1}^{k_2-1} \frac{(-1)^{j-1}}{(2n-1)^j} T(k_1 + 1 - j) S_n(j) + \frac{(-1)^{k_2-1}}{(2n-1)^k_2} \int_0^1 A(k_1, 1; x) \, dx.$$  

According to the definitions of A-functions, MTVs and MSVs, on the one hand, we have

$$\int_0^1 \frac{A(k_1, k_2; x)A(l; x)}{x} \, dx$$

$$= 2 \sum_{n=1}^{\infty} \frac{1}{(2n-1)^l} \int_0^1 x^{2n-2} A(k_1, k_2; x) \, dx$$

$$= \sum_{j=1}^{k_2-1} (-1)^{j-1} T(k_1, k_2 + 1 - j) T(l + j) - (-1)^{k_2}$$

$$\times \sum_{j=1}^{k_2-1} (-1)^{j-1} T(k_1 + 1 - j) S(j, k_2 + l)$$

$$- (-1)^{k_2} T(k_2 + l) \int_0^1 A(k_1, 1; x) \, dx + (-1)^{k_1+k_2} T(1, k_1, k_2 + l).$$

On the other hand,

$$\int_0^1 \frac{A(k_1, k_2; x)A(l; x)}{x} \, dx$$

$$= 2 \sum_{n=1}^{\infty} \frac{T_n(k_1)}{(2n)^k_2} \int_0^1 x^{2n-1} A(l; x) \, dx$$

$$= \sum_{j=1}^{l-1} (-1)^{j-1} T(l + 1 - j) T(k_1, k_2 + j) + (-1)^{l-1} T((k_1, k_2) \circ (1, l)).$$

Hence, combining the two identities above, we obtain the desired evaluation. \qed
THEOREM 2.14. For positive integers $k_1$, $k_2$ and $l_1$, $l_2$, we have
\[
(-1)^{k_1+k_2} T((l_1, l_2) \oplus (1, k_1, k_2)) - (-1)^{l_1+l_2} T((k_1, k_2) \oplus (1, l_1, l_2))
\]
\[
= \sum_{j=1}^{k_2-1} (-1)^j T(k_1, k_2 + 1 - j) T(l_1, l_2 + j) - \sum_{j=1}^{l_2-1} (-1)^j T(l_1, l_2 + 1 - j) T(k_1, k_2 + j)
\]
\[
- (-1)^{k_2} \sum_{j=1}^{k_1} (-1)^j T(k_1 + 1 - j) T((l_1, l_2) \oplus (j, k_2))
\]
\[
+ (-1)^{l_2} \sum_{j=1}^{l_1} (-1)^j T(l_1 + 1 - j) T((k_1, k_2) \oplus (j, l_2)),
\]
where $T(1) := 2 \log(2)$.

Proof. Consider the integral
\[
\int_0^1 \frac{A(k_1, k_2; x)A(l_1, l_2; x)}{x} \, dx.
\]
By a similar argument as used in the proof of Theorem 2.13, we can prove Theorem 2.14. □

Moreover, according to the definitions of the Kaneko–Tsumura $\psi$-function and the Kaneko–Tsumura $\Lambda$-function (which is a single-variable multiple polylogarithm function of level two) [6, 7],
\[
\psi(k_1, \ldots, k_r; s) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{r^{s-1}}{\sinh(r)} A(k_1, \ldots, k_r; \tanh(t/2)) \, dt \quad (\Re(s) > 0) \quad (34)
\]
and
\[
A(k_1, \ldots, k_r; z) := 2^r \sum_{1 \leq n_1 < \cdots < n_r \atop n_i \equiv i \mod 2} \frac{z^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}}, \quad z \in [-1, 1]. \quad (35)
\]
Setting $\tanh(t/2) = x$ and $s = p + 1 \in \mathbb{N}$, we have
\[
\psi(k_1, \ldots, k_r; p + 1) = \frac{(-1)^p}{p!} \int_0^1 \frac{\log^p((1 - x)/(1 + x)) A(k_1, \ldots, k_r; x)}{x} \, dx
\]
\[
= \int_0^1 \frac{A([1]_p; x)A(k_1, \ldots, k_r; x)}{x} \, dx, \quad (36)
\]
where we have used the relation
\[
A([1]_r; x) = \frac{1}{r!} (A(1; x))^r = \frac{(-1)^r}{r!} \log^r \left( \frac{1 - x}{1 + x} \right).
\]

We remark that the Kaneko–Tsumura $\psi$-values can be regarded as a special case of the integral $I_A(k; I)$. So, one can prove [13, Theorem 3.3] by considering the integrals $I_A(k; I)$. 
2.3. Multiple integrals associated with 2-labeled posets

According to iterated integral expressions, we know that \( L_k(x) \) and \( A(k; x) \) satisfy the shuffle product relation. In this subsection, we will express integrals \( I_L(k; l) \) and \( I_A(k; l) \) in terms of multiple integrals associated with 2-labeled posets, which implies that the integrals \( I_L(k; l) \) and \( I_A(k; l) \) can be expressed in terms of linear combination of MZVs (or MTVs). The key properties of these integrals were first studied by Yamamoto in [14].

**Definition 2.1.** Let \( X = (X, \leq) \) be any partially ordered finite set. A label map \( \delta_X \) is a map from \( X \to \{0, 1\} \). We say the pair \((X, \delta_X)\) is a 2-poset. We often call \( X \) itself a 2-poset if no confusion can arise.

A 2-poset \((X, \delta_X)\) is called admissible if \( \delta_X(x) = 0 \) for all maximal elements \( x \in X \) and \( \delta_X(x) = 1 \) for all minimal elements \( x \in X \).

**Definition 2.2.** For an admissible 2-poset \( X \), we define the associated integral

\[
I_j(X) = \int_{\Delta_X} \prod_{x \in X} \omega_{\delta_X(x)}^{(j)}(t_x), \quad j = 1, 2,
\]

where \( \Delta_X = \{(t_x)_x \in [0, 1]^X \mid t_x < t_y \text{ if } x < y \} \)

and

\[
\omega_0^{(1)}(t) = \omega_0^{(2)}(t) = \frac{dt}{t}, \quad \omega_1^{(1)}(t) = \frac{dt}{1-t}, \quad \omega_1^{(2)}(t) = \frac{2dt}{1-t^2}.
\]

For the empty 2-poset, denoted by \( \emptyset \), we put \( I_j(\emptyset) := 1 \) (\( j = 1, 2 \)).

**Proposition 2.15.** For non-comparable elements \( a \) and \( b \) of a 2-poset \( X \), \( X_{ab}^b \) denotes the 2-poset that is obtained from \( X \) by adjoining the relation \( a < b \). If \( X \) is an admissible 2-poset, then the 2-posets \( X_{ab}^b \) and \( X_{ba}^a \) are admissible and

\[
I_j(X) = I_j(X_{ab}^b) + I_j(X_{ba}^a) \quad (j = 1, 2).
\]

Note that the admissibility of a 2-poset corresponds to the convergence of the associated integral. We use Hasse diagrams to indicate 2-posets, with vertices \( \circ \) and \( \bullet \) corresponding to \( \delta(x) = 0 \) and \( \delta(x) = 1 \), respectively. For example, the diagram

\[\text{•} \quad \text{◦} \quad \text{•} \quad \text{◦} \quad \text{◦}\]

represents the 2-poset \( X = \{x_1, x_2, x_3, x_4, x_5\} \) with order \( x_1 < x_2 > x_3 < x_4 < x_5 \) and label \( (\delta_X(x_1), \ldots, \delta_X(x_5)) = (1, 0, 1, 0, 0) \). This 2-poset is admissible. To describe the corresponding diagram, we introduce an abbreviation: For a sequence \( k_r = (k_1, \ldots, k_r) \) of positive integers, we write

\[k_r\]
for the following vertical diagram.

Hence, for admissible composition \( k \), using this notation of multiple associated integral, one can verify that

\[
\zeta(k) = I_1 \left( \begin{array}{c} \circ \cr k \end{array} \right) \quad \text{and} \quad T(k) = I_2 \left( \begin{array}{c} \circ \cr k \end{array} \right).
\]

Therefore, according to the definitions of \( I_L(k; l) \) and \( I_A(k; l) \), and using this notation of multiple associated integral, we can get the following theorem.

**Theorem 2.16.** For compositions \( k \) and \( l \), we have

\[
I_L(k; l) = I_1 \left( \begin{array}{c} k \\
\circ \cr \circ \cr \cdots \\
\circ \\
l \end{array} \right) \quad \text{and} \quad I_A(k; l) = I_2 \left( \begin{array}{c} k \\
\circ \cr \circ \cr \cdots \\
\circ \\
l \end{array} \right).
\]

**Proof.** This follows immediately from the definitions of \( I_L(k; l) \) and \( I_A(k; l) \). We leave the detail to the interested reader. \(\square\)

It is clear that using Theorem 2.16 we can express the integrals \( I_L(k; l) \) (or \( I_A(k; l) \)) in terms of MZVs or MTVs. In particular, for any positive integer \( s \) the integrals \( I_L(k; \{1\}_s) \) and \( I_A(k; \{1\}_s) \) become the Arakawa–Kaneko zeta values and Kaneko–Tsumura \( \psi \)-values, respectively. Moreover, Kawasaki and Ohno [9] and Xu and Zhao [13] have used the multiple integrals associated with 2-posets to prove explicit formulas for all Arakawa–Kaneko zeta values and Kaneko–Tsumura \( \psi \)-values.

Now, we end this section with the following duality relations. For any \( n \in \mathbb{N} \) and composition \( k = (k_1, \ldots, k_r) \), set

\[
k_{+n} := (k_1, \ldots, k_{r-1}, k_r + n).
\]

**Theorem 2.17.** For any \( p \in \mathbb{N} \) and compositions of positive integers \( k, l \), we have

\[
I_L(k_{+(p-1)}; l) + (-1)^p I_L(k; l_{+(p-1)}) = \sum_{j=1}^{p-1} (-1)^{j-1} \zeta(k_{+(p-j)}) \zeta(l_{+j})
\]

and

\[
I_A(k_{+(p-1)}; l) + (-1)^p I_A(k; l_{+(p-1)}) = \sum_{j=1}^{p-1} (-1)^{j-1} T(k_{+(p-j)}) T(l_{+j}).
\]

**Proof.** This follows easily from the definitions of \( I_L(k; l) \) and \( I_A(k; l) \) by using integration by parts. We leave the detail to the interested reader. \(\square\)
variants. The first one is the star version:

\[ \xi((1)_{r-1}, p; s + 1) + (-1)^p \xi((1)_{s-1}, p; r + 1) = \sum_{j=0}^{p-2} (-1)^j \xi((1)_{r-1}, p - j) \xi((1)_j, s + 1) \]

and

\[ \psi((1)_{r-1}, p; s + 1) + (-1)^p \psi((1)_{s-1}, p; r + 1) = \sum_{j=0}^{p-2} (-1)^j T((1)_{r-1}, p - j) T((1)_j, s + 1). \]

3. Alternating variant of Kaneko–Yamamoto MZVs

3.1. Integrals of multiple polylogarithm function with r-variable

For any composition \( k_r = (k_1, \ldots, k_r) \in \mathbb{N}^r \), we define the classical multiple polylogarithm function with r-variable by

\[ \text{Li}_{k_r}(x_1, \ldots, x_r) := \sum_{0 < n_1 < n_2 < \cdots < n_r} \frac{x_1^{n_1} \cdots x_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}}, \]

which converges if \( |x_j \cdots x_r| < 1 \) for all \( j = 1, \ldots, r \). It can be analytically continued to a multi-valued meromorphic function on \( \mathbb{C}^r \) (see [17]). We also consider the following two variants. The first one is the star version:

\[ \text{Li}^\star_{k_r}(x_1, \ldots, x_r) := \sum_{0 < n_1 \leq n_2 \leq \cdots \leq n_r} \frac{x_1^{n_1} \cdots x_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}}. \]

The second is the most useful when we need to apply the technique of iterated integrals:

\[ \lambda_{k_r}(x_1, \ldots, x_{r-1}, x_r) := \text{Li}_{k_r}(x_1/x_2, \ldots, x_{r-1}/x_r, x_r) = \sum_{0 < n_1 < n_2 < \cdots < n_r} \frac{(x_1/x_2)^{n_1} \cdots (x_{r-1}/x_r)^{n_{r-1}} x_r^{n_r}}{n_1^{k_1} \cdots n_{r-1}^{k_{r-1}} n_r^{k_r}}, \]

which converges if \( |x_j| < 1 \) for all \( j = 1, \ldots, r \). Namely,

\[ \lambda_{k_r}(x_1, \ldots, x_r) = \int_0^1 \left( \frac{x_1}{1 - x_1 t} \right) \left( \frac{dt}{t} \right)^{k_1-1} \cdots \left( \frac{x_r}{1 - x_r t} \right) \left( \frac{dt}{t} \right)^{k_r-1}. \]

Similarly, the parametric multiple harmonic sums and parametric multiple harmonic star sums with r-variable are defined by

\[ \zeta_n(k_1, \ldots, k_r; x_1, \ldots, x_r) := \sum_{0 < m_1 < \cdots < m_r \leq n} \frac{x_1^{m_1} \cdots x_r^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}} \]
and
\[ \zeta_n^*(k_1, \ldots, k_r; x_1, \ldots, x_r) := \sum_{0 < m_1 \leq \cdots \leq m_r \leq n} \frac{x_1^{m_1} \cdots x_r^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}}, \]
respectively. Obviously,
\[ \lim_{n \to \infty} \zeta_n(k_1, \ldots, k_r; x_1, \ldots, x_r) = \text{Li}_{k_1, \ldots, k_r}(x_1, \ldots, x_r) \]
and
\[ \lim_{n \to \infty} \zeta_n^*(k_1, \ldots, k_r; x_1, \ldots, x_r) = \text{Li}_{k_1, \ldots, k_r}^*(x_1, \ldots, x_r). \]

**Definition 3.1.** For any two compositions of positive integers \( k = (k_1, \ldots, k_r) \), \( l = (l_1, \ldots, l_s) \), \( \sigma := (\sigma_1, \ldots, \sigma_r) \in \{\pm 1\}^r \) and \( \varepsilon := (\varepsilon_1, \ldots, \varepsilon_s) \in \{\pm 1\}^s \), define
\[ \zeta((k; \sigma) \oplus (l; \varepsilon)^*) \]
\[ \equiv \zeta((k_1, \ldots, k_r; \sigma_1, \ldots, \sigma_r) \oplus (l_1, \ldots, l_s; \varepsilon_1, \ldots, \varepsilon_s)^*) \]
\[ := \sum_{n=1}^{\infty} \frac{\zeta_{n-1}(k_1, \ldots, k_{r-1}; \sigma_1, \ldots, \sigma_{r-1}) \zeta_n^*(l_1, \ldots, l_{s-1}; \varepsilon_1, \ldots, \varepsilon_{s-1})}{n^{k_1+l_1}} (\sigma_r \varepsilon_s)^n. \] (41)

We call them **alternating Kaneko–Yamamoto MZVs**.

**Theorem 3.1.** For \( n \in \mathbb{N} \), \( k_r = (k_1, \ldots, k_r) \in \mathbb{N}^r \) and \( \sigma_r := (\sigma_1, \ldots, \sigma_r) \in \{\pm 1\}^r \), we have
\[ \int_0^1 x^{n-1} \kappa_{k_1, \ldots, k_r}(\sigma_1 x, \ldots, \sigma_r x) \, dx \]
\[ = \sum_{j=1}^{k_r} \frac{(-1)^{j-1}}{n^{j-1}} \kappa_{k_r-j, k_r+1-j}(\sigma_r) + \frac{(-1)^{k_r}}{n^{k_r}} (\sigma_r^n - 1) \kappa_{k_r, k_r+1}(\sigma_r) \]
\[ - \frac{\sigma_r}{n^{k_r}} \sum_{j=1}^{k_r-1} (-1)^j \zeta_n^*(k_r-j; \sigma_r-j+1, (\sigma_r \sigma_{r-1})^{j-1}) \kappa_{k_{r-j}, k_{r+1-j}}(\sigma_{r-j}) \]
\[ - \frac{\sigma_r}{n^{k_r}} \sum_{j=1}^{k_r-1} (-1)^j \zeta_n^*(k_r-j; \sigma_r-j+1, (\sigma_r \sigma_{r-1})^{j-1}) \kappa_{k_{r-j}, k_{r+1-j}}^*(\sigma_{r-j}) \]
\[ - \zeta_n^*(k_r; \sigma_r - 1, (\sigma_r \sigma_{r-1})^{j-1}) + (-1)^{k_r-j} \frac{\sigma_r}{n^{k_r}} \zeta_n^*(1, k_r; \sigma_1, (\sigma_r \sigma_{r-1})^{j-1}), \]
where \( (\sigma_r \sigma_{r-1})^j := (\sigma_r \sigma_{r+2} \sigma_{r-1}, \ldots, \sigma_r \sigma_{r-1}) \) and \( (\sigma_r \sigma_{r-1})^0 := \emptyset \). If \( \sigma_r = 1 \)
then \( (\sigma_r^n - 1) \kappa_{k_r, k_r+1}(\sigma_r) := 0 \), and if \( \sigma_{r-1} = 1 \) then
\[ \kappa_{k_{r-1}, k_{r}}(\sigma_{r-1}, (\sigma_r \sigma_{r-1})^{j-1}) - \zeta_n^*(k_r; \sigma_r - 1, (\sigma_r \sigma_{r-1})^{j-1}) \]
\[ \equiv 0. \]

**Proof.** According to definition,
\[ \frac{d}{dx} \kappa_{k_1, \ldots, k_r}(\sigma_1 x, \ldots, \sigma_r x) \]
\[ = \begin{cases} \frac{1}{x} \kappa_{k_1, \ldots, k_{r-1}, k_r-1}(\sigma_1 x, \ldots, \sigma_{r-1} x, \sigma_r x), & \text{if } k_r \geq 2, \\ \frac{\sigma_r}{1 - \sigma_r x} \kappa_{k_1, \ldots, k_{r-1}, k_r}(\sigma_1 x, \ldots, \sigma_{r-1} x), & \text{if } k_r = 1. \end{cases} \]
Hence, using the identity above, we can get the following recurrence relation:

\[
\int_0^1 x^{n-1} \lambda_{k_1, \ldots, k_r}(\sigma_1 x, \sigma_2 x, \ldots, \sigma_r x) \, dx \\
= \sum_{j=1}^{k_r-1} \frac{(-1)^{j-1} n^j}{j} \lambda_{k_1, \ldots, k_{r-1}, k_r+1-j}(\sigma_1, \sigma_2, \ldots, \sigma_r) \\
+ \frac{(-1)^{k_r}}{n^{k_r}} (\sigma_r^n - 1) \lambda_{k_1, \ldots, k_{r-1}, 1}(\sigma_1, \ldots, \sigma_r) \\
- \frac{(-1)^{k_r}}{n^{k_r}} \sum_{k=1}^{n-1} \sigma_r^k \int_0^1 x^{k-1} \lambda_{k_1, k_2, \ldots, k_{r-1}}(\sigma_1 x, \sigma_2 x, \ldots, \sigma_{r-1} x) \, dx.
\]

Thus, we obtain the desired formula by using the recurrence relation above.

Letting \( r = 1 \) and \( 2 \), we can get the following two corollaries.

**Corollary 3.2.** For positive integers \( n, k \) and \( \sigma \in \{\pm 1\} \),

\[
\int_0^1 x^{n-1} \lambda_k(\sigma x) \, dx = \frac{(-1)^k}{n^k} (\sigma^n - 1) \lambda_1(\sigma) - \frac{(-1)^k}{n^k} \sigma^n(1; \sigma) - \frac{k-1}{n^k} \lambda_{k+1}(\sigma).
\]

**Corollary 3.3.** For positive integers \( n, k_1, k_2 \) and \( \sigma_1, \sigma_2 \in \{\pm 1\} \),

\[
\int_0^1 x^{n-1} \lambda_{k_1, k_2}(\sigma_1 x, \sigma_2 x) \, dx \\
= \sum_{j=1}^{k_2-1} \frac{(-1)^{j-1} n^j}{j} \lambda_{k_1, k_2+1-j}(\sigma_1, \sigma_2) + \frac{(-1)^{k_1}}{n^{k_1}} \sum_{j=1}^{k_2-1} (-1)^j \lambda_{k_1+1-j}(\sigma_1) \lambda_{k_2} \left( \frac{\sigma_1^n}{n^{k_1}} \right) - \frac{(-1)^{k_1}}{n^{k_1}} \lambda_{k_1+1}(\sigma_1, \sigma_2) \\
+ \frac{(-1)^{k_1+k_2}}{n^{k_1+k_2}} \lambda_{k_1, 1}(\sigma_1, \sigma_2) + \frac{(-1)^{k_1+k_2}}{n^{k_1+k_2}} \lambda_{k_1}(\sigma_1) \lambda_{k_2} \left( \frac{\sigma_2^n}{n^{k_2}} \right) - \frac{(-1)^{k_1+k_2}}{n^{k_1+k_2}} \lambda_{k_1, 1}(\sigma_1, \sigma_2) \\
+ \frac{(-1)^{k_1+k_2}}{n^{k_1+k_2}} \lambda_{k_1} \left( \frac{\sigma_1^n}{n^{k_1}} \right) \lambda_{k_2} \left( \frac{\sigma_2^n}{n^{k_2}} \right) - \frac{(-1)^{k_1+k_2}}{n^{k_1+k_2}} \lambda_{k_1} \left( \frac{\sigma_1^n}{n^{k_1}} \right) \lambda_{k_2} \left( \frac{\sigma_2^n}{n^{k_2}} \right).
\]

Clearly, setting \( \sigma_1 = \sigma_2 = \cdots = \sigma_r = 1 \) gives the formula (16).

### 3.2. Explicit formulas for alternating Kaneko–Yamamoto MZVs

Obviously, we can consider the integral

\[
I_2((k; \sigma), (l; \epsilon)) := \int_0^1 \frac{\lambda_{k_1}(\sigma_1 x, \ldots, \sigma_r x) \lambda_{l_1}(\epsilon_1 x, \ldots, \epsilon_r x)}{x} \, dx
\]

to find some explicit relations of \( \Omega((k; \sigma) \oplus (l; \epsilon)^*) \). We have the following theorems.

**Theorem 3.4.** For positive integers \( k, l \) and \( \sigma, \epsilon \in \{\pm 1\} \),

\[
(-1)^k \operatorname{Li}_{k+l}^*(\sigma, \epsilon) - (-1)^l \operatorname{Li}_{l+k}^*(\epsilon, \sigma) \\
= \sum_{j=1}^{k+l-1} (-1)^j \lambda_{k+l-j}(\sigma) \lambda_{l+j}(\epsilon) - \sum_{j=1}^{l+k-1} (-1)^l \lambda_{l+j}(\epsilon) \lambda_{k+l-j}(\sigma) \\
+ \frac{(-1)^l}{\lambda_{l+j}(\epsilon)} (\lambda_{k+l} - \lambda_{k+l}(\epsilon)) - \frac{(-1)^k}{\lambda_{k+l}(\epsilon)} (\lambda_{k+l}(\sigma) - \lambda_{k+l}(\epsilon)),
\]

(42)
where if \( \sigma = 1 \) then \( \lambda_1(\sigma)(\lambda_{k+l}(\epsilon) - \lambda_{k+l}(\sigma \epsilon)) := 0 \). Similarly, if \( \epsilon = 1 \) then \( \lambda_1(\epsilon)(\lambda_{k+l}(\sigma) - \lambda_{k+l}(\sigma \epsilon)) := 0 \).

**Proof.** Considering the integral \( \int_0^1 (\lambda_k(\sigma x) \lambda_l(\epsilon x)) / x \) dx and using Corollary 3.2 with an elementary calculation, we obtain the formula. \( \square \)

**Theorem 3.5.** For positive integers \( k_1, k_2, l \) and \( \sigma_1, \sigma_2, \epsilon \in \{ \pm 1 \}, \)

\[
\sum_{j=1}^{l-1} (-1)^{j-1} \lambda_{l+1-j}(\epsilon) \text{Li}_{k_1,k_2+j}(\sigma_1\sigma_2, \sigma_2) - (-1)^l \xi((k_1, k_2; \sigma_1\sigma_2, \sigma_2) \oplus (1, l; \epsilon, \epsilon)^*)
\]

\[
- (-1)^l \lambda_1(\epsilon)(\text{Li}_{k_1,k_2+l}(\sigma_1\sigma_2, \sigma_2) - \text{Li}_{k_1,k_2+l}(\sigma_1\sigma_2, \sigma_2\epsilon))
\]

\[
= \sum_{j=1}^{k_1-1} (-1)^{j-1} \lambda_{k_1,k_2+1-j}(\sigma_1, \sigma_2) \lambda_{l+j}(\epsilon) - (-1)^{k_2}
\]

\[
\times \sum_{j=1}^{k_1-1} (-1)^{j-1} \lambda_{k_1+1-j}(\sigma_1) \text{Li}_{j, k_2+j}(\sigma_2, \epsilon \sigma_2)
\]

\[
- (-1)^{k_2} \lambda_{k_1,1}(\sigma_1, \sigma_2)(\lambda_{k_2}(\epsilon) - \lambda_{k_2}(\epsilon \sigma_2)) + (-1)^{k_1+k_2} \text{Li}_{k_1, k_2+l}(\sigma_1, \sigma_2 \sigma_1, \sigma_2 \epsilon),
\]

\[
+ (-1)^{k_1+k_2} \lambda_1(\sigma_1)(\text{Li}_{k_1,k_2+l}(\sigma_2, \epsilon \sigma_2) - \text{Li}_{k_1,k_2+l}(\sigma_2 \sigma_1, \epsilon \sigma_2)), \tag{43}
\]

where

\[
\lambda_1(\sigma_1)(\text{Li}_{k_1,k_2+l}(\sigma_2, \epsilon \sigma_2) - \text{Li}_{k_1,k_2+l}(\sigma_2 \sigma_1, \epsilon \sigma_2)) = 0 \quad \text{if } \sigma_1 = 1,
\]

\[
\lambda_{k_1,1}(\sigma_1, \sigma_2)(\lambda_{k_2}(\epsilon) - \lambda_{k_2}(\epsilon \sigma_2)) = 0 \quad \text{if } \sigma_2 = 1,
\]

and

\[
\lambda_1(\epsilon)(\text{Li}_{k_1,k_2+l}(\sigma_1\sigma_2, \sigma_2) - \text{Li}_{k_1,k_2+l}(\sigma_1\sigma_2, \sigma_2\epsilon)) = 0 \quad \text{if } \epsilon = 1.
\]

**Proof.** Similarly, considering the integral \( \int_0^1 (\lambda_{k_1,k_2}(\sigma_1 x, \sigma_1 x) \lambda_l(\epsilon x)) / x \) dx and using Corollary 3.3 with an elementary calculation, we prove the formula. \( \square \)

On the other hand, according to definition, we have

\[
\xi((k_1, k_2; \sigma_1\sigma_2, \sigma_2) \oplus (1, l; \epsilon, \epsilon)^*)
\]

\[
= \sum_{n=1}^{\infty} \frac{\zeta_{n-1}(k_1; \sigma_1\sigma_2) \zeta_n^*(1; \epsilon)}{n^{k_2+l}} (\sigma_2 \epsilon)^n
\]

\[
= \text{Li}_{k_1,1,k_2+l}^*(\sigma_1\sigma_2, \epsilon, \sigma_2 \epsilon) + \text{Li}_{k_1,k_2+l}^*(\epsilon, \sigma_1\sigma_2, \sigma_2 \epsilon) - \text{Li}_{k_1+1,k_2+l}^*(\sigma_1\sigma_2\epsilon, \sigma_2 \epsilon)
\]

\[
- \text{Li}_{k_1+1,k_2+l}^*(\epsilon, \sigma_1 \epsilon).
\]

Substituting it into (43) yields the following corollary.
COROLLARY 3.6. For positive integers $k_1$, $k_2$, $l$ and $\sigma_1$, $\sigma_2$, $\varepsilon \in \{\pm 1\}$,

\[
(-1)^l \text{Li}_{k_1,1,k_2+l}^*(\sigma_1\sigma_2,\varepsilon,\sigma_2\varepsilon) + (-1)^l \text{Li}_{k_1,k_2+l}^*(\sigma_1,\sigma_2) \varepsilon \\
+ (-1)^{k_1+k_2} \text{Li}_{k_1,k_2+l}^*(\sigma_1,\sigma_2,\varepsilon\sigma_2) \\
= \sum_{j=1}^{k_2-1} (-1)^j \lambda_{k_1,k_2+1-j}(\sigma_1,\sigma_2) \lambda_{l+j}(\varepsilon) - (-1)^j \lambda_{k_1+1-j}(\sigma_1) \text{Li}_{k_1,k_2+l}^*(\sigma_2,\varepsilon\sigma_2) \\
- \sum_{j=1}^{l-1} (-1)^j \lambda_{l+1-j}(\varepsilon) \text{Li}_{k_1,k_2+j}(\sigma_1\sigma_2,\sigma_2) + (-1)^j \lambda_{k_1,1}(\sigma_1,\sigma_2)(\lambda_{k_2}(\varepsilon) - \lambda_{k_2}(\varepsilon\sigma_2)) \\
- (-1)^{k_1+k_2} \lambda_1(\sigma_1)(\text{Li}_{k_1,k_2+l}^*(\sigma_2,\varepsilon\sigma_2) - \text{Li}_{k_1,k_2+l}^*(\sigma_2\sigma_1,\varepsilon\sigma_2)) \\
- (-1)^l \lambda_1(\varepsilon)(\text{Li}_{k_1,k_2+l}(\sigma_1\sigma_2,\sigma_2) - \text{Li}_{k_1,k_2+l}(\sigma_1\sigma_2,\sigma_2\varepsilon)) \\
+ (-1)^l \text{Li}_{k_1+1,k_2+l}^*(\sigma_1\sigma_2\varepsilon,\varepsilon\sigma_2) + (-1)^l \text{Li}_{k_1+1,k_2+l}^*(\sigma_1,\varepsilon\sigma_2). \\
\]

Clearly, setting $\sigma_1 = \sigma_2 = \varepsilon = 1$ in Corollary 3.6 gives the formula (20). We also find numerous explicit relations involving alternating MZVs. For example, letting $k_1 = k_2 = l = 2$ and $\sigma_1 = \varepsilon = -1$, $\sigma_2 = 1$, we have

\[
\zeta^*(\hat{2},\hat{1},\hat{4}) + 2\zeta^*(\hat{1},\hat{2},\hat{4}) = 3 \text{Li}_4\left(\frac{1}{2}\right) \zeta(3) - \frac{7\pi^4 \zeta(3)}{128} + \frac{61\pi^2 \zeta(5)}{192} - \frac{105\zeta(7)}{128} \\
+ \frac{1}{8} \zeta(3) \log^4(2) - \frac{1}{8} \pi^2 \zeta(3) \log^2(2) + \frac{63}{16} \zeta(3)^2 \log(2) \\
- \frac{61\pi^6 \log(2)}{10080},
\]

where we used Au’s Mathematica package [2].

3.3. Multiple integrals associated with 3-labeled posets

In this subsection, we introduce the multiple integrals associated with 3-labeled posets, and express the integrals $I_{\lambda}(k;\sigma_1,\lambda;\varepsilon)$ in terms of multiple integrals associated with 3-labeled posets.

**Definition 3.2.** A 3-labeled poset is a pair $(X, \delta_X)$, where $X = (X, \preceq)$ is a finite partially ordered set and the label map $\delta_X$ is a map $X \to \{-1, 0, 1\}$. We often omit $\delta_X$ and simply say a ‘3-labeled poset $X$’.

Similar to 2-labeled posets, a 3-labeled poset $(X, \delta_X)$ is called admissible if $\delta_X(x) \neq 1$ for all maximal elements and $\delta_X(x) \neq 0$ for all minimal elements $x \in X$.

**Definition 3.3.** For an admissible 3-labeled poset $X$, we define the associated integral

\[
I(X) = \int_{\Delta_X} \prod_{x \in X} \omega_{\delta_X(x)}(t_x),
\]

where

\[
\Delta_X = \{(t_x)_x \in [0, 1]^X \mid t_x < t_y \text{ if } x < y\}
\]
Kaneko–Yamamoto type multiple zeta values 395

\[ \omega^{-1}(t) = \frac{dt}{1+t}, \quad \omega_0(t) = \frac{dt}{t}, \quad \omega_1(t) = \frac{dt}{1-t}. \]

For the empty 3-labeled poset, denoted by \( \emptyset \), we put \( I(\emptyset) := 1 \).

**Proposition 3.7.** For non-comparable elements \( a \) and \( b \) of a 3-labeled poset \( X \), \( X^b_a \) denotes the 3-labeled poset that is obtained from \( X \) by adjoining the relation \( a < b \) if \( X \) is an admissible 3-labeled poset, then the 3-labeled posets \( X^b_a \) and \( X^a_b \) are admissible and

\[ I(X) = I(X^b_a) + I(X^a_b). \quad (45) \]

Note that the admissibility of a 3-labeled poset corresponds to the convergence of the associated integral. We use Hasse diagrams to indicate 3-labeled posets, with vertices \( \circ \) and \( \bullet \) corresponding to \( \delta(x) = 0 \) and \( \delta(x) = \sigma \) (\( \sigma \in \{\pm 1\} \)), respectively. For convenience, if \( \sigma = 1 \), replace \( \bullet \) by \( \circ \), and if \( \sigma = -1 \), replace \( \bullet \) by \( \bar{\bullet} \). For example, the diagram

![Hasse diagram](image)

represents the 3-labeled poset \( X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\} \) with order \( x_1 < x_2 > x_3 < x_4 < x_5 > x_6 < x_7 < x_8 \) and label \( (\delta_X(x_1), \ldots, \delta_X(x_8)) = (1, 0, -1, 0, 0, -1, 0, 0) \). For a composition \( k = (k_1, \ldots, k_r) \) and \( \sigma \in \{\pm 1\}^r \) (admissible or not), we write

![Diagram with label](image)

for the following ‘totally ordered’ diagram.

If \( k_i = 1 \), we understand the notation \( \circ \) as a single \( \bullet \). We see from (40) that

\[ I\left( \tilde{\sigma} \right) (k, \sigma) = \frac{\lambda_{k_1, \ldots, k_r} (\sigma_1, \sigma_2, \ldots, \sigma_r)}{\sigma_1 \sigma_2 \cdots \sigma_r}. \quad (46) \]

Therefore, according to the definition of \( I_k((k; \sigma), (I; \epsilon)) \), and using this notation of multiple associated integral, we can get the following theorem.
Theorem 3.8. For any compositions \( k \equiv k_r \) and \( l \equiv l_s \) with \( \sigma \in \{\pm 1\}^r \) and \( \varepsilon \in \{\pm 1\}^s \),

\[
I_\lambda((k; \sigma), (l; \varepsilon)) = I \left( \frac{\zeta((k_1; \sigma_1) \oplus (l_{s-1}; \{1\}_s^*)}{\sigma_1 \sigma_2 \cdots \sigma_r} \right),
\]

Proof. This follows immediately from the definition of \( I_\lambda((k; \sigma), (l; \varepsilon)) \). We leave the detail to the interested reader. \( \square \)

Finally, we end this section with the following theorem, which extends [8, Theorem 4.1] to level two.

Theorem 3.9. For any non-empty compositions \( k_r, l_s \) and \( \sigma_r \in \{\pm 1\}^r \), we have

\[
I \left( \frac{\zeta((k_1; \sigma_1) \oplus (l_{s-1}; \{1\}_s^*)}{\sigma_1 \sigma_2 \cdots \sigma_r} \right) = \zeta((k_r; \sigma_r) \oplus (l_s; \{1\}_s^*)},
\]

where \( \sigma_r' = \sigma_{j+1} \cdots \sigma_r \), and \( \bullet \sigma_r' \) corresponding to \( \delta(x) = \sigma_r' \).

Proof. The proof is done straightforwardly by computing the multiple integral on the left-hand side of (47) as a repeated integral ‘from left to right’ using the key ideas in the proof of (40) and [14, Corollary 3.1]. \( \square \)

Letting \( \sigma_1 = \sigma_2 = \cdots = \sigma_r = 1 \), we obtain the ‘integral-series’ relation of Kaneko and Yamamoto [8].

From Proposition 3.7 and (46), it is clear that the left-hand side of (47) can be expressed in terms of a linear combination of alternating multiple zeta values. Hence, we can find many linear relations of alternating multiple zeta values from (47). For example, \( \quad 2\lambda_{1,1,3}(\sigma_1', \sigma_2', 1) + 2\lambda_{1,1,3}(\sigma_2', 1, \sigma_1') + 2\lambda_{1,1,3}(1, \sigma_1', \sigma_2') \\
+ \lambda_{1,2,2}(\sigma_1', 1, \sigma_2') + \lambda_{1,2,2}(1, \sigma_1', \sigma_2') + \lambda_{2,1,2}(1, \sigma_1', \sigma_2') \\
= \zeta(2, 1, 2; 1, \sigma_1, \sigma_2) + \zeta(1, 2, 2; \sigma_1, 1, \sigma_2) + \zeta(3, 2; \sigma_1, \sigma_2) + \zeta(1, 4; \sigma_1, \sigma_2). \quad (48) \)

If \( (\sigma_1, \sigma_2) = (1, 1) \) and \( (1, -1) \), then we get the following two cases:

\[
6\zeta(1, 1, 3) + 2\zeta(1, 2, 2) + \zeta(2, 1, 2) = \zeta(1, 2, 2) + \zeta(2, 1, 2) + \zeta(3, 2) + \zeta(1, 4),
\]

and

\[
2\zeta(1, \tilde{1}, 3) + 2\zeta(\tilde{1}, \tilde{1}, \tilde{3}) + 2\zeta(\tilde{1}, 1, \tilde{3}) + \zeta(\tilde{1}, 2, \tilde{2}) + \zeta(\tilde{1}, 2, \tilde{2}) = \zeta(2, 1, \tilde{2}) + \zeta(1, 2, \tilde{2}) + \zeta(3, \tilde{2}) + \zeta(1, \tilde{4}).
\]
4. Integrals about multiple $t$-harmonic (star) sums

Similar to MHSs and MHSSs, we can define their following $t$-versions.

**Definition 4.1.** For any $n, r \in \mathbb{N}$ and any composition $k := (k_1, \ldots, k_r) \in \mathbb{N}^r$,

\[
t_n(k_1, \ldots, k_r) := \sum_{0 < n_1 < n_2 < \cdots < n_r \leq n} \frac{1}{(2n_1 - 1)^{k_1}(2n_2 - 1)^{k_2} \cdots (2n_r - 1)^{k_r}}, \]

\[
t^*_n(k_1, \ldots, k_r) := \sum_{0 < n_1 \leq n_2 \leq \cdots \leq n_r \leq n} \frac{1}{(2n_1 - 1)^{k_1}(2n_2 - 1)^{k_2} \cdots (2n_r - 1)^{k_r}},
\]

where we call (49) and (50) multiple $t$-harmonic sums and multiple $t$-harmonic star sums, respectively. If $n < r$ then $t_n(k) := 0$ and $t_n(\varnothing) = t^*_n(\varnothing) := 1$.

For a composition $k := (k_1, \ldots, k_r)$, define

\[
L(k_1, \ldots, k_r; x) := \frac{1}{2^{k_1+\cdots+k_r}} \text{Li}_{k_1,\ldots,k_r}(x^2), \quad L(\varnothing; x) := 1,
\]

where $x \in [-1, 1]$ and $(k_r, |x|) \neq (1, 1)$. Set $L(k_1, \ldots, k_r) := L(k_1, \ldots, k_r; 1)$. Similarly, define

\[
t(k_1, \ldots, k_r; x) := \sum_{0 < n_1 < n_2 < \cdots < n_r} \frac{x^{2n_r - 1}}{(2n_1 - 1)^{k_1}(2n_2 - 1)^{k_2} \cdots (2n_r - 1)^{k_r}}
\]

\[
= \sum_{n=1}^{\infty} \frac{t_{n-1}(k_1, \ldots, k_{r-1})}{(2n - 1)^{k_r}} x^{2n-1},
\]

where $t(\varnothing; x) := 1/x$. Note that $t(k_1, \ldots, k_r; 1) = t(k_1, \ldots, k_r)$.

**Theorem 4.1.** For any composition $k := (k_1, \ldots, k_r)$ and any positive integer $n$, we have

\[
\int_0^1 x^{2n-2} L(k_r; x) \, dx
\]

\[
= \sum_{j=1}^{k_r-1} \frac{(-1)^{j-1}}{(2n-1)^j} L(k_r-1, k_r+1-j) + \frac{(-1)^{|k|-r}}{(2n-1)^{k_r}} t^*_n(1, k_{r-1})
\]

\[
+ \frac{1}{(2n-1)^{k_r}} \sum_{l=1}^{r-1} \sum_{j=1}^{k_{r-l}-1} (-1)^{l-1} L(k_{r-l}, k_{r-l}+1-j) t^*_n(j, k_{r-l-1})
\]

\[
- \frac{1}{(2n-1)^{k_r}} \sum_{l=0}^{r-1} (-1)^{|k|+l-1} \left( \int_0^1 L(k_{r-l-1}, 1; x) \frac{dx}{x^2} \right) t^*_n(k_{r-l-1}). \tag{51}
\]

**Proof.** By the simple substitution $t \to t^2/x^2$ in (40) we see quickly that

\[
L(k_1, \ldots, k_r; x) = \int_0^x \frac{t \, dt}{1-t^2} \left( \frac{dt}{t} \right)^{k_1-1} \cdots \frac{t \, dt}{1-t^2} \left( \frac{dt}{t} \right)^{k_r-1}.
\]
By an elementary calculation, we deduce the recurrence relation
\[
\int_0^1 x^{2n-2} L(k_r; x) \, dx
= \frac{k_r^{r-1}}{(2n-1)^j} \left( -1 \right)^j \frac{L(k_{r-1}, k_r + 1 - j)}{(2n-1)^{k_r}} \int_0^1 L(k_{r-1}, 1; x) \, dx
+ \frac{(-1)^{k_r-1}}{(2n-1)^{k_r}} \sum_{j=1}^n \int_0^1 x^{2l-2} L(k_{r-1}; x) \, dx.
\]

Hence, using the recurrence relation, we obtain the desired evaluation by direct calculations.

**Theorem 4.2.** For any composition \( k := (k_1, \ldots, k_r) \) and any positive integer \( n \), we have
\[
\int_0^1 x^{2n-2} t(k_r; x) \, dx
= \sum_{j=1}^{k_r-1} \frac{(-1)^{j-1}}{(2n-1)^j} t(k_{r-1}, k_r + 1 - j) + \frac{(-1)^{|k|-r}}{(2n-1)^{k_r}} s_n^*(1, k_{r-1})
\]
\[
+ \frac{1}{(2n-1)^{k_r}} \sum_{l=1}^{k_r-1} (-1)^{k_r-l} \sum_{j=1}^{k_r-l-1} (-1)^j \int_0^1 t(k_{r-l-1}, k_{r-l} + 1 - j) \tau_n^*(j, k_{r-l-1}) \, dx
\]
\[
+ \frac{1}{(2n-1)^{k_r}} \sum_{l=0}^{k_r-1} (-1)^{k_r+l-1} \left( \int_0^1 t(k_{r-l-1}, 1; x) \, dx \right) \tau_n^*(k_{r-l-1}), \tag{52}
\]

where
\[
\tau_n^*(k_1, \ldots, k_r) := \sum_{2 \leq n_1 \leq n_2 \leq \cdots \leq n_r \leq n} \frac{1}{(2n_1-1)^{k_1}(2n_2-1)^{k_2} \cdots (2n_r-1)^{k_r}},
\]
\[
s_n^*(k_1, \ldots, k_r) := \sum_{2 \leq n_1 \leq n_2 \leq \cdots \leq n_r \leq n} \frac{1}{(2n_1-2)^{k_1}(2n_2-1)^{k_2} \cdots (2n_r-1)^{k_r}}.
\]

**Proof.** By definition we have
\[
\frac{d}{dx} t(k_1, \ldots, k_{r-1}, k_r; x) = \begin{cases} 
\frac{1}{x} t(k_1, \ldots, k_{r-1}, k_r - 1; x) & (k_r \geq 2), \\
\frac{x}{x^2 - 1} t(k_1, \ldots, k_{r-1}; x) & (k_r = 1), 
\end{cases}
\]
where \( t(\emptyset; x) := 1/x \). Hence, we obtain the iterated integral
\[
t(k_1, \ldots, k_r; x) = \int_0^x \frac{1}{1-t^2} \left( \frac{dt}{t} \right)^{k_1-1} t \frac{dt}{1-t^2} \left( \frac{dt}{t} \right)^{k_2-1} \cdots t \frac{dt}{1-t^2} \left( \frac{dt}{t} \right)^{k_r-1}.
\]
By an elementary calculation, we deduce the recurrence relation
\[
\int_0^1 x^{2n-2} t(k_r; x) \, dx = \sum_{j=1}^{k_r-1} \frac{(-1)^{j-1}}{(2n-1)^j} t(k_{r-1}, k_r + 1 - j) + \frac{(-1)^{k_r-1}}{(2n-1)^{k_r}} \int_0^1 t(k_{r-1}, 1; x) \, dx
\]

\[
+ \frac{(-1)^{k_r-1}}{(2n-1)^{k_r}} \sum_{l=2}^n \int_0^1 x^{2l-2} t(k_{r-1}; x) \, dx.
\]

Hence, using the recurrence relation, we obtain the desired evaluation by direct calculations.

**Theorem 4.3.** For any positive integer \( r \), \( \int_0^1 (L([1]_r; x)/x^2) \, dx \) can be expressed as a \( \mathbb{Q} \)-linear combinations of products of \( \log 2 \) and Riemann zeta values. More precisely, we have

\[
1 - \sum_{r \geq 1} \left( \int_0^1 \frac{L([1]_r; x)}{x^2} \, dx \right) u^r = \exp \left( \sum_{n=1}^{\infty} \frac{\pi(n)}{n} u^n \right)
\]

\[
= \exp \left( - \log(2)u - \sum_{n=2}^{\infty} \frac{1 - 2^{1-n}}{n} \zeta(n)u^n \right).
\]

**Proof.** Consider the generating function

\[
F(u) := 1 - \sum_{r=1}^{\infty} 2^r \left( \int_0^1 \frac{L([1]_r; x)}{x^2} \, dx \right) u^r.
\]

By definition

\[
F(u) = 1 - \sum_{r=1}^{\infty} u^r \int_0^1 \int_0^1 x^2 \left( \frac{dt}{1-t} \right)^r dx
\]

\[
= 1 - \sum_{r=1}^{\infty} \frac{u^r}{r!} \int_0^1 \left( \int_0^1 \frac{dt}{1-t} \right)^r dx
\]

\[
= 1 - \int_0^1 \left( \sum_{r=1}^{\infty} \frac{(-u \log(1-x^2))^r}{r!} \right) dx
\]

\[
= 1 + \int_0^1 ((1-x^2)^{-u} - 1) \, dx
\]

by integration by parts followed by the substitution \( x = \sqrt{t} \). Using the expansion

\[
\Gamma(1-u) = \exp \left( \gamma u + \sum_{n=2}^{\infty} \frac{\pi(n)}{n} u^n \right) \quad (|u| < 1)
\]

and setting \( x = 1/2 - u \) in the duplication formula \( \Gamma(x)\Gamma(x+1/2) = 2^{1-2x}\sqrt{\pi} \Gamma(2x) \), we obtain

\[
\log \Gamma(1/2 - u) = \frac{\log \pi}{2} + \gamma u + 2u \log(2) + \sum_{n=2}^{\infty} \frac{(2^n - 1)\xi(n)}{n} u^n \quad (|u| < 1/2).
\]
Therefore
\[ F(u) = \exp\left(-2 \log(2)u - \sum_{n=2}^{\infty} \frac{2^n - 2}{n} \zeta(n)u^n\right) = \exp\left(\sum_{n=1}^{\infty} \frac{2^n}{n} \zeta(\bar{n})u^n\right) \]

by the facts that \( \zeta(\bar{1}) = -\log 2 \) and \( 2^n \zeta(\bar{n}) = (2 - 2^n)\zeta(n) \) for \( n \geq 2 \). The theorem follows immediately. \( \Box \)

Clearly, \( \int_0^1 (L(\{1\}_r; x)/x^2) \, dx \in \mathbb{Q}[\log(2), \zeta(2), \zeta(3), \zeta(4), \ldots] \). For example,

\[
\int_0^1 \frac{L(1; x)}{x^2} \, dx = \log(2), \\
\int_0^1 \frac{L(1, 1; x)}{x^2} \, dx = \frac{1}{4} \zeta(2) - \frac{1}{2} \log^2(2), \\
\int_0^1 \frac{L(1, 1, 1; x)}{x^2} \, dx = \frac{1}{4} \zeta(3) + \frac{1}{6} \log^3(2) - \frac{1}{4} \zeta(2) \log(2). \\
\int_0^1 \frac{L(k_1, \ldots, k_r, 1; x)}{x^n} \, dx = 0 \quad (0 \leq n \leq 2r + 2) \] (53)

More generally, using a similar argument as in the proof of Theorem 2.9, we can prove the following more general results.

**Theorem 4.4.** Let \( r \) and \( n \) be two non-negative integers and \( k_r = (k_1, \ldots, k_r) \in \mathbb{N}^r \) with \( k_0 = \emptyset \). Then one can express all of the integrals

\[ \int_0^1 \frac{L(k_1, \ldots, k_r, 1; x)}{x^n} \, dx \quad (0 \leq n \leq 2r + 2) \]

and

\[ \int_0^1 \frac{t(k_1, \ldots, k_r, 1; x)}{x^n} \, dx \quad (0 \leq n \leq 2r + 1) \]

as \( \mathbb{Q} \)-linear combinations of alternating MZVs (and number 1 for \( \int_0^1 L(k_r, 1; x) \, dx \)).

**Proof.** The case \( n = 1 \) is trivial, as both integrals are clearly already MMVs after the integration.

If \( n = 0 \) then we have

\[
\int_0^1 t(k_1, \ldots, k_r, 1; x) \, dx = \int_0^1 \frac{x^{2m-1} \, dx}{(2n_1 - 1)^{k_1} \cdots (2n_r - 1)^{k_r}(2m - 1)} = \lim_{N \to \infty} c_N \frac{2^{r+1}}{2^{r+1}}
\]
Hence, Proposition 14.2.5. If by (18) we have

\[ \int_1^1 \left(1 + \sum_{n=1}^{N} \frac{1}{n(n+1)} \sum_{n=1}^{m} \frac{(-1)^n - (-1)^m}{nm} \right) = \int_0^1 t(1, 1; x) \, dx = \frac{1}{4} (\zeta(\bar{1}, \bar{1}) - 2\zeta(1, 1) + 4 \log 2) = \log 2 - \frac{1}{4} \xi(2) \quad (56) \]

By Kaneko–Yamamoto type multiple zeta values 401
Taking \( m = n_{r-1} + 1 \) and \( n = n_r \) in (54) we get

\[
c_N = -2 \sum_{0 < n_1 < \ldots < n_r < m \leq 2N} \frac{(1 - (-1)^{n_1}) \cdots (1 - (-1)^{n_r})(-1)^m}{n_1^{k_1} \cdots n_r^{k_r} m} \\
+ 2 \sum_{j=2}^{k_r} (-1)^{k_r-j} \sum_{0 < n_1 < \ldots < n_{r-1} < m \leq 2N} \frac{(1 - (-1)^{n_1}) \cdots (1 - (-1)^{n_{r-1}})}{n_1^{k_1} \cdots n_{r-1}^{k_{r-1}} n_r^j} \\
- 4(-1)^{k_r} \sum_{0 < n_1 < \ldots < n_r < 2N} \frac{(1 - (-1)^{n_1}) \cdots (1 - (-1)^{n_{r-1}})}{n_1^{k_1} \cdots n_{r-1}^{k_{r-1}}} \\
\times \left( \frac{1}{n_{r-1}+1} - \sum_{n_r = n_{r-1}+1}^{2N} \frac{(-1)^{n_r}}{n_r} \right).
\]

Here, when \( r = 1 \) the last line above degenerates to \( 4(-1)^{k_r} \sum_{n_{r-1}=1}^{2N} (-1)^{n_{r-1}}/n_{r-1} \). Taking \( N \rightarrow \infty \) and using induction on \( r \), we see that the claim for \( \int_0^1 t(k_r, 1; x) \, dx \) in the theorem follows.

The computation of \( \int_0^1 L(k_r, 1; x) \, dx \) is completely similar to that of \( \int_0^1 t(k_r, 1; x) \, dx \). Thus we can get

\[
\int_0^1 L(k_r, 1; x) \, dx = \frac{1}{2^r} \sum_{0 < n_1 < \ldots < n_r < m} \frac{(1 + (-1)^{n_1}) \cdots (1 + (-1)^{n_r})(-1)^m}{n_1^{k_1} \cdots n_r^{k_r} m} \\
+ \frac{1}{2^r} \sum_{j=2}^{k_r} (-1)^{k_r-j} \sum_{0 < n_1 < \ldots < n_r} \frac{(1 + (-1)^{n_1}) \cdots (1 + (-1)^{n_{r-1}})}{n_1^{k_1} \cdots n_{r-1}^{k_{r-1}} n_r^j} \\
- \frac{2(-1)^{k_r}}{2^r} \sum_{0 < n_1 < \ldots < n_r} \frac{(1 + (-1)^{n_1}) \cdots (1 + (-1)^{n_{r-1}})(-1)^{n_r}}{n_1^{k_1} \cdots n_{r-1}^{k_{r-1}} n_r} \\
- \frac{2(-1)^{k_r}}{2^r} \sum_{0 < n_1 < \ldots < n_r} \frac{(1 + (-1)^{n_1}) \cdots (1 + (-1)^{n_{r-1}})}{n_1^{k_1} \cdots n_{r-1}^{k_{r-1}} (n_{r-1}+1)}.
\]

Here, when \( r = 1 \) the last line above degenerates to \(-(-1)^{k_1}\). So by induction on \( r \) we see that the claim for \( \int_0^1 L(k_r, 1; x) \, dx \) is true.

Similarly, if \( n = 2 \) then we can apply the same technique as above to get

\[
\int_0^1 \frac{L(k_1, \ldots, k_r, 1; x)}{x^2} \, dx = \int_0^1 \frac{1}{2^{k_1+\cdots+k_r+1}} \sum_{0 < n_1 < \ldots < n_r < m} \frac{x^{2m-2} \, dx}{n_1^{k_1} \cdots n_r^{k_r} m} = \lim_{N \rightarrow \infty} \frac{d_N}{2^{r+1}},
\]
where

\[
d_N = \sum_{0 < n_1 < \cdots < n_r < m \leq N} \frac{2^{r+1}}{(2n_1)^{k_1} \cdots (2n_r)^{k_r} 2m(2m - 1)}
\]

\[
= \sum_{0 < n_1 < \cdots < n_r < m \leq 2N} \frac{(1 + (-1)^{n_1}) \cdots (1 + (-1)^{n_r})(-1)^m}{n_1^{k_1} \cdots n_r^{k_r}} \left( 1 + \frac{(-1)^m}{m-1} - 1 \right) \left( 1 - \frac{(-1)^m}{m} \right)
\]

\[
= \sum_{0 < n_1 < \cdots < n_r < m \leq 2N} \frac{(1 + (-1)^{n_1}) \cdots (1 + (-1)^{n_r})(-1)^m}{n_1^{k_1} \cdots n_r^{k_r}} \left( \frac{1}{m-1} - \frac{(-1)^m}{m} \right)
\]

\[
= -2 \sum_{0 < n_1 < \cdots < n_r < m \leq 2N} \frac{(1 + (-1)^{n_1}) \cdots (1 + (-1)^{n_r})(-1)^m}{n_1^{k_1} \cdots n_r^{k_r} m} + \sum_{0 < n_1 < \cdots < n_r < 2N} \frac{(1 + (-1)^{n_1}) \cdots (1 + (-1)^{n_r})(-1)^m}{n_1^{k_1} \cdots n_r^{k_r} m} \sum_{m=n_r+1}^{2N} \left( \frac{1}{m-1} - \frac{(-1)^m}{m} \right)
\]

\[
= -2 \sum_{0 < n_1 < \cdots < n_r < m \leq 2N} \frac{(1 + (-1)^{n_1}) \cdots (1 + (-1)^{n_r})(-1)^m}{n_1^{k_1} \cdots n_r^{k_r}} \left( 1 - \frac{(-1)^{n_r}}{n_r} \right)
\]

\[
= -2 \sum_{0 < n_1 < \cdots < n_r < m \leq 2N} \frac{(1 + (-1)^{n_1}) \cdots (1 + (-1)^{n_r})(-1)^m}{n_1^{k_1} \cdots n_r^{k_r} m}
\]

\[
\rightarrow -2 \sum_{\epsilon_j=\pm1, 1 \leq j \leq r} \zeta(k_1, \ldots, k_r, 1; \epsilon_1, \ldots, \epsilon_r, -1)
\]

as \( N \to \infty \). Hence,

\[
\int_{0}^{1} \frac{L(k_1, \ldots, k_r, 1; x)}{x^2} dx = -\frac{1}{2^r} \sum_{\epsilon_j=\pm1, 1 \leq j \leq r} \zeta(k_1, \ldots, k_r, 1; \epsilon_1, \ldots, \epsilon_r, -1). \quad (57)
\]

By exactly the same approach as above, we find that

\[
\int_{0}^{1} \frac{t(k_1, \ldots, k_r, 1; x)}{x^2} dx = -\frac{1}{2^r} \sum_{0 < n_1 < \cdots < n_r < m} \frac{(1 - (-1)^{n_1}) \cdots (1 - (-1)^{n_r})(-1)^m}{n_1^{k_1} \cdots n_r^{k_r} m}
\]

\[
= -\frac{1}{2^r} \sum_{\epsilon_j=\pm1, 1 \leq j \leq r} \epsilon_1 \cdots \epsilon_r \zeta(k_1, \ldots, k_r, 1; \epsilon_1, \ldots, \epsilon_r, -1).
\] (58)

More generally, for any larger values of \( n \) we may use the partial fraction technique and similar argument as above to express the integrals in Theorem 4.4 as \( \mathbb{Q} \)-linear combinations of alternating MZVs. So we leave the details to the interested reader. This finishes the proof of the theorem.
Example 4.5. For a positive integer $k > 1$, by the computation in the $n = 0$ case in the proof of Theorem 4.4, we get

\[
\int_0^1 t(k, 1; x) \, dx = \frac{1}{2} \zeta(\bar{k}, \bar{1}) - \zeta(k, \bar{1}) - (-1)^k \log 2 + \frac{1}{2} \sum_{j=2}^k (-1)^{k-j} (\zeta(j) - \zeta(\bar{j})).
\]

\[
\int_0^1 L(k, 1; x) \, dx = \frac{1}{2} (\zeta(\bar{k}, \bar{1}) + \zeta(k, \bar{1})) - (-1)^k + (-1)^k \log 2 + \frac{1}{2} \sum_{j=2}^k (-1)^{k-j} (\zeta(j) + \zeta(\bar{j})).
\]

\[
= \frac{1}{2} (\zeta(\bar{k}, \bar{1}) + \zeta(k, \bar{1})) - (-1)^k + (-1)^k \log 2 + \frac{1}{2} \sum_{j=2}^k (-1)^{k-j} \zeta(j).
\]

Example 4.6. For a positive integer $k > 1$, we see from (57) and (58) that

\[
\int_0^1 \frac{L(k, 1; x)}{x^2} \, dx = -\frac{1}{2} (\zeta(k, \bar{1}) + \zeta(\bar{k}, \bar{1})), \tag{59}
\]

\[
\int_0^1 \frac{t(k, 1; x)}{x^2} \, dx = \frac{1}{2} (\zeta(k, \bar{1}) - \zeta(\bar{k}, \bar{1})).
\]

Taking $r = 2$ in (57) and (58) we get

\[
\int_0^1 \frac{L(k_1, k_2, 1; x)}{x^2} \, dx = -\frac{1}{4} (\zeta(k_1, k_2, \bar{1}) + \zeta(\bar{k}_1, \bar{k}_2, \bar{1}) + \zeta(\bar{k}_1, k_2, \bar{1}) + \zeta(k_1, \bar{k}_2, \bar{1})),
\]

\[
\int_0^1 \frac{t(k_1, k_2, 1; x)}{x^2} \, dx = -\frac{1}{4} (\zeta(k_1, k_2, \bar{1}) - \zeta(\bar{k}_1, \bar{k}_2, \bar{1}) - \zeta(\bar{k}_1, k_2, \bar{1}) + (\zeta(k_1, \bar{k}_2, \bar{1})).
\]

Remark 4.7. It is possible to give an induction proof of Theorem 4.4 using the regularized values of MMVs as defined by [15, Definition 3.2]. However, the general formula for the integral of $L(k, 1; x)$ would be implicit. To illustrate the idea for computing $\int_0^1 t(k, 1; x) \, dx$, we consider the case $k = k \in \mathbb{N}$. Notice that

\[
\sum_{0 < m < n < N} \frac{1}{(2m-1)(2n-1)2n} = \sum_{0 < m < n < 2N} \frac{1}{m(n - 1)} \frac{1}{m(n + 1)}.
\]

\[
= \sum_{0 < m < n < 2N} \frac{1}{mn} - \sum_{0 < m < n \leq 2N} \frac{1}{mn} + \sum_{0 < m < 2N} \frac{1}{mn} + \sum_{0 < m < 2N} \frac{1}{m(m + 1)}.
\]

By using regularized values (with the notation from [13, Section 2.2]), we see that

\[
\int_0^1 t(1, 1; x) \, dx = \sum_{0 < m < n} \frac{1}{(2m-1)(2n-1)2n} = \frac{1}{4} (M_*(\bar{1}, \bar{1}) - M_*(\bar{1}, 1)) + \log 2.
\]

We have

\[
M_*(\bar{1}, \bar{1}) = \frac{1}{2} (M_*(\bar{1})^2 - 2M_*(\bar{2})) = \frac{1}{2} ((T + 2 \log 2)^2 - 2M(\bar{2})).
\]
Since \(2M(\tilde{2}) = 3\zeta(2)\),
\[
M_{\tilde{\mathfrak{u}}}(\tilde{1}, \tilde{1}) = \frac{1}{2} \rho((T + 2 \log 2)^2 - 3\zeta(2)) = \frac{1}{2}((T + \log 2)^2 - 2\zeta(2))
\]
by [18, Theorem 2.7]. On the other hand,
\[
\rho(M_*(\tilde{1}, 1)) = M_{\tilde{\mathfrak{u}}}(\tilde{1}, 1) = \frac{1}{2}M_{\tilde{\mathfrak{u}}}((\tilde{1})^2 = \frac{1}{2}(T + \log 2)^2.
\]
Since \(\rho\) is an \(\mathbb{R}\)-linear map,
\[
\int_0^1 t(1, 1; x) \, dx = \log 2 + \frac{1}{4} \rho(M_*(1, \tilde{1}) - M_*(1, 1)) = \log 2 - \frac{1}{4}\zeta(2),
\]
which agrees with (56).

Similarly, by considering some related integrals, we can establish many relations involving multiple \(\tilde{t}\)-star values. For example, from (51) we have
\[
\int_0^1 x^{2n-2} L(k_1, k_2; x) \, dx
= \sum_{j=1}^{k_2-1} \frac{(-1)^{j-1}}{(2n-1)^j} L(k_1, k_2 + 1 - j) + \frac{(-1)^{k_2}}{(2n-1)^{k_2}} \int_0^1 \frac{L(k_1, 1; x)}{x^2} \, dx
- \frac{(-1)^{k_2}}{(2n-1)^{k_2}} \sum_{j=1}^{k_1-1} (-1)^{j-1} L(k_1 + 1 - j) t_1^*(j)
- \frac{(-1)^{k_1+k_2}}{(2n-1)^{k_2}} \log(2) t_1^*(k_1) + \frac{(-1)^{k_1+k_2}}{(2n-1)^{k_2}} t_1^*(1, k_1).
\]
Hence, considering the integral
\[
\int_0^1 \frac{A(l; x)L(k_1, k_2; x)}{x} \, dx \quad \text{or} \quad \int_0^1 \frac{t(l; x)L(k_1, k_2; x)}{x} \, dx,
\]
we can get the following theorem.

**Theorem 4.8.** For positive integers \(k_1, k_2\) and \(l\),
\[
\sum_{j=1}^{k_2-1} (-1)^{j-1} L(k_1, k_2 + 1 - j) T(l + j) + (-1)^{k_2} T(k_2 + l) \int_0^1 \frac{L(k_1, 1; x)}{x^2} \, dx
- (-1)^{k_2} \sum_{j=1}^{k_1-1} (-1)^{j-1} L(k_1 + 1 - j) t_1^*(j, k_2 + l)
- (-1)^{k_1+k_2} \log(2) t_1^*(k_1, k_2 + l) + (-1)^{k_1+k_2} 2t_1^*(1, k_1, k_2 + l)
= \frac{1}{2^{k_1+k_2}} \sum_{j=1}^{l-1} (-1)^{j-1} T(l + 1 - j) \zeta(k_1, k_2 + j) - (-1)^{l-1} \sum_{n=1}^{\infty} \frac{\zeta_{n-1}(k_1)T_n(1)}{n^{k_2+l}}.
\]
where \(\int_0^1 (L(k_1, 1; x)/x^2) \, dx\) is given by (59).
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