A STIELTJES SEPARATION PROPERTY OF ZEROS OF EISENSTEIN SERIES

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Abstract. For \( k < \ell \), let \( E_k(\zeta) \) and \( E_\ell(\zeta) \) be Eisenstein series of weights \( k \) and \( \ell \), respectively, for \( \text{SL}_2(\mathbb{Z}) \). We prove that between any two zeros of \( E_k(e^{i\theta}) \) there is a zero of \( E_\ell(e^{i\theta}) \) on the interval \( \pi/2 < \theta < 2\pi/3 \).

1. Introduction
The Eisenstein series of weight \( k \geq 4 \) for \( \text{SL}_2(\mathbb{Z}) \) is defined as
\[
E_k(z) = \frac{1}{2} \sum_{(c,d) \in \mathbb{Z}^2} \frac{1}{(cz+d)^k}.
\]
This also has the Fourier expansion
\[
E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,
\]
where \( B_k \) is the \( k \)th Bernoulli number, \( \sigma_{k-1}(n) = \sum_{d|n} d^{k-1} \) is the divisor function, and \( q = e^{2\pi i z} \) with \( z \) in the upper half-plane.

Rankin and Swinnerton-Dyer \[9\] proved that all the zeros of \( E_k(z) \) in the fundamental domain for \( \text{SL}_2(\mathbb{Z}) \) are simple and lie on the open arc \( \{e^{i\theta}: \pi/2 < \theta < 2\pi/3\} \), except for the zeros \( i = e^{\pi i/2} \) and \( \rho = e^{2\pi i/3} \), by examining the function
\[
F_k(\theta) = e^{ik\theta/2} E_k(e^{i\theta}) = \frac{1}{2} \sum_{(c,d) \in \mathbb{Z}} (ce^{i\theta/2} + de^{-i\theta/2})^{-k},
\]
a real function over \( (\pi/2, 2\pi/3) \) whose zeros are exactly the zeros of \( E_k(e^{i\theta}) \). We call the zeros of \( F_k(\theta) \) in \( (\pi/2, 2\pi/3) \) the non-trivial zeros.

Nozaki \[8\] showed that the zeros of \( E_k(e^{i\theta}) \) and \( E_{k+12}(e^{i\theta}) \) interlace, a result originally conjectured by Gekeler in \[4\]. Nozaki compared the location of the zeros of \( F_k(\theta) \) with the location of the zeros of 2 cos\((k\theta/2)\) and bounded the error. More precisely, the approach

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in [8] is based on the following approximation of $F_k(\theta)$ [8, Section 2]:

\[ F_k(\theta) = 2 \cos \left( \frac{k\theta}{2} \right) + T_k(\theta) = 2 \cos \left( \frac{k\theta}{2} \right) + \left( 2 \cos \left( \frac{\theta}{2} \right) \right)^{-k} + R_k(\theta), \]

where $R_k(\theta)$ is such that

\[ |R_k(\theta)| < 5 \left( \sqrt{2} \right)^{-k}. \tag{1.2} \]

Expanding on Nozaki’s method, Griffin et al [5] further refined Nozaki’s methods and extended their results in [6] to establish necessary and sufficient conditions under which the non-trivial zeros of $E_k(e^{i\theta})$ and $E_\ell(e^{i\theta})$ interlace.

Others have investigated the interlacing of zeros of similar modular forms. Jermann [7] studied the interlacing of zeros of other modular functions; Jermann proved that the zeros of $j_n(\tau)$ and $j_{n+1}(\tau)$ interlace. Here, $j(\tau)$ is the usual $j$-invariant and $j_n(\tau)$ is its image under the $n$th Hecke operator. Additionally, Saha and Saradha [10] explored the similar interlacing property of zeros of modular forms within a certain family.

Gekeler’s conjecture is partially motivated by the separation property of the zeros of orthogonal polynomials $p_m(x), m \in \mathbb{N}$; see Szegő’s book [12] on orthogonal polynomials. More precisely, the zeros of $p_m(x)$ and $p_{m+1}(x)$ interlace for each $m \geq 1$. The same book also stated the following separation property between the zeros of orthogonal polynomials, which is due to Stieltjes.

**Theorem 1.1.** [12, Theorem 3.3.3] Between two zeros of $p_m(x)$ there is at least one zero of $p_n(x)$ for $m < n$.

Inspired by Theorem 1.1 and the analogy between zeros of orthogonal polynomials and Eisenstein series, we propose and prove the following theorem on the separation property of zeros of Eisenstein series.

**Theorem 1.2.** Let $\ell > k \geq 24$ be positive even integers with $k \neq 26$. Then between any two zeros of $F_k(\theta)$ on the interval $(\pi/2, 2\pi/3)$ there exists at least one zero of $F_\ell(\theta)$.

Following [5, Section 3.1], we introduce the following notation on relevant zeros of $\cos(k\theta/2)$.

**Definition 1.3.** [5] Let $n_k$ be the number of zeros of $F_k(\theta)$ on $(\pi/2, 2\pi/3)$. Let $\beta_{k,i}$ be the $i$th zero of $\cos(k\theta/2)$ on $(\pi/2, 2\pi/3)$ counting from $2\pi/3$, where $0 \leq i \leq n_k - 1$. Note that

\[ \beta_{k,i} = \pi \left( \frac{2}{3} - \frac{6i + 2k + 3}{3k} \right), \]

where $0 \leq \bar{k} \leq 2$ is the value of $k$ modulo 3. We denote $\beta_{k,i}^*$ as the zero of $F_k(\theta)$ associated with $\beta_{k,i}$.

We also have an equivalent definition following [5, (2.1)], [6, Definition 2.1], and [8, (2.3)–(2.6)]. We replace $\bar{k}$ in Griffin et al’s definition [5] with $\hat{k}$.

**Definition 1.4.** [5, 6, 8] Let $\alpha_{k,i}$ be the $i$th zero of $\cos(k\theta/2)$ on $(\pi/2, 2\pi/3)$ counting from $\pi/2$, where $1 \leq i \leq n_k$. We have that

\[ \alpha_{k,i} = \pi \left( \frac{1}{2} + \frac{2i + (\hat{k}/2) - 1}{\hat{k}} \right). \]
where \( \tilde{k} = 0 \) or 2 is the value of \( k \) modulo 4 (since \( k \) is even). We denote \( \alpha_{k,i}^* \) as the zero of \( F_k(\theta) \) associated with \( \alpha_{k,i} \).

Keep in mind the difference in indexing between \( \alpha \) and \( \beta \); for \( \alpha \), we start counting from \( i = 1 \), but for \( \beta \), we start counting from \( i = 0 \). For the rest of this paper, we generally use \( \beta_{k,i} \), but there are a few situations where \( \alpha_{k,i} \) is more helpful for calculations. It is not hard to check that 
\[
\beta_{k,i} = \alpha_{k,n_k-i} \quad \text{for} \quad i = 0, \ldots, n_k - 1.
\]

The common theme of all the previous work is that the location of a non-trivial zero of \( F_k(\theta) \) is very close to that of the corresponding zero of \( \cos(k\theta/2) \). So, the following simple observation serves as the starting point as well as a motivation for Theorem 1.2.

**Proposition 1.5.** Let \( \ell > k \geq 24 \) be positive even integers with \( k \neq 26 \). Suppose we consider only the zeros of \( \cos(k\theta/2) \) and \( \cos(\ell\theta/2) \) corresponding to the non-trivial zeros of \( F_k(\theta) \) and \( F_\ell(\theta) \), respectively. Then strictly between any two distinct zeros of \( \cos(k\theta/2) \) there exists a zero of \( \cos(\ell\theta/2) \).

We will postpone the proof of Proposition 1.5 until Section 2.

Let us give a brief summary of the paper. Section 2 establishes some technical lemmas. We will prove Theorem 1.2 by dividing it into multiple cases. The case when \( \ell/k \) is large is addressed in Section 3. In Section 4, we will cover the case when the zeros are close to \( \pi/2 \), or equivalently when \( i/k \) is large. The case when \( \ell/k \) is small and the zeros are close to \( 2\pi/3 \), or equivalently when \( \ell/k \) is small and \( i/k \) is small, is treated in Section 5. We conclude the proof of Theorem 1.2 by combining all of these cases in Section 6. In Section 7, we apply Theorem 1.2 to show indivisibility between certain Eisenstein series and thus establish linear independence of certain vectors consisting of special values of \( L \)-functions associated to modular forms.

### 2. Preliminary lemmas

We start by providing some basic lemmas. They will mostly be applied in Section 5, but a few are also applied in other sections as well. Note throughout this section that \( k \) and \( \ell \) are even.

We start by restating a result proved in [8, Lemma 3.1, Lemma 4.1] (which was mentioned in Section 3.1 of [5]) and [5, Lemma 3.5]. We include the result here explicitly to facilitate future proofs.

**Lemma 2.1.** [8, 5] For \( k \geq 12 \) and \( 0 \leq i \leq n_k - 1 \), \( |\beta_{k,i}^* - \beta_{k,i}| \leq \pi/(3k) \).

Similarly, the following lemma restates a result proved in [9].

**Lemma 2.2.** [9] The zeros \( \beta_{k,i}^* \) remain in the same order as the zeros \( \beta_{k,i} \). In particular, the zeros \( \beta_{k,i}^* \) are decreasing in \( i \).

We care about the interval between two consecutive zeros. We classify when zeros belong in specific intervals.

**Lemma 2.3.** We have \( \beta_{k,i} > \beta_{\ell,j} > \beta_{k,i+1} \) if and only if

\[
\frac{6j + 2\ell + 3}{6i + 2\ell + 9} < \frac{\ell}{k} < \frac{6j + 2\ell + 3}{6i + 2k + 9}.
\]
Proof. By Definition 1.3, we have that
\[
\beta_{k,i} > \beta_{\ell,j} \iff \frac{6j + 2\ell + 3}{3\ell} > \frac{6i + 2k + 3}{3k} \iff \frac{6j + 2\ell + 3}{6i + 2k + 3} > \frac{\ell}{k}
\]
and
\[
\beta_{\ell,j} > \beta_{k,i+1} \iff \frac{6i + 2k + 9}{3k} > \frac{6j + 2\ell + 3}{3\ell} \iff \frac{\ell}{k} > \frac{6j + 2\ell + 3}{6i + 2k + 9},
\]
as desired. \(\square\)

We want to know when two zeros of \(\cos(k\theta/2)\) and \(\cos(\ell\theta/2)\) and the two associated zeros of \(F_k(\theta)\) and \(F_\ell(\theta)\) are ordered differently. We develop a lemma to characterize when this happens.

We define the following interval for ease of notation:
\[
I_{k,i,\ell,j} = \left[ \frac{6j + 2\ell + 3 - e^{-8(3j+\ell+1)/9} - 5(\sqrt{2})^{-\ell}}{6i + 2k + 3 + e^{-8(3i+k+1)/9} + 5(\sqrt{2})^{-k}}, \frac{6j + 2\ell + 3 + e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-k}}{6i + 2k + 3 - e^{-8(3i+k+1)/9} - 5(\sqrt{2})^{-k}} \right].
\]

**Lemma 2.4.** If \((\beta_{k,i} - \beta_{\ell,j})(\beta_{k,i}^* - \beta_{\ell,j}^*) \leq 0\), then

\[
\frac{\ell}{k} \in I_{k,i,\ell,j}.
\]

**Proof.** We recall by Definition 1.3 that
\[
\beta_{k,i} = \pi \left( \frac{2}{3} - \frac{6i + 2k + 3}{3k} \right) \quad \text{and} \quad \beta_{\ell,j} = \pi \left( \frac{2}{3} - \frac{6j + 2\ell + 3}{3\ell} \right).
\]

First, suppose \(\beta_{k,i} < \beta_{\ell,j}\). Then \(\beta_{k,i}^* \geq \beta_{\ell,j}^*\). We now have three cases.

**Case 1.** First, suppose \(\beta_{\ell,j}^* \leq \beta_{k,i}^*\). Then
\[
\beta_{\ell,j} - \beta_{k,i} \leq \beta_{\ell,j} - \beta_{\ell,j}^* \leq |\beta_{\ell,j} - \beta_{\ell,j}^*| \leq |\beta_{\ell,j} - \beta_{\ell,j}^*| + |\beta_{k,i} - \beta_{k,i}^*|.
\]

**Case 2.** Next, suppose \(\beta_{k,i}^* \geq \beta_{\ell,j}^*\). Then
\[
\beta_{\ell,j} - \beta_{k,i} \leq \beta_{k,i}^* - \beta_{k,i} \leq |\beta_{k,i}^* - \beta_{k,i}| \leq |\beta_{k,i}^* - \beta_{k,i}| + |\beta_{\ell,j} - \beta_{\ell,j}^*|.
\]

**Case 3.** Finally, suppose \(\beta_{k,i}^* < \beta_{\ell,j}^*\) and \(\beta_{k,i}^* > \beta_{k,i}\). Then
\[
\beta_{\ell,j} - \beta_{k,i} = (\beta_{\ell,j} - \beta_{k,i}^*) + (\beta_{k,i}^* - \beta_{k,i}) \leq (\beta_{\ell,j} - \beta_{k,i}^*) + (\beta_{k,i}^* - \beta_{k,i}) = |\beta_{\ell,j} - \beta_{k,i}^*| + |\beta_{k,i} - \beta_{k,i}^*|.
\]

In all three cases, we conclude that
\[
\beta_{\ell,j} - \beta_{k,i} \leq |\beta_{\ell,j} - \beta_{\ell,j}^*| + |\beta_{k,i} - \beta_{k,i}^*|.
\]
By [5, Lemma 3.5], we know that
\[ |\beta_{\ell, j} - \beta_{\ell, j}^*| + |\beta_{k, i}^* - \beta_{k, i}| \leq \frac{\pi}{3\ell} \left( e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-\ell} \right) + \frac{\pi}{3k} \left( e^{-8(3i+\ell+1)/9} + 5(\sqrt{2})^{-k} \right), \]
which implies that
\[ \pi \left( \frac{6i + 2k + 3}{3\ell} - \frac{6j + 2\ell + 3}{3k} \right) = \beta_{\ell, j} - \beta_{k, i} \leq \frac{\pi}{3\ell} \left( e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-\ell} \right) + \frac{\pi}{3k} \left( e^{-8(3i+\ell+1)/9} + 5(\sqrt{2})^{-k} \right). \]
Rearranging gives us
\[ \frac{\pi}{3\ell} \left( 6j + 2\ell + 3 - e^{-8(3i+\ell+1)/9} - 5(\sqrt{2})^{-\ell} \right) \leq \frac{(\pi/3)(6i + 2k + 3) - e^{-8(3j+\ell+1)/9} - 5(\sqrt{2})^{-\ell}}{\ell}, \]
so
\[ \frac{\ell}{k} \leq \frac{6j + 2\ell + 3 + e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-\ell}}{6i + 2k + 3 - e^{-8(3i+\ell+1)/9} - 5(\sqrt{2})^{-k}}. \]
Moreover, by Lemma 2.3, we know that, since \( \beta_{k, i} \neq \beta_{\ell, j} \),
\[ \frac{\ell}{k} > \frac{6j + 2\ell + 3}{6(i - 1) + 2\ell + 9} = \frac{6j + 2\ell + 3}{6i + 2k + 3} > \frac{6j + 2\ell + 3 - e^{-8(3j+\ell+1)/9} - 5(\sqrt{2})^{-\ell}}{6i + 2k + 3 + e^{-8(3i+\ell+1)/9} + 5(\sqrt{2})^{-k}}, \]
so the lower bound of (2.1) holds.

Now suppose that \( \beta_{k, i} > \beta_{\ell, j} \). We must have that \( \beta_{k, i}^* \leq \beta_{\ell, j}^* \). Similarly, we have
\[ \beta_{k, i} - \beta_{\ell, j} \leq |\beta_{\ell, j} - \beta_{\ell, j}^*| + |\beta_{k, i} - \beta_{k, i}^*|. \]
By [5, Lemma 3.5], we know that
\[ |\beta_{\ell, j} - \beta_{\ell, j}^*| + |\beta_{k, i}^* - \beta_{k, i}| \leq \frac{\pi}{3\ell} \left( e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-\ell} \right) + \frac{\pi}{3k} \left( e^{-8(3i+\ell+1)/9} + 5(\sqrt{2})^{-k} \right), \]
which implies that
\[ \pi \left( \frac{6i + 2k + 3}{3\ell} - \frac{6j + 2\ell + 3}{3k} \right) = \beta_{k, i} - \beta_{\ell, j} \leq \frac{\pi}{3\ell} \left( e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-\ell} \right) + \frac{\pi}{3k} \left( e^{-8(3i+\ell+1)/9} + 5(\sqrt{2})^{-k} \right). \]
Rearranging gives us
\[ \frac{(\pi/3)(6j + 2\ell + 3 - e^{-8(3j+\ell+1)/9} - 5(\sqrt{2})^{-\ell})}{\ell} \leq \frac{(\pi/3)(6i + 2k + 3 + e^{-8(3i+\ell+1)/9} + 5(\sqrt{2})^{-k})}{k}. \]
Proof. Some lemmas to help us with this task. Since

\[
\frac{6j + 2\ell + 3 - e^{-8(3j+\ell+1)/9} - 5(\sqrt{2})^{-\ell}}{6i + 2\overline{k} + 3 + e^{-8(3j+\overline{k}+1)/9} + 5(\sqrt{2})^{-\overline{k}}} \leq \frac{\ell}{k},
\]

Moreover, by Lemma 2.3, since \( \beta_{k,i} > \beta_{\ell,j} \),

\[
\frac{\ell}{k} < \frac{6j + 2\ell + 3}{6i + 2\overline{k} + 3} < \frac{6j + 2\ell + 3 + e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-\ell}}{6i + 2\overline{k} + 3 + e^{-8(3j+\overline{k}+1)/9} + 5(\sqrt{2})^{-\overline{k}}},
\]

so the upper bound of (2.1) holds.

Finally, suppose \( \beta_{k,i} = \beta_{\ell,j} \). Then

\[
\frac{6i + 2\overline{k} + 3}{3k} = \frac{6j + 2\ell + 3}{3\ell},
\]

so

\[
\frac{\ell}{k} = \frac{6j + 2\ell + 3}{6i + 2\overline{k} + 3},
\]

and

\[
\frac{6j + 2\ell + 3 - e^{-8(3j+\ell+1)/9} - 5(\sqrt{2})^{-\ell}}{6i + 2\overline{k} + 3 + e^{-8(3j+\overline{k}+1)/9} + 5(\sqrt{2})^{-\overline{k}}} < \frac{6j + 2\ell + 3}{6i + 2\overline{k} + 3} = \frac{\ell}{k}
\]

\[
< \frac{6j + 2\ell + 3 + e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-\ell}}{6i + 2\overline{k} + 3 - e^{-8(3j+\overline{k}+1)/9} - 5(\sqrt{2})^{-\overline{k}}},
\]

so the lemma holds as desired. \( \square \)

We now prove a lemma that is related to [5, Lemma 2.1a] but uses \( \beta_{k,i} \). This will help us prove Lemma 2.6.

**Lemma 2.5.** If \( \beta_{k,i} > \beta_{\ell,j} \), for any \( 0 \leq w \leq \min(i, j) \), we have that \( \beta_{k,i-w} > \beta_{\ell,j-w} \).

**Proof.** Using Definition 1.3, we have that

\[
\beta_{k,i-w} - \beta_{\ell,j-w} = \pi \left( \frac{2}{3} - \frac{6(i-w) + 2\overline{k} + 3}{3k} \right) - \pi \left( \frac{2}{3} - \frac{6(j-w) + 2\ell + 3}{3\ell} \right)
\]

\[
= 2\pi \left( \frac{1}{k} - \frac{1}{\ell} \right) + \pi \left( \frac{6j + 2\ell + 3}{3\ell} - \frac{6i + 2\overline{k} + 3}{3k} \right)
\]

\[
= 2\pi \left( \frac{1}{k} - \frac{1}{\ell} \right) + \beta_{k,i} - \beta_{\ell,j}
\]

\[
> 0,
\]

since \( \ell > k \) and \( w \geq 0 \). \( \square \)

In Section 5, we examine the interval between consecutive zeros \( \beta_{k,i}, \beta_{k,i+1} \) and \( \beta_{i,j}^*, \beta_{i,j+1}^* \) to see if there is a zero \( \beta_{\ell,j} \) or \( \beta_{l,j}^* \) that lies in these intervals, respectively. We develop some lemmas to help us with this task.

**Lemma 2.6.** If \( \beta_{k,i} > \beta_{\ell,j} > \beta_{k,i+1} \), then \( j \geq i \); furthermore, if \( \ell \leq k \), then \( j > i \).
Proof. Using Definition 1.3, we have that
\[
\beta_\ell,0 - \beta_{k,0} = \pi \left( \frac{2\ell + 3}{3k} - \frac{2\ell + 3}{3\ell} \right) = \pi \left( \frac{\ell/k - (2\ell + 3)/(2\ell + 3)}{3\ell/(2\ell + 3)} \right)
\]
and
\[
\beta_\ell,0 - \beta_{k,1} = \pi \left( \frac{2\ell + 9}{3k} - \frac{2\ell + 3}{3\ell} \right) = \pi \left( \frac{\ell/k - (2\ell + 3)/(2\ell + 9)}{3\ell/(2\ell + 9)} \right).
\]

There are two cases. If \(\ell > k\), then
\[
\frac{\ell}{k} > 1 \geq \frac{2\ell + 3}{2\ell + 9}, \quad (2.2)
\]
since there are only finitely many possible values for \(\ell\) and \(k\), and \(\frac{7}{9}\) is the largest ratio of all of the possible values. Thus, \(\beta_\ell,0 > \beta_{k,1}\). We show that \(j \geq i\). Suppose on the contrary that \(j < i\). By Lemma 2.5, if \(\beta_{k,i} > \beta_{\ell,j} > \beta_{k,i+1}\), then \(\beta_{k,i-j} > \beta_\ell,0\). But then \(\beta_{k,i-j} > \beta_\ell,0 > \beta_{k,1}\), which is a contradiction, since for \(i - j \geq 1\), \(\beta_{k,1} \geq \beta_{k,i-j}\) by Definition 1.3. Thus, \(j \geq i\) in this case, as desired.

Now, note that, if \(\ell \leq k\), then
\[
\frac{\ell}{k} > 1 \geq \frac{2\ell + 3}{2\ell + 3},
\]
so \(\beta_\ell,0 > \beta_{k,0}\). We show that \(j > i\). Suppose on the contrary that \(j \leq i\). By Lemma 2.5, if \(\beta_{k,i} > \beta_{\ell,j} > \beta_{k,i+1}\), then \(\beta_{k,i-j} > \beta_\ell,0\). But then \(\beta_{k,i-j} > \beta_\ell,0 > \beta_{k,0}\), which is a contradiction, since for \(i - j \geq 0\), \(\beta_{k,0} \geq \beta_{k,i-j}\) by Definition 1.3. Thus, \(j > i\) in this case, as desired.

Note that (2.2) holds for all \(\ell\) and \(k\). Thus, we have the following corollary.

**Corollary 2.7.** If \(\ell > k \geq 24\) with \(k \neq 26\), then \(\beta_\ell,0 > \beta_{k,1}\).

We require the condition that \(k \geq 24\) and \(k \neq 26\), as otherwise there is only one non-trivial zero of \(F_k(\theta)\).

A similar result applies for the zeros of \(F_k(\theta)\) and \(F_\ell(\theta)\).

**Lemma 2.8.** If \(\ell > k \geq 24\) with \(k \neq 26\), then \(\beta^*_{\ell,0} > \beta^*_{k,1}\).

**Proof.** Suppose for contradiction that \(\beta^*_{\ell,0} \leq \beta^*_{k,1}\). As a result, \(\ell/k \in I_{k,1,\ell,0}\) by Lemma 2.4 and Corollary 2.7. Then,
\[
\frac{\ell}{k} \leq \frac{6 \cdot 0 + 2\ell + 3 + e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-\ell}}{6 \cdot 1 + 2\ell + 3 - e^{-8(3j+\ell+1)/9} - 5(\sqrt{2})^{-\ell}} < \frac{2\ell + 3 + e^{-8/9} + 5(\sqrt{2})^{-24}}{2\ell + 9 - e^{-32/9} - 5(\sqrt{2})^{-24}} < \frac{7 + 0.45}{9 - 0.15} < 1,
\]
using the bounds
\[
e^{-8/9} + 5(\sqrt{2})^{-24} < 0.45 \quad \text{and} \quad e^{-32/9} + 5(\sqrt{2})^{-24} < 0.15.
\]
This contradicts the fact that \(\ell > k\), so \(\beta^*_{\ell,0} > \beta^*_{k,1}\) as desired. \(\square\)
We now prove results similar to Lemmas 2.7 and 2.8 for the first zeros starting from \( \pi/2 \).

Here, we use the \( \alpha \) notation and the \( \beta \) notation for zeros to simplify calculations.

**Lemma 2.9.** If \( \ell > k \geq 24 \) with \( k \neq 26 \), then \( \alpha_{\ell,1} < \alpha_{k,2} \).

**Proof.** By Definition 1.4, we have that

\[
\alpha_{\ell,1} = \pi \left( \frac{1}{2} + \frac{1 + (\ell/2)}{\ell} \right) \quad \text{and} \quad \alpha_{k,2} = \pi \left( \frac{1}{2} + \frac{3 + (k/2)}{k} \right).
\]

Note that

\[
\frac{1 + (\ell/2)}{3 + (k/2)} \leq \frac{2}{3} < 1 < \frac{\ell}{k}.
\]

which implies that

\[
\frac{1 + (\ell/2)}{\ell} < \frac{3 + (k/2)}{k},
\]

and thus \( \alpha_{\ell,1} < \alpha_{k,2} \), as desired. \( \square \)

**Lemma 2.10.** If \( \ell > k \geq 24 \) with \( k \neq 26 \), then \( \alpha_{\ell,1}^* < \alpha_{k,2}^* \).

**Proof.** First, note that \( \alpha_{\ell,1} = \beta_{\ell,n_\ell-1}, \alpha_{k,2} = \beta_{k,n_k-2}, \alpha_{\ell,1}^* = \beta_{\ell,n_\ell-1}^*, \) and \( \alpha_{k,2}^* = \beta_{k,n_k-2}^* \).

Thus, we know that \( \beta_{\ell,n_\ell-1} < \beta_{k,n_k-2} \) from Lemma 2.9. We wish to show that \( \beta_{\ell,n_\ell-1}^* < \beta_{k,n_k-2}^* \).

Suppose on the contrary that \( \beta_{\ell,n_\ell-1}^* \geq \beta_{k,n_k-2}^* \). Then by Lemma 2.4, \( \ell/k \) is in \( I_{k,n_k-2,\ell,n_\ell-1} \). Thus,

\[
\frac{\ell}{k} \geq \frac{6(n_\ell - 1) + 2\ell + 3 - e^{-8(3(n_\ell-1)+\ell+1)/9} - 5(\sqrt{2})^{-\ell}}{6(n_k - 2) + 2k + 3 + e^{-8(3(n_k-2)+k+1)/9} + 5(\sqrt{2})^{-k}}.
\]

Since \( \alpha_{\ell,1} = \beta_{\ell,n_\ell-1} \), by Definition 1.4 and Definition 1.3,

\[
\pi \left( \frac{1}{2} + \frac{1 + (\ell/2)}{\ell} \right) = \pi \left( \frac{2}{3} - \frac{6(n_\ell - 1) + 2\ell + 3}{3\ell} \right),
\]

so

\[
6(n_\ell - 1) + 2\ell + 3 = \ell/2 - 3(1 + (\ell/2)).
\]

Similarly, since \( \alpha_{k,2} = \beta_{k,n_k-2} \), we have that

\[
\pi \left( \frac{1}{2} + \frac{3 + (k/2)}{\ell} \right) = \pi \left( \frac{2}{3} - \frac{6(n_k - 2) + 2k + 3}{3k} \right),
\]

so

\[
6(n_k - 2) + 2k + 3 = k/2 - 3(3 + (k/2)).
\]

We conclude that

\[
\frac{\ell}{k} > \frac{\ell/2 - 3(1 + (\ell/2)) - e^{-8(3(n_\ell-1)+\ell+1)/9} - 5(\sqrt{2})^{-\ell}}{k/2 - 3(3 + (k/2)) + e^{-8(3(n_k-2)+k+1)/9} + 5(\sqrt{2})^{-k}} > \frac{\ell/2 - 3(1 + (\ell/2)) - e^{-8/9} - 5(\sqrt{2})^{-24}}{k/2 - 3(3 + (k/2)) + e^{-8/9} + 5(\sqrt{2})^{-24}} > \frac{\ell/2 - 3(1 + (\ell/2)) - 0.45}{k/2 - 3(3 + (k/2)) + 0.45}.
\]

Proof. Since $\beta_k < \ell$, we are done.

Pay extra attention to non-trivial zeros near the endpoints of the interval $(\pi/2, 2\pi/3)$; this sort of treatment is necessary for future proofs.

Proof of Proposition 1.5. The distance between every consecutive pair of zeros of $\cos(k\theta/2)$ is $2\pi/k$. Likewise, for $\cos(\ell\theta/2)$, the distance between every pair of zeros is $2\pi/\ell$. Since $k < \ell$, $2\pi/\ell < 2\pi/k$. Hence the distance between any two consecutive zeros of $\cos(\ell\theta/2)$ is less than the distance between any two consecutive zeros of $\cos(k\theta/2)$. Moreover, note that by Corollary 2.7, we have that $\beta_{\ell,0} > \beta_{k,1}$, and by Lemma 2.9, we have that $\alpha_{\ell,1} < \alpha_{k,2}$, so we are done.

We now combine the results of Lemmas 2.3, 2.4, and 2.6 to get a sufficient condition for when $\beta_{k,i} > \beta_{\ell,j} > \beta_{k,i+1}$.

Lemma 2.11. Suppose $\ell > k \geq 24$. If $\ell/k$ lies in the gap between $I_{k,i+1,\ell,j}$ and $I_{k,i,\ell,j}$, or in other words, if

$$\frac{6j + 2\ell + 3 + e^{-8(3j+7+1)/9} + 5(\sqrt{2})^{-\ell}}{6i + 2\ell + 3 + e^{-8(3j+7+1)/9} + 5(\sqrt{2})^{-k}} < \frac{\ell}{k} < \frac{6j + 2\ell + 3 + e^{-8(3j+7+1)/9} + 5(\sqrt{2})^{-\ell}}{6i + 2\ell + 3 + e^{-8(3j+7+1)/9} + 5(\sqrt{2})^{-k}},$$

then $\beta_{k,i}^* > \beta_{\ell,j}^* > \beta_{k,i+1}^*$.

Proof. Since

$$\frac{6j + 2\ell + 3}{6i + 2\ell + 9} < \frac{6j + 2\ell + 3 + e^{-8(3j+7+1)/9} + 5(\sqrt{2})^{-\ell}}{6i + 2\ell + 9 + e^{-8(3j+7+1)/9} + 5(\sqrt{2})^{-k}}$$

and

$$\frac{6j + 2\ell + 3}{6i + 2\ell + 3} > \frac{6j + 2\ell + 3 + e^{-8(3j+7+1)/9} + 5(\sqrt{2})^{-\ell}}{6i + 2\ell + 3 + e^{-8(3j+7+1)/9} + 5(\sqrt{2})^{-k}},$$

by Lemma 2.3, we have that $\beta_{k,i} > \beta_{\ell,j} > \beta_{k,i+1}$.

Moreover, by Lemma 2.4, since $\ell/k \notin I_{k,i,\ell,j}$ and $\ell/k \notin I_{k,i+1,\ell,j}$, we conclude that $(\beta_{k,i} - \beta_{\ell,j})(\beta_{k,i}^* - \beta_{\ell,j}^*) > 0$ and $(\beta_{k,i+1} - \beta_{\ell,j})(\beta_{k,i+1}^* - \beta_{\ell,j}^*) > 0$. Since $\beta_{k,i} > \beta_{\ell,j} > \beta_{k,i+1}$, we get that $\beta_{k,i}^* > \beta_{\ell,j}^* > \beta_{k,i+1}^*$.
as desired.

We now check that the gap stated in the lemma is non-empty; in other words, we show that
\[
\frac{6j + 2\ell + 3 + e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-\ell}}{6i + 2\ell + 9 - e^{-8(3i+\ell+1)/9} - 5(\sqrt{2})^{-\ell}} < \frac{6j + 2\ell + 3 - e^{-8(3j+\ell+1)/9} - 5(\sqrt{2})^{-\ell}}{6i + 2\ell + 9 + e^{-8(3i+\ell+1)/9} + 5(\sqrt{2})^{-\ell}}.
\]
Note that
\[
e^{-8/9} + 5(\sqrt{2})^{-24} < 0.45 \quad \text{and} \quad e^{-32/9} + 5(\sqrt{2})^{-24} < 0.15.
\]
Since \(\ell > k \geq 24\), we have
\[
(6i + 2\ell + 9 - e^{-8(3i+\ell+1)/9} - 5(\sqrt{2})^{-k})(6j + 2\ell + 3 - e^{-8(3j+\ell+1)/9} - 5(\sqrt{2})^{-\ell})
- (6i + 2\ell + 3 + e^{-8(3i+\ell+1)/9} + 5(\sqrt{2})^{-k})(6j + 2\ell + 3 + e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-\ell})
= (6i + 2\ell + 9)(6j + 2\ell + 3) - (6i + 2\ell + 3)(e^{-8(3i+\ell+1)/9} + 5(\sqrt{2})^{-k})
- (6i + 2\ell + 9)(e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-\ell})
+ (e^{-8(3i+\ell+1)/9} + 5(\sqrt{2})^{-k})(e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-\ell})
- (6i + 2\ell + 3)(6j + 2\ell + 3 - e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-k})
- (6i + 2\ell + 3)(e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-\ell})
- (e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-k})(e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-\ell})
> 6(6j + 2\ell + 3) - 0.15(6j + 2\ell + 3) - 0.45(6i + 2\ell + 9)
- 0.45(6i + 2\ell + 3) - 0.45(6i + 2\ell + 3) - 0.45^2
= 32.4j - 5.4i + 10.5975 + 12\ell - 1.8k
> 0
\]
using the fact that \(j \geq i\) from Lemma 2.6 and that \(0 \leq \ell, \ell \leq 2.\) \(\square\)

3. The case when \(\ell/k\) is large

When \(\ell/k\) is large, the distance between any two consecutive zeros of \(\cos(\ell \theta/2)\) is much smaller than the distance between any two consecutive zeros of \(\cos(k \theta/2)\). This serves as a starting point for the following proposition.

**Proposition 3.1.** Let \(\ell > k \geq 24\) be positive even integers with \(k \neq 26\) such that \(\ell/k > 1.16\). Then strictly between any two zeros of \(F_k(\theta)\) there exists a zero of \(F_\ell(\theta)\).

**Proof.** Since the zeros of \(\cos(k \theta/2)\) are spaced \(2\pi/k\) apart, we have that
\[
\beta_{k,i} - \beta_{k,i+1} = \frac{2\pi}{k} \quad \text{and} \quad \beta_{\ell,j} - \beta_{\ell,j+1} = \frac{2\pi}{\ell}.
\]
Additionally, by [5, Lemma 3.5], we see that
\[
|\beta_{k,i}^* - \beta_{k,i}| \leq \frac{\pi}{3k}(e^{-(8/9)(3i+\ell+1)} + 5(\sqrt{2})^{-\ell}) \leq \frac{\pi}{3k}(e^{-(8/9)} + 5(\sqrt{2})^{-k})
\]
and
\[ |\beta_{k,i+1}^n - \beta_{k,i+1}| \leq \frac{\pi}{3k} \left( e^{-\left(\frac{8}{9}\right)\left(3i+1\right) + \frac{\pi}{6}} + 5(\sqrt{2})^{-k} \right) \leq \frac{\pi}{3k} \left( e^{-\left(\frac{32}{9}\right)} + 5(\sqrt{2})^{-k} \right). \]

Therefore, by Lemma 2.2,
\[
\beta_{k,i}^n - \beta_{k,i+1}^n \geq \frac{2\pi}{k} - \frac{e^{-\frac{8}{9}} + 5(\sqrt{2})^{-k}}{3k} \pi - \frac{e^{-\frac{32}{9}} + 5(\sqrt{2})^{-k}}{3k} \pi \geq \frac{\pi}{3k} \left( 6 - e^{-\frac{8}{9}} - e^{-\frac{32}{9}} - 10(\sqrt{2})^{-24} \right).
\]

Also,
\[
\beta_{\ell,j}^n - \beta_{\ell,j+1}^n \leq |\beta_{\ell,j}^n - \beta_{\ell,j+1}| + |\beta_{\ell,j}^n - \beta_{\ell,j+1}| \leq \frac{e^{-\frac{8}{9}} + 5(\sqrt{2})^{-\ell}}{3\ell} \pi + \frac{e^{-\frac{32}{9}} + 5(\sqrt{2})^{-\ell}}{3\ell} \pi \leq \frac{\pi}{3\ell} \left( e^{-\frac{8}{9}} + e^{-\frac{32}{9}} + 6 + 10(\sqrt{2})^{-24} \right).
\]

Using that
\[
\frac{e^{-\frac{8}{9}} + e^{-\frac{32}{9}} + 6 + 10(\sqrt{2})^{-24}}{-e^{-\frac{8}{9}} - e^{-\frac{32}{9}} + 6 - 10(\sqrt{2})^{-24}} < 1.16 \quad \text{and} \quad \ell > 1.16k,
\]
we conclude that
\[
\frac{e^{-\frac{8}{9}} + e^{-\frac{32}{9}} + 6 + 10(\sqrt{2})^{-24}}{-e^{-\frac{8}{9}} - e^{-\frac{32}{9}} + 6 - 10(\sqrt{2})^{-24}} < \ell
\]

\[
\implies \frac{\pi}{3\ell} \left( e^{-\frac{8}{9}} + e^{-\frac{32}{9}} + 6 + 10(\sqrt{2})^{-24} \right) < \frac{\pi}{3k} \left( e^{-\frac{8}{9}} - e^{-\frac{32}{9}} + 6 - 10(\sqrt{2})^{-24} \right)
\]

\[
\implies \beta_{\ell,j}^n - \beta_{\ell,j+1}^n < \beta_{k,i}^n - \beta_{k,i+1}^n.
\]

Therefore, the distance between any two consecutive zeros of \(F_{\ell}(\theta)\) is strictly smaller than the distance between any two consecutive zeros of \(F_k(\theta)\). Combined with Lemma 2.8 and Lemma 2.10, we conclude that between any two zeros of \(F_k(\theta)\) there must be a zero of \(F_{\ell}(\theta)\). \(\square\)

4. **Zeros close to \(\pi/2\)**

As [8, Lemma 3.1], [6, Lemma 2.2], and [5, Lemma 3.5] suggest, the distance between a zero of \(\cos(k\theta/2)\) and its associated zero of \(F_k(\theta)\) is particularly small when the zero is close to \(\pi/2\). We refine their results.

**Lemma 4.1.** Let \(k \geq 304\) be an even integer. If \(\beta_{k,i} + \pi/(2k^2) \leq 60\pi/91\), then \(\beta_{k,i}^n - \beta_{k,i} < \pi/(2k^2)\).
Proof. We follow the same idea as in [8, Lemma 3.1], [6, Lemma 2.2], and [5, Lemma 3.5]. First, we see that for \( \theta \in (\pi/2, 60\pi/91) \),

\[
2 \sin \left( \frac{\pi}{4k} \right) - |T_k(\theta)| \geq 2 \sin \left( \frac{\pi}{4k} \right) - \left( \frac{1}{2 \cos(\theta/2)} \right)^k - |R_k(\theta)| \\
> \frac{\pi}{4k} - \left( \frac{1}{2 \cos((1/2)(60\pi/91))} \right)^k - 5(\sqrt{2})^{-k} \\
> 0,
\]

by (1.2), \( k \geq 304 \), and the inequality \( 2 \sin(x) > x \) (since \( \pi/(4k) \) is sufficiently small for the inequality to hold).

Note that

\[
2 \sin \left( \frac{\pi}{4k} \right) = \left| 2 \cos \left( \frac{k}{2} \left( \beta_{k,i} + \frac{\pi}{2k^2} \right) \right) \right| = \left| 2 \cos \left( \frac{k}{2} \left( \beta_{k,i} - \frac{\pi}{2k^2} \right) \right) \right|,
\]

using the cosine addition formula and the fact that \( \beta_{k,i} \) is a zero of \( 2 \cos(k\theta/2) \). Thus,

\[
\left| 2 \cos \left( \frac{k}{2} \left( \beta_{k,i} + \frac{\pi}{2k^2} \right) \right) \right| = \left| 2 \cos \left( \frac{k}{2} \left( \beta_{k,i} - \frac{\pi}{2k^2} \right) \right) \right| > |T_k(\theta)|.
\]

This implies that

\[
F_k \left( \frac{k}{2} \left( \beta_{k,i} - \frac{\pi}{2k^2} \right) \right) \cdot F_k \left( \frac{k}{2} \left( \beta_{k,i} + \frac{\pi}{2k^2} \right) \right) < 0.
\]

Therefore, \( F_k(\theta) \) has a zero in the interval

\[
\left( \beta_{k,i} - \frac{\pi}{2k^2}, \beta_{k,i} + \frac{\pi}{2k^2} \right).
\]

By Definition 1.3 and Lemma 2.1 this gives us

\[
|\beta^*_{k,i} - \beta_{k,i}| < \frac{\pi}{2k^2}
\]
as desired. \( \square \)

With Lemma 4.1, we can show that the distance between any two consecutive zeros of \( F_k \) is much larger than the distance between any two consecutive zeros of \( F_\ell \).

PROPOSITION 4.2. Let \( \ell > k \geq 304 \) be positive even integers. In the interval \((\pi/2, 0.658\pi)\), strictly between any two zeros of \( F_k \) there exists a zero of \( F_\ell \).

Proof. It suffices to show the proposition for consecutive zeros of \( F_k \). We know that

\[
\beta_{k,i} - \beta_{k,i+1} = \frac{2\pi}{k},
\]

and by Lemma 4.1, for \( \beta_{k,i} \leq 60\pi/91 - \pi/(2k^2) \),

\[
|\beta^*_{k,i} - \beta_{k,i}| \leq \frac{\pi}{2k^2}.
\]
First, we show that if $\beta_{k,i}^* < 0.658\pi$, then $|\beta_{k,i}^* - \beta_{k,i}| \leq \pi/(2k^2)$. This is clear if $\beta_{k,i} \leq 60\pi/91 - \pi/(2k^2)$. It therefore suffices to show it is impossible that $\beta_{k,i} > 60\pi/91 - \pi/(2k^2)$ and $\beta_{k,i}^* < 0.658\pi$.

By Lemma 2.1, we have that $|\beta_{k,i}^* - \beta_{k,i}| \leq \pi/(3k)$. Therefore, if $\beta_{k,i} > 60\pi/91 - \pi/(2k^2)$, then $\beta_{k,i}^* > 60\pi/91 - \pi/(2k^2) - \pi/(3k) > 0.658\pi$, which is a contradiction. Thus, when $\beta_{k,i}^* < 0.658\pi$, we have that

$$|\beta_{k,i}^* - \beta_{k,i}| \leq \frac{\pi}{2k^2}. \quad (4.1)$$

Therefore, for $\beta_{k,i}^* < 0.658\pi$, and noting that $\beta_{k,i+1}^* < 0.658\pi$ as well by Lemma 2.2, we have the bound

$$\beta_{k,i} - \beta_{k,i+1}^* \geq |\beta_{k,i+1} - \beta_{k,i} - |\beta_{k,i+1}^* - \beta_{k,i+1}| - |\beta_{k,i}^* - \beta_{k,i}|$$

$$\geq \frac{2\pi}{k} - \frac{\pi}{2k^2} = \frac{2\pi}{k} - \frac{\pi}{k^2}.$$ \hspace{1cm} (4.2)

Similarly, if $\beta_{\ell,j}^* < 0.658\pi$, then

$$\beta_{\ell,j}^* - \beta_{\ell,j+1}^* \leq |\beta_{\ell,j+1} - \beta_{\ell,j}| + |\beta_{\ell,j+1} - \beta_{\ell,j} + |\beta_{\ell,j} - \beta_{\ell,j}^*|$$

$$\leq \frac{\pi}{2\ell^2} + \frac{2\pi}{\ell} + \frac{\pi}{2\ell^2} = \frac{2\pi}{\ell} + \frac{\pi}{\ell^2}.$$ \hspace{1cm} (4.3)

Recalling that $\ell \geq k + 2$,

$$4k^2 + 8k > 2k^2 + 4k + 4$$

$$\Rightarrow \frac{4}{k(k+2)} > \frac{1}{k^2} + \frac{1}{(k+2)^2}$$

$$\Rightarrow \frac{2\pi}{k} - \frac{\pi}{k^2} > \frac{2\pi}{\ell} + \frac{\pi}{\ell^2}$$

$$\Rightarrow \beta_{k,i}^* - \beta_{k,i+1}^* > \beta_{\ell,j}^* - \beta_{\ell,j+1}^*. \quad (4.4)$$

Therefore, in the interval $(\pi/2, 0.658\pi)$, the distance between any two consecutive zeros of $F_\ell$ is strictly smaller than the distance between any two consecutive zeros of $F_k$.

Finally, similar to the proof of Proposition 1.5, we need to take extra care with the zeros close to the endpoints of the interval $(\pi/2, 0.658\pi)$. More precisely, we show that there is a zero of $F_\ell$ in between the two consecutive zeros of $F_k$ that are closest to $0.658\pi$ while still being in the interval $(\pi/2, 0.658\pi)$. Let $\beta_{k,m}^*$ and $\beta_{k,m+1}^*$ be the two zeros of $F_k$ closest to $0.658\pi$ and let $\beta_{\ell,n+1}^*$ be the zero of $F_\ell$ closest to $\beta_{k,m+1}^*$ such that $\beta_{\ell,n+1}^* \leq \beta_{k,m+1}^*$. We note that the existences of $\beta_{k,m}^*$ and $\beta_{k,m+1}^*$ are guaranteed since $k$ is large. Thus we have $\beta_{\ell,n+1}^* \leq \beta_{k,m+1}^* < \beta_{k,m}^* < 0.658\pi$. We wish to show that $\beta_{\ell,n}^* < \beta_{k,m}^*$. We note that $\beta_{\ell,n}^*$ must exist by Lemma 2.8.

As shown above in (4.2),

$$\beta_{k,m}^* - \beta_{k,m+1}^* \geq \frac{2\pi}{k} - \frac{\pi}{k^2}. \quad (4.5)$$
Thus,
\[ \beta^*_{\ell,n+1} \leq 0.658\pi - (\beta^*_{k,m} - \beta^*_{k,m+1}) \leq 0.658\pi - \frac{2\pi}{k} + \frac{\pi}{k^2}. \]

Since \( \beta^*_{\ell,n+1} < 0.658\pi \), we also know that \( |\beta^*_{\ell,n+1} - \beta^*_{\ell,n}| \leq \pi/(2\ell^2) \) by (4.1). Using that \( \beta^*_{\ell,n} - \beta^*_{\ell,n+1} = 2\pi/\ell \),
\[ \beta^*_{\ell,n} = \beta^*_{\ell,n+1} + \frac{2\pi}{\ell} \leq \beta^*_{\ell,n+1} + \frac{\pi}{2\ell^2} + \frac{2\pi}{\ell} \leq 0.658\pi - \frac{2\pi}{k} + \frac{\pi}{k^2} + \frac{\pi}{2\ell^2} + \frac{2\pi}{\ell} . \]

Because \( \ell \geq k + 2 \geq 306 \), then \( -2\pi/k + \pi/k^2 + \pi/(2\ell^2) + 2\pi/\ell < 0 \). Thus, \( \beta^*_{\ell,n} < 0.658\pi \), implying that \( |\beta^*_{\ell,n} - \beta^*_{\ell,n}| < \pi/(2\ell^2) \) by Lemma 4.1 since \( 0.658\pi < 60\pi/91 - \pi/(2k^2) \) for \( k \geq 304 \). Applying the same strategy as in (4.3),
\[ \beta^*_{\ell,n} - \beta^*_{\ell,n+1} \leq \frac{2\pi}{\ell} + \frac{\pi}{\ell^2}. \] (4.6)

Using (4.4), we see that the right-hand side of (4.5) is always larger than the right-hand side of (4.6), so
\[ \beta^*_{k,m} - \beta^*_{k,m+1} > \beta^*_{\ell,n} - \beta^*_{\ell,n+1} . \]

Thus, \( \beta^*_{\ell,n} < \beta^*_{k,m} \) as desired because \( \beta^*_{\ell,n+1} \leq \beta^*_{k,m+1} \). We must then conclude that \( \beta^*_{\ell,n+1} < \beta^*_{k,m+1} \). In conclusion, in the interval \((\pi/2, 0.658\pi)\), there is a zero of \( F_\ell \) in between the two zeros of \( F_k \) that are closest to 0.658\pi. Note that the two zeros in the leftmost end of the interval are covered by Lemma 2.10. We can then conclude that, since the distance between any two consecutive zeros of \( F_\ell \) is strictly smaller than the distance between any two consecutive zeros of \( F_k \), the proposition is proved.

\[ \square \]

5. The case when \( \ell/k \) is small and zeros are far from \( \pi/2 \)

It remains to consider the case where \( \ell/k \) is small and the zeros are far from \( \pi/2 \). By Proposition 1.5, between any two consecutive zeros \( \beta_{k,i} \) and \( \beta_{k,i+1} \) of \( \cos(k\theta/2) \), there is a zero \( \beta_{\ell,j} \) of \( \cos(\ell\theta/2) \). We need to show that some \( \beta^*_{\ell} \) lies between \( \beta^*_{k,i} \) and \( \beta^*_{k,i+1} \), so if \( \beta^*_{\ell,j} \) does not lie between \( \beta^*_{k,i} \) and \( \beta^*_{k,i+1} \), we need to find another \( \beta^*_{\ell} \) that lies between \( \beta^*_{k,i} \) and \( \beta^*_{k,i+1} \). In Lemmas 5.2 and 5.3, we will show that if this happens, then either \( \beta^*_{\ell,j-1} \) or \( \beta^*_{\ell,j+1} \) lies between \( \beta^*_{k,i} \) and \( \beta^*_{k,i+1} \). In most cases, the lemmas in Section 2 suffice to prove this fact. However, in the case where \( \tilde{\ell} = \tilde{k} \) and \( j = i + 1 \), we use the methods in Section 4 in [8] and Section 4 in [5].

Our main goal in this section is to prove the following proposition.

**Proposition 5.1.** Suppose \( 1 < \ell/k \leq 1.16 \) with \( k \geq 28 \), and consider any index \( i \) such that \( i < k/52 \). Then strictly between \( \beta^*_{k,i} \) and \( \beta^*_{k,i+1} \) there is a zero of \( F_\ell(\theta) \).

We first prove Proposition 5.1 assuming Lemmas 5.2 and 5.3, then prove the lemmas.

**Lemma 5.2.** Suppose \( 1 < \ell/k \leq 1.16 \) with \( k \geq 28 \), and consider any index \( i \) such that \( i < k/52 \).
If \( \beta_{k,i} > \beta_{\ell,j} > \beta_{k,i+1} \) and \( \beta^*_{k,i} \geq \beta^*_{k,j} \), then \( \beta^*_{k,i} \geq \beta^*_{k,j} > \beta^*_{k,i+1} \).

**Lemma 5.3.** Suppose \( 1 < \ell/k \leq 1.16 \) with \( k \geq 28 \), and consider any index \( i \) such that \( i < k/52 \).
If \( \beta_{k,i} > \beta_{\ell,j} > \beta_{k,i+1} \) and \( \beta^*_{k,i} \leq \beta^*_{k,j} \), then \( \beta^*_{k,i} \geq \beta^*_{k,j} > \beta^*_{k,i+1} \).


Proof of Proposition 5.1. Let $i < k/52$. It follows by Proposition 1.5 that there exists a zero $\beta_{i,j}$ such that $\beta_{i,j} > \beta_{k,i} > \beta_{k,i+1}$. Moreover, by Lemma 2.2, $\beta_{k,i} > \beta_{k,i+1}^*$. If $\beta_{k,i} > \beta_{i,j}^*$, then $\beta_{k,i} > \beta_{k,i+1}^*$, we are done. By Lemma 5.2, if $\beta_{i,j}^* \geq \beta_{k,i}^*$, then $\beta_{k,i}^* > \beta_{k,i+1}^*$, and by Lemma 5.3, if $\beta_{k,i}^* \leq \beta_{k,i+1}^*$, then $\beta_{k,i}^* > \beta_{k,i+1}^*$. Thus we are done if we assume the proofs of these lemmas.

We prove Lemma 5.2 in Section 5.1 and Lemma 5.3 in Section 5.2.

Throughout this section, we use the bounds

\[ e^{-8/9} + 5(\sqrt{2})^{-28} < 0.45 \]

and

\[ e^{-56/9} + 5(\sqrt{2})^{-28} < e^{-32/9} + 5(\sqrt{2})^{-28} < 0.15. \]

5.1. Proof of Lemma 5.2

Proof of Lemma 5.2. Since $\beta_{i,j}^* \geq \beta_{k,i}^*$ and $\beta_{k,i} > \beta_{i,j}$, it follows from Lemma 2.4 that

\[ \frac{\ell}{k} \in I_{k,i,\ell,j}. \]

Recall from (2.1) that

\[ I_{k,i,\ell,j} = \left[ \frac{6j + 2\ell + 3 - e^{-8(3j+\ell+1)/9} - 5(\sqrt{2})^{-\ell}}{6i + 2k + 3 + e^{-8(3i+k+1)/9} + 5(\sqrt{2})^{-k}}, \right. \]

\[ \left. \frac{6j + 2\ell + 3 + e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-\ell}}{6i + 2k + 3 - e^{-8(3i+k+1)/9} - 5(\sqrt{2})^{-k}} \right]. \]

We wish to show that $\beta_{k,i}^* > \beta_{k,i+1}^*$. By Lemma 5.4, given below, we have that

\[ \frac{6j + 2\ell + 9 + e^{-8(3j+\ell+4)/9} + 5(\sqrt{2})^{-\ell}}{6i + 2k + 9 - e^{-8(3i+k+4)/9} - 5(\sqrt{2})^{-k}} < \frac{6j + 2\ell + 3 - e^{-8(3j+\ell+1)/9} - 5(\sqrt{2})^{-\ell}}{6i + 2k + 3 + e^{-8(3i+k+1)/9} + 5(\sqrt{2})^{-k}}. \]

Likewise, Lemma 5.5, given below, implies that

\[ \frac{6j + 2\ell + 9 - e^{-8(3j+\ell+4)/9} - 5(\sqrt{2})^{-\ell}}{6i + 2k + 9 - e^{-8(3i+k+4)/9} + 5(\sqrt{2})^{-k}} > \frac{6j + 2\ell + 3 + e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-\ell}}{6i + 2k + 3 - e^{-8(3i+k+1)/9} - 5(\sqrt{2})^{-k}}. \]

Since the upper bound of (5.1) is the same as the lower bound of $I_{k,i,\ell,j}$ and the lower bound of (5.2) is the same as the upper bound of $I_{k,i,\ell,j}$, we see that if $\ell/k \in I_{k,i,\ell,j}$, then

\[ \frac{6j + 2\ell + 9 + e^{-8(3j+\ell+4)/9} + 5(\sqrt{2})^{-\ell}}{6i + 2k + 9 - e^{-8(3i+k+4)/9} - 5(\sqrt{2})^{-k}} < \frac{\ell}{k} < \frac{6j + 2\ell + 9 - e^{-8(3j+\ell+4)/9} - 5(\sqrt{2})^{-\ell}}{6i + 2k + 3 + e^{-8(3i+k+1)/9} + 5(\sqrt{2})^{-k}}. \]

Thus, we are done by Lemma 2.11.
We now prove the two lemmas assumed in the previous proof.

**Lemma 5.4.** Suppose \(1 < \ell/k \leq 1.16\) with \(k \geq 28\), and suppose \(i < k/52\). If \(\beta_{k,i} > \beta_{\ell,j} > \beta_{k,i+1}\) and \(\beta_{k,i}^* \geq \beta_{\ell,j}^*\), then (5.1) holds.

**Remark.** The condition \(\beta_{k,i}^* \geq \beta_{\ell,j}^*\) is not necessary in the proof, but we state it here since it is in the assumptions of Lemma 5.2. We may replace the condition \(\beta_{k,i} > \beta_{\ell,j} > \beta_{k,i+1}\) with the condition \(j > i \geq 0\), and the lemma still holds. For Case 2 in the proof of this lemma, we may also replace the condition \(\beta_{k,i} > \beta_{\ell,j} > \beta_{k,i+1}\) with the condition \(j \geq i \geq 0\), and the case still holds.

**Proof of Lemma 5.4.** By Lemma 2.6, \(j \geq i\). Thus,

\[
(6j + 2\tilde{\ell} + 3 - e^{-8(3j+\tilde{\ell}+1)/9} - 5(\sqrt{2})^{-\ell})(6i + 2\tilde{k} + 9 - e^{-8(3i+\tilde{k}+4)/9} - 5(\sqrt{2})^{-k})
- (6j + 2\tilde{\ell} + 9 + e^{-8(3j+\tilde{\ell}+4)/9} + 5(\sqrt{2})^{-\ell})(6i + 2\tilde{k} + 3 + e^{-8(3i+\tilde{k}+1)/9} + 5(\sqrt{2})^{-k})
= (6j + 2\tilde{\ell} + 3)(6i + 2\tilde{k} + 9) - (6i + 2\tilde{k} + 9)(e^{-8(3j+\tilde{\ell}+1)/9} + 5(\sqrt{2})^{-\ell})
- (6j + 2\tilde{\ell} + 3)(e^{-8(3i+\tilde{k}+4)/9} + 5(\sqrt{2})^{-k})
+ (e^{-8(3i+\tilde{k}+1)/9} + 5(\sqrt{2})^{-k})(e^{-8(3j+\tilde{\ell}+1)/9} + 5(\sqrt{2})^{-\ell})
- (6j + 2\tilde{\ell} + 9)(6i + 2\tilde{k} + 3) - (6i + 2\tilde{k} + 3)(e^{-8(3j+\tilde{\ell}+4)/9} + 5(\sqrt{2})^{-\ell})
- (6j + 2\tilde{\ell} + 9)(e^{-8(3i+\tilde{k}+1)/9} + 5(\sqrt{2})^{-k})
- (e^{-8(3i+\tilde{k}+1)/9} + 5(\sqrt{2})^{-k})(e^{-8(3j+\tilde{\ell}+4)/9} + 5(\sqrt{2})^{-\ell})
> 6(6j + 2\tilde{\ell} - 6i - 2\tilde{k}) - 0.15 \cdot 0.45 - 6 \cdot 0.15(j - i) - 6 \cdot 0.45(j - i)
- (6i + 2\tilde{k} + 3)(e^{-8(3j+\tilde{\ell}+4)/9} + 5(\sqrt{2})^{-\ell}) - (6i + 2\tilde{k} + 9)(e^{-8(3j+\tilde{\ell}+1)/9} + 5(\sqrt{2})^{-\ell})
- (6i + 2\tilde{k} + 3)(e^{-8(3i+\tilde{k}+4)/9} + 5(\sqrt{2})^{-k}) - (6i + 2\tilde{k} + 9)(e^{-8(3i+\tilde{k}+1)/9} + 5(\sqrt{2})^{-k})
= 32.4(j - i) + 12\tilde{\ell} - 12\tilde{k} - 0.0675 - (6i + 2\tilde{k} + 3)(e^{-8(3j+\tilde{\ell}+4)/9} + 5(\sqrt{2})^{-\ell})
- (6i + 2\tilde{k} + 9)(e^{-8(3j+\tilde{\ell}+1)/9} + 5(\sqrt{2})^{-\ell}) - (6i + 2\tilde{k} + 3)(e^{-8(3i+\tilde{k}+4)/9} + 5(\sqrt{2})^{-k})
- (6i + 2\tilde{k} + 9)(e^{-8(3i+\tilde{k}+1)/9} + 5(\sqrt{2})^{-k})
.

We consider two cases, each with two subcases, to show that the last expression is positive.

**Case 1:** \(\tilde{\ell} \leq \tilde{k}\). By Lemma 2.6, we know that \(j > i\), so combined with the fact that \(\ell > k\), we have

\[
32.4(j - i) + 12\tilde{\ell} - 12\tilde{k} - 0.0675 - (6i + 2\tilde{k} + 3)(e^{-8(3j+\tilde{\ell}+4)/9} + 5(\sqrt{2})^{-\ell})
- (6i + 2\tilde{k} + 9)(e^{-8(3j+\tilde{\ell}+1)/9} + 5(\sqrt{2})^{-\ell}) - (6i + 2\tilde{k} + 3)(e^{-8(3i+\tilde{k}+4)/9} + 5(\sqrt{2})^{-k})
- (6i + 2\tilde{k} + 9)(e^{-8(3i+\tilde{k}+1)/9} + 5(\sqrt{2})^{-k})
.
\]
so the desired positivity has been shown in this case, so (5.1) holds.

Since \( 93 - 3e^{-32/9} - 9e^{-8/9} > 0 \), we check all three to find that \((2\overline{k} + 3)(e^{-8(\overline{k}+4)/9}) + (2\overline{k} + 9)(e^{-8(\overline{k}+1)/9}) \) is maximized when \( \overline{k} = 0 \). Then

\[
7.93 - (2\overline{k} + 3)(e^{-8(\overline{k}+4)/9}) - (2\overline{k} + 9)(e^{-8(\overline{k}+1)/9}) \geq 7.93 - 3e^{-32/9} - 9e^{-8/9} > 0,
\]

so the desired positivity has been shown in this case, so (5.1) holds.

**Subcase 1.2:** \( i > 0 \). This is the same as \( i \geq 1 \). Then since \( j > i \) and \( k \leq 52i \), we have

\[
8.3325 - (6i + 2\overline{k} + 3)(e^{-8(3j+4)/9} + 5(\sqrt{2})^{-k})
\]

\[
- (6i + 2\overline{k} + 9)(e^{-8(3j+1)/9} + 5(\sqrt{2})^{-k}) - (6i + 2\overline{k} + 3)(e^{-8(3i+\overline{k}+4)/9} + 5(\sqrt{2})^{-k})
\]

\[
- (6i + 2\overline{k} + 9)(e^{-8(3i+\overline{k}+1)/9} + 5(\sqrt{2})^{-k})
\]

\[
> 8.3325 - (6i + 7)(e^{-8(3i+1)/9} + 5(\sqrt{2})^{-52i}) - (6i + 13)(e^{-8(3i+1)/9} + 5(\sqrt{2})^{-52i})
\]

\[
- (6i + 7)(e^{-8(3i+1)/9} + 5(\sqrt{2})^{-52i}) - (6i + 13)(e^{-8(3i+1)/9} + 5(\sqrt{2})^{-52i})
\]

\[
= 8.3325 - 8(3i + 5)(e^{-8(3i+1)/9} + 5(\sqrt{2})^{-52i}).
\]

Note that the derivative of \( 8(3i + 5)(e^{-8(3i+1)/9} + 5(\sqrt{2})^{-52i}) \) with respect to \( i \) is

\[
8(e^{-8(3i+1)/9}(3 - 24/5)(3i + 5)) + 5(\sqrt{2})^{-52i}(3 - 52 \ln(\sqrt{2})(3i + 5))
\]

which is negative for \( i \geq 1 \). Moreover,

\[
8.3325 - 8(3 \cdot 1 + 5)(e^{-8(3\cdot 1+1)/9} + 5(\sqrt{2})^{-52\cdot 1}) > 0.
\]
so
\[ 8.3325 - 8(3i + 5)(e^{-8(3i+1)/9} + 5(\sqrt{2})^{-52i}) > 0 \quad (5.3) \]
for all \( i \geq 1 \), as desired.

We have thus proven that (5.1) holds in all cases where \( k \geq \ell \).

**Case 2:** \( k < \ell \). By Lemma 2.6, \( j \geq i \). Then since \( \ell > k \),

\[
\begin{align*}
32.4(j - i) + 12\ell - 12k - 0.0675 & - (6i + 2\ell + 3)(e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-\ell}) \\
& - (6i + 2\ell + 9)(e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-\ell}) \\
& - (6i + 2\ell + 3)(e^{-8(3i+\ell+4)/9} + 5(\sqrt{2})^{-k}) \\
& - (6i + 2\ell + 9)(e^{-8(3i+\ell+1)/9} + 5(\sqrt{2})^{-k}) \\
& > 11.9 - (6i + 2\ell + 3)(e^{-8(3i+\ell+4)/9} + 5(\sqrt{2})^{-k}) \\
& - (6i + 2\ell + 9)(e^{-8(3i+\ell+1)/9} + 5(\sqrt{2})^{-k}) \\
& - (6i + 2\ell + 3)(e^{-8(3i+\ell+4)/9} + 5(\sqrt{2})^{-k}) \\
& - (6i + 2\ell + 9)(e^{-8(3i+\ell+1)/9} + 5(\sqrt{2})^{-k}). \\
& (5.4)
\end{align*}
\]

We again divide this into two subcases to show positivity of this final term.

**Subcase 2.1:** \( i = 0 \). Then since \( k \geq 28 \),

\[
(5.4) \geq 11.9 - (2\ell + 3)(e^{-8(\ell+4)/9} + 5(\sqrt{2})^{-28}) - (2\ell + 9)(e^{-8(\ell+1)/9} + 5(\sqrt{2})^{-28}) \\
- (2\ell + 3)(e^{-32/9} + 5(\sqrt{2})^{-28}) - (2\ell + 9)(e^{-8/9} + 5(\sqrt{2})^{-28}).
\]

There are only two values for \( \ell \); 1 and 2. For both values of \( \ell \), the last expression is greater than 0, as desired.

**Subcase 2.2:** \( i > 0 \). This is equivalent to \( i \geq 1 \). Then since \( i < k/52 \), or \( k > 52i \),

\[
(5.4) > 11.9 - (6i + 7)(e^{-8(3i+1)/9} + 5(\sqrt{2})^{-52i}) - (6i + 13)(e^{-8(3i+1)/9} + 5(\sqrt{2})^{-52i}) \\
- (6i + 7)(e^{-8(3i+1)/9} + 5(\sqrt{2})^{-52i}) - (6i + 13)(e^{-8(3i+1)/9} + 5(\sqrt{2})^{-52i}) \\
= 11.9 - 8(3i + 5)(e^{-8(3i+1)/9} + 5(\sqrt{2})^{-52i}) \\
> 0,
\]
by (5.3).

Thus, we have shown that (5.1) holds in all cases and the proof of Lemma 5.4 is complete.

We now prove the second lemma required for Lemma 5.2.

**Lemma 5.5.** Suppose \( 1 < \ell/k \leq 1.16 \) with \( k \geq 28 \), and consider any \( i \) such that \( i < k/52 \). If \( \beta_{k,i} > \beta_{k,j} > \beta_{k,i+1} \) and \( \beta_{k,i}^* > \beta_{k,j}^* \), then (5.2) holds.
Remark. The condition $\beta^*_{k,i,j} \geq \beta^*_{k,i}$ is not necessary in the proof, but we state it here since it is in the assumptions of Lemma 5.2. Moreover, we may replace the condition $\beta^*_{k,i,j} > \beta^*_{k,i} > \beta^*_{k,i+1}$ with the conditions $(6j + 2\ell + 3)/(6i + 2\overline{k} + 9) < \ell/k$, $j \geq 0$, and $i \geq 0$, and the lemma still holds.

Proof of Lemma 5.5. Proceeding similarly to the proof of Lemma 5.4,

$$
(6j + 2\ell + 9 - e^{-8(3j+\ell+4)/9} - 5(\sqrt{2})^{-\ell})(6i + 2\overline{k} + 3 - e^{-8(3i+\overline{k}+1)/9} - 5(\sqrt{2})^{-k})
$$

$$
-(6j + 2\ell + 3 + e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-\ell})(6i + 2\overline{k} + 3 + e^{-8(3i+\overline{k}+1)/9} + 5(\sqrt{2})^{-k})
$$

$$
\geq 6(6i + 2\overline{k} + 3) - (6i + 2\overline{k} + 3)(e^{-8(3j+\ell+4)/9} + 5(\sqrt{2})^{-28})
$$

$$
-(6j + 2\ell + 9)(e^{-8(3i+\overline{k}+1)/9} + 5(\sqrt{2})^{-28})
$$

$$
-(6i + 2\overline{k} + 3)(e^{-8(3j+\ell+1)/9} + 5(\sqrt{2})^{-28}) - (6j + 2\ell + 3)(e^{-8(3i+\overline{k}+1)/9} + 5(\sqrt{2})^{-28})
$$

$$
-(e^{-8(3i+\overline{k}+1)/9} + 5(\sqrt{2})^{-28})(e^{-8(3j+\ell+4)/9} + 5(\sqrt{2})^{-28})
$$

$$
\geq (6 - 0.15 - 0.45)(6i + 2\overline{k} + 3) - (0.45 \cdot 2)(6j + 2\ell + 3) - 6 \cdot 0.45 - 0.45^2
$$

$$
= (6i + 2\overline{k} + 9)
\left[
5.4 - 0.9 \left(\frac{6j + 2\ell + 3}{6i + 2\overline{k} + 9} - \frac{35.3025}{6i + 2\overline{k} + 9}\right)
\right]
$$

$$
> (6i + 2\overline{k} + 9)
\left[
1.4775 - 0.9 \left(\frac{6j + 2\ell + 3}{6i + 2\overline{k} + 9}\right)
\right]
$$

By Lemma 2.3, $(6j + 2\ell + 3)/(6i + 2\overline{k} + 9) < \ell/k$. Since $\ell/k \leq 1.16$, we have that

$$(6i + 2\overline{k} + 9)
\left[
1.4775 - 0.9 \left(\frac{6j + 2\ell + 3}{6i + 2\overline{k} + 9}\right)
\right] > (6i + 2\overline{k} + 9)(1.4775 - 0.9 \cdot 1.16) > 0,$$

so we conclude that (5.2) holds as desired.

Combining Lemmas 5.4 and 5.5, we have now completed the proof of Lemma 5.2.

5.2. Proof of Lemma 5.3

Proof of Lemma 5.3. We know from the assumptions that $\beta^*_{k,i,j} > \beta^*_{k,i+1}$ and $\beta^*_{k,i,j} < \beta^*_{k,i+1}$. It follows from Lemma 2.4 that

$$
\frac{\ell}{k} \in I_{k,i+1, \ell,j}
$$

Note that $j > 0$ by Lemma 2.8.
We wish to show that $\beta_{k,i}^* > \beta_{\ell,i-1}^* > \beta_{k,i+1}^*$. We do so by proving that

$$\frac{6j + 2\ell - 3 + e^{-8(3j + \ell - 2)/9} + 5(\sqrt{2})^{-\ell}}{6i + 2k + 9 - e^{-8(3i + k + 4)/9} - 5(\sqrt{2})^{-k}} < \frac{6j + 2\ell + 3 - e^{-8(3j + \ell + 1)/9} - 5(\sqrt{2})^{-\ell}}{6i + 2k + 9 + e^{-8(3i + k + 4)/9} + 5(\sqrt{2})^{-k}}$$

and

$$\frac{6j + 2\ell - 3 - e^{-8(3j + \ell - 2)/9} - 5(\sqrt{2})^{-\ell}}{6i + 2k + 3 + e^{-8(3i + k + 1)/9} + 5(\sqrt{2})^{-k}} < \frac{6j + 2\ell - 3 + e^{-8(3j + \ell - 2)/9} - 5(\sqrt{2})^{-\ell}}{6i + 2k + 3 + e^{-8(3i + k + 1)/9} + 5(\sqrt{2})^{-k}}$$

Note that the upper bound of (5.5) is the lower bound of $I_{k,i+1,\ell,j}$, and the lower bound of (5.6) is the upper bound of $I_{k,i+1,\ell,j}$, so if $\ell/k \in I_{k,i+1,\ell,j}$, assuming (5.5) and (5.6) are true, then

$$\frac{6j + 2\ell - 3 + e^{-8(3j + \ell - 2)/9} + 5(\sqrt{2})^{-\ell}}{6i + 2k + 9 - e^{-8(3i + k + 4)/9} - 5(\sqrt{2})^{-k}} < \frac{6j + 2\ell - 3 - e^{-8(3j + \ell - 2)/9} - 5(\sqrt{2})^{-\ell}}{6i + 2k + 3 + e^{-8(3i + k + 1)/9} + 5(\sqrt{2})^{-k}}$$

and we are done by Lemma 2.11. It remains to show that inequalities (5.5) and (5.6) hold.

To show (5.5), let $j = j' + 1$ and $i = i' - 1$. Then we get inequality (5.2), but with $i'$ and $j'$ instead of $i$ and $j$. By Lemma 5.5 (and the following remark), we see that (5.5) holds.

To prove (5.6), let $j = j' + 1$. Then we get the same inequality as in (5.1), but with $j'$ instead of $j$. If $j - 1 > i$, then by Lemma 5.4 (and the following remark), we see that (5.6) holds.

It remains to consider the case when $j - 1 \leq i$. By Lemma 2.6, we have $j \geq i$, so either $j = i$ or $j = i + 1$.

First, suppose $j = i$. From Lemma 2.4, we have that $\ell/k \in I_{k,i+1,\ell,j}$, so

$$\frac{\ell}{k} < \frac{6j + 2\ell + 3 + e^{-8(3j + \ell + 1)/9} + 5(\sqrt{2})^{-\ell}}{6i + 2k + 9 - e^{-8(3i + k + 4)/9} - 5(\sqrt{2})^{-k}} < \frac{6i + 7 + e^{-8/9} + 5(\sqrt{2})^{-28}}{6i + 9 - e^{-32/9} - 5(\sqrt{2})^{-28}}$$

which gives a contradiction, since $\ell > k$. Thus, $j = i$ is not possible.

Now, suppose $j = i + 1$. There are three cases to consider.

Case 1: $\ell > k$. Then (5.6) holds, since $j' = j - 1 \geq i$, and this corresponds to Case 2 of Lemma 5.4, which was proven above.

Case 2: $\ell < k$. Then similar to the $j = i$ case, we have that

$$\frac{\ell}{k} < \frac{6j + 2\ell + 3 + e^{-8(3j + \ell + 1)/9} + 5(\sqrt{2})^{-\ell}}{6i + 2k + 9 - e^{-8(3i + k + 4)/9} - 5(\sqrt{2})^{-k}} < \frac{6i + 9 + 2(k - 1) + e^{-8/9} + 5(\sqrt{2})^{-28}}{6i + 9 + 2k - e^{-32/9} - 5(\sqrt{2})^{-28}} < 1,$$

which gives a contradiction since $\ell > k$. Thus, this case is not possible.
Case 3: \( \ell = \bar{k} \). We will show that this case is impossible by applying the methods in Section 4 of [8] and in Section 4 in [5], which are different from our previous proof methods. In this case we will assume that \( k \geq 114 \). For \( k \leq 112 \), since \( 1 < \ell/k \leq 1.16 \) and \( \ell = \bar{k} \), we get that \( \ell - k = 6 \) or \( 12 \), so this case is impossible by [8, Theorem 1.1] and [5, Theorems 1.1 and 1.2].

We will first prove that the way that the cosine zeros shift corresponds to Types 2 and 3 in [8]. Recall that \( i < k/52 \), so

\[
\beta_{k,i} > \beta_{k,i+1} = \frac{2\pi}{3} - \frac{\pi(6i + 2\bar{k} + 9)}{3k} > \frac{2\pi}{3} - \frac{\pi(6k/52 + 13)}{3k} > \frac{7\pi}{12},
\]

Moreover,

\[
\beta_{k,i+1} < \beta_{k,i} \leq \frac{2\pi}{3} - \frac{\pi}{k} < \frac{2\pi}{3} - \frac{2\pi}{3k}.
\]

Similarly, by Lemma 2.1,

\[
\beta_{k,i}^* > \beta_{k,i+1}^* \geq \beta_{k,i+1} - \frac{\pi}{3k} = \frac{2\pi}{3} - \frac{\pi(6i + 2\bar{k} + 10)}{3k} > \frac{2\pi}{3} - \frac{\pi(6k/52 + 14)}{3k} > \frac{7\pi}{12},
\]

and

\[
\beta_{k,i+1}^* < \beta_{k,i}^* \leq \frac{2\pi}{3} - \frac{\pi(6i + 2\bar{k} + 2)}{3k} \leq \frac{2\pi}{3} - \frac{2\pi}{3k},
\]

so for all \( 0 \leq i < k/52 \), \( \beta_{k,i}^*, \beta_{k,i+1}^*, \beta_{k,i}^*, \) and \( \beta_{k,i+1}^* \) lie on the interval \([7\pi/12, 2\pi/3 - 2\pi/(3k)]\), and hence the results from Section 4 of [8] apply.

Recall from (1.1) that

\[
F_k(\theta) = 2 \cos\left(\frac{k\theta}{2}\right) + T_k(\theta).
\]

By [8, Lemma 4.1], \( 0 < T_k(\theta) < 1 \) on \([7\pi/12, 2\pi/3 - 2\pi/(3k)]\). Therefore, we can think of the zeros of \( F_k(\theta) \) on this interval as the zeros of \( 2\cos(k\theta/2) \) shifted horizontally as a result of \( 2\cos(k\theta/2) \) being moved upwards by a small amount (specifically, \( T_k(\theta) \)).

We define

\[
D_k(\theta) = \frac{d}{d\theta} \cos\left(\frac{k\theta}{2}\right) = -\frac{k}{2} \sin\left(\frac{k\theta}{2}\right),
\]

and evaluation at \( \beta_{k,0} \) gives

\[
-\frac{k}{2} \sin\left(\frac{k\beta_{k,0}}{2}\right) = -\frac{k}{2} \sin\left(\frac{k}{2} \left(\frac{2\pi}{3} - \frac{\pi(2\bar{k} + 3)}{3k}\right)\right) = -\frac{k}{2} \sin\left(\frac{(k - \bar{k})\pi}{3} - \frac{\pi}{2}\right)
\]

\[
= \frac{k}{2} \cos\left(\frac{(k - \bar{k})\pi}{3}\right).
\]

the sign of which is determined by \( k \mod 6 \). Specifically, the sign of \( D_k(\beta_{k,0}) \) is equal to \((-1)^{(k-\bar{k})/3}\). Since \( \bar{k} = \ell \) and both \( k \) and \( \ell \) are even, \( k \) and \( \ell \) have the same value mod 6. Thus, \( D_k(\beta_{k,0}) \) and \( D_\ell(\beta_{\ell,0}) \) have the same sign, so \( \beta_{k,0} \) and \( \beta_{\ell,0} \) shift in the same direction towards \( \beta_{k,0}^* \) and \( \beta_{\ell,0}^* \), respectively. Because the cosine curve is moved by a positive amount upwards
for all $\theta \in [7\pi/12, 2\pi/3 - 2\pi/(3k)]$, consecutive zeros of $2 \cos(k\theta/2)$ on this interval must shift in opposite directions. Thus, $\beta_{k,i}$ and $\beta_{\ell,i}$ shift in the same direction as well, since their indices are the same. Note that the zeros shifting in the same direction implies that we have either Type 2 or Type 3 in [8] (see Figure 5.1). In other words, for all $0 \leq i < k/52$, we have

$$\beta_{k,i} - \beta_{k,i+1}^* > 0$$

We will break into two subcases and show that both are impossible.

**Subcase 3.1: $\beta_{k,i+1} < \beta_{k,i+1}^*$ and $\beta_{\ell,i+1} < \beta_{\ell,i+1}^*$.** This corresponds to Type 2.

We follow Section 4.2 in [8] and Section 4.2 in [5]. Recall that $\ell/k \in I_{k,i+1,\ell,i+1}$, $\beta_{k,i} > \beta_{\ell,i+1} > \beta_{k,i+1}$, and $\beta_{\ell,i+1}^* \leq \beta_{k,i+1}$ by the assumptions of Lemma 5.3. We will show that

$$\beta_{\ell,i+1}^* > \beta_{k,i+1}^*$$

and thus derive a contradiction. Since $k \geq 114$, $|\beta_{k,i+1}^* - \beta_{k,i+1}| \leq \pi/(3k)$ by Lemma 2.1. Thus, if $\beta_{\ell,i+1} > \beta_{k,i+1} + \pi/(3k)$, then (5.8) clearly holds because $\beta_{\ell,i+1}^* > \beta_{\ell,i+1}$. Thus, it suffices to show (5.8) for $\beta_{\ell,i+1} \leq \beta_{k,i+1} + \pi/(3k)$. To do so, we will show that

$$F_\ell(\theta) - F_k(\theta) > 0$$

for $\theta \in [\beta_{k,i+1}, \beta_{k,i+1} + \pi/(3k)]$. Note that $[\beta_{k,i+1}, \beta_{k,i+1}^*] \subset [\beta_{k,i+1}, \beta_{k,i+1} + \pi/(3k)]$. Following the notation in Section 4.2 of [5], let $\theta = \beta_{k,i+1} + \delta$, where $0 \leq \delta \leq \pi/(3k)$.

First, we compute a lower bound for $2 \cos(\ell\theta/2) - 2 \cos(k\theta/2)$. Rewriting,

$$2 \cos\left(\frac{\ell\theta}{2}\right) - 2 \cos\left(\frac{k\theta}{2}\right) = -4 \sin\left(\frac{k + \ell}{4}\theta\right) \sin\left(\frac{\ell - k}{4}\theta\right)$$

$$= -4 \sin\left(\frac{k + \ell}{4}(\beta_{k,i+1} + \delta)\right) \sin\left(\frac{\ell - k}{4}(\beta_{k,i+1} + \delta)\right)$$

$$= -4 \sin\left(\frac{\ell - k}{4}(\beta_{k,i+1} + \delta) + \frac{k}{2}(\beta_{k,i+1} + \delta)\right)$$

$$\times \sin\left(\frac{\ell - k}{4}(\beta_{k,i+1} + \delta)\right).$$

![Figure 5.1. Movement of zeros.](image-url)
We simplify the expressions. To ease notation define
\[ c_i = \frac{\pi (\ell - k)(6i + 2\bar{k} + 9)}{12k}. \tag{5.9} \]

By Definition 1.3,
\[ \frac{\ell - k}{4} \beta_{k,i+1} = \frac{\ell - k}{4} \pi \left( \frac{2}{3} - \frac{6(i + 1) + 2\bar{k} + 3}{3k} \right) = \frac{\ell - k}{6} - c_i \]
where \( \pi (\ell - k)/6 \) is an integer multiple of \( \pi \) since \( k \) and \( \ell \) are the same modulo 6.

Likewise,
\[ \frac{k}{2} \beta_{k,i+1} = \frac{k}{2} \pi \left( \frac{2}{3} - \frac{6(i + 1) + 2\bar{k} + 3}{3k} \right) = \frac{(k - \bar{k})\pi}{3} - (i + 2)\pi + \frac{\pi}{2}, \]
where \( (k - \bar{k})\pi/3 \) is an integer multiple of \( \pi \) since \( k - \bar{k} \) is divisible by 3.

Thus,
\begin{align*}
-4 \sin \left( \frac{\ell - k}{4} \beta_{k,i+1} + \delta \right) + k \frac{k}{2} \beta_{k,i+1} + \delta) \sin \left( \frac{\ell - k}{4} \beta_{k,i+1} + \delta \right) \\
= -(-1)^{(\ell-k)/6}(-1)^{(\ell-k)/6}(-1)^{(k-\bar{k})/3-(i+2)}4 \sin \left( \frac{\pi}{2} - c_i + \frac{\ell - k}{4} \delta + \frac{k}{2} \delta \right) \\
\times \sin \left( -c_i + \frac{\ell - k}{4} \delta \right) \\
= (-1)^{(k-\bar{k})/3-i}4 \cos \left( c_i - \frac{\ell - k}{4} \delta - \frac{k}{2} \delta \right) \sin \left( c_i - \frac{\ell - k}{4} \delta \right) \\
= (-1)^{(k-\bar{k})/3-i} \left( 4 \cos \left( c_i - \frac{\ell - k}{4} \delta \right) \sin \left( c_i - \frac{\ell - k}{4} \delta \right) \cos \left( \frac{k}{2} \delta \right) \right. \\
+ 4 \sin^2 \left( c_i - \frac{\ell - k}{4} \delta \right) \sin \left( \frac{k}{2} \delta \right) \right) \\
= (-1)^{(k-\bar{k})/3-i} \left( 2 \sin \left( 2 \left( c_i - \frac{\ell - k}{4} \delta \right) \right) \cos \left( \frac{k}{2} \delta \right) \right) \\
+ 4 \sin^2 \left( c_i - \frac{\ell - k}{4} \delta \right) \sin \left( \frac{k}{2} \delta \right). \tag{5.10}
\end{align*}

Because \( T_k(\theta) > 0 \) on the relevant interval, the sign of \( D_k(\beta_{k,0}) \) is the opposite of the sign of \( \beta_{k,0}^* - \beta_{k,0} \) and consecutive zeros of \( 2\cos(k\theta/2) \) shift in opposite directions. Consequently, the sign of \( \beta_{k,i+1}^* - \beta_{k,i+1} \) is the same as the sign of \( (-1)^{i+1}(\beta_{k,0}^* - \beta_{k,0}) \). Thus, the sign of \( \beta_{k,i+1}^* - \beta_{k,i+1} \) must be
\[ (-1)^{i+1}(-(-1)^{(k-\bar{k})/3}) = (-1)^{(k-\bar{k})/3-i}. \]

Since we are in Type 2, \( \beta_{k,i+1}^* - \beta_{k,i+1} > 0 \), so \( (-1)^{(k-\bar{k})/3-i} = 1 \).

We now show that the trigonometric functions in (5.10) are non-negative and that the expression as a whole is positive. By Lemma 2.4,
\[ \frac{\ell}{k} < \frac{6(i + 1) + 2\bar{k} + 3 + e^{-8(3(i+1)+\bar{k}+1)/9} + 5(\sqrt{2})^{-\ell}}{6(i + 1) + 2\bar{k} + 3 - e^{-8(3(i+1)+\bar{k}+1)/9} - 5(\sqrt{2})^{-\ell}} < \frac{6i + 2\bar{k} + 9.15}{6i + 2\bar{k} + 8.85}. \]
From this,

\[
c_i - \frac{\ell - k}{4} \delta \leq c_i = \frac{(\ell - k)(6i + 2\overline{k} + 9)\pi}{12k} < \left( \frac{6i + 2\overline{k} + 9.15}{6i + 2\overline{k} + 8.85} - 1 \right) \frac{(6i + 2\overline{k} + 9)\pi}{12} = \left( \frac{0.3}{6i + 2\overline{k} + 8.85} \right) \frac{(6i + 2\overline{k} + 9)\pi}{12} \leq \left( \frac{9}{8.85} \right) \frac{0.3\pi}{12} < \frac{\pi}{30},
\]

and

\[
c_i - \frac{\ell - k}{4} \delta \geq \frac{\pi(\ell - k)(6i + 2\overline{k} + 9) - \pi(\ell - k)}{12k} = \frac{\pi(\ell - k)(3i + \overline{k} + 4)}{6k} > 0,
\]

so \( 0 < c_i - ((\ell - k)/4)\delta < \pi/30 \). Thus, the minimum value of \( \sin^2(c_i - ((\ell - k)/4)\delta) \) and \( \sin(2(c_i - ((\ell - k)/4)\delta)) \) occur when \( c_i - ((\ell - k)/4)\delta \) is closest to 0, or equivalently when \( i = 0 \) and \( \delta = \pi/(3k) \).

Moreover, since \( 0 \leq \delta \leq \pi/(3k) \), or equivalently \( 0 \leq (k/2)\delta \leq \pi/6 \),

\[
\cos\left( \frac{k\delta}{2} \right) \geq \cos\left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2}.
\]

Since \( 0 < c_i - ((\ell - k)/4)\delta < \pi/30 \) and \( 0 \leq (k/2)\delta \leq \pi/6 \), all of the sines and cosines are non-negative. Thus,

\[
2 \sin\left( 2\left( c_i - \frac{\ell - k}{4} \delta \right) \right) \cos\left( \frac{k\delta}{2} \right) + 4 \sin^2\left( c_i - \frac{\ell - k}{4} \delta \right) \sin\left( \frac{k\delta}{2} \right) \geq 2 \sin\left( 2\left( c_0 - \frac{\ell - k}{4} \frac{\pi}{3k} \right) \right) \cdot \frac{\sqrt{3}}{2} = \sqrt{3} \sin\left( 2\left( \frac{\pi(\ell - k)(2\overline{k} + 9) - \pi(\ell - k)}{12k} \right) \right) = \sqrt{3} \sin\left( \frac{\pi(\ell - k)(\overline{k} + 4)}{3k} \right) \geq \sqrt{3} \sin\left( \frac{4(\ell - k)\pi}{3k} \right) > 0,
\]

where the last two inequalities hold because, for \( 1 < \ell/k \leq 1.16 \),

\[
0 < \frac{(\ell - k)(\overline{k} + 4)\pi}{3k} \leq \frac{2(\ell - k)\pi}{k} < \frac{\pi}{2}.
\]

Note that for \( 1 < \ell/k \leq 1.16 \),

\[
0 < \frac{4(\ell - k)\pi}{3k} \leq \frac{4(0.16)\pi}{3} < 0.22\pi.
\]
Since \( \sin(x) \) is concave down on the interval \((0, 0.22\pi)\),

\[
\frac{\sin(x)}{x} > \frac{d}{dx} \sin(x) \bigg|_{x=0.22\pi} = \cos(0.22\pi),
\]

so

\[
\sqrt{3} \sin\left(\frac{4(\ell - k)\pi}{3k}\right) \geq \sqrt{3} \cos(0.22\pi) \frac{4(\ell - k)\pi}{3k} > \frac{1.75(\ell - k)\pi}{k}.
\]

Thus, we conclude that for \( \theta \in [\beta_{k,i+1}, \beta_{k,i+1} + \pi/(3k)] \),

\[
2 \cos\left(\frac{\ell\theta}{2}\right) - 2 \cos\left(\frac{k\theta}{2}\right) > \frac{1.75(\ell - k)\pi}{k}.
\]

We now compute a lower bound for \((2 \cos(\theta/2))^{-\ell} - (2 \cos(\theta/2))^{-k}\) over the interval \([7\pi/12, 2\pi/3] \supset [7\pi/12, 2\pi/3 - 2\pi/(3k)]\) using the same analysis as \([5, (4.2)]\) and \([8, (4.24)]\). Let

\[
g(\theta) = \left(2 \cos\left(\frac{\theta}{2}\right)\right)^{-\ell} - \left(2 \cos\left(\frac{\theta}{2}\right)\right)^{-k}
\]

over the interval \([7\pi/12, 2\pi/3)\) and let \(\theta'\) be a value in this interval such that \(g'(\theta) = 0\). We will show that \(\theta'\) uniquely exists and that the minimum of \(g(\theta)\) occurs at \(\theta'\).

We start with the existence and uniqueness of \(\theta'\). We calculate

\[
g'(\theta) = \sin\left(\frac{\theta}{2}\right) \left(\ell \left(2 \cos\left(\frac{\theta}{2}\right)\right)^{-\ell-1} - k \left(2 \cos\left(\frac{\theta}{2}\right)\right)^{-k-1}\right).
\]

For \(\theta \in [7\pi/12, 2\pi/3)\), \(\sin(\theta/2) > 0\), so to find the zeros of \(g'\) it suffices to only consider

\[
\ell \left(2 \cos\left(\frac{\theta}{2}\right)\right)^{-\ell-1} - k \left(2 \cos\left(\frac{\theta}{2}\right)\right)^{-k-1}.
\]

Rearranging (5.11) we see that \(\theta'\) is precisely the value that satisfies

\[
\left(2 \cos\left(\frac{\theta'}{2}\right)\right)^{-\ell+k} = \frac{k}{\ell}.
\]

On the interval \([7\pi/12, 2\pi/3)\), \(2 \cos(\theta/2)\) is decreasing, so the left-hand side of (5.12) is strictly increasing and takes precisely the values \((2 \cos(7\pi/24))^{-\ell+k}, 1)\).

Since \(1 < \ell/k \leq 1.16\),

\[
\left(2 \cos\left(\frac{7\pi}{24}\right)\right)^{-\ell+k} < \frac{1}{1.21^{\ell-k}} < \frac{1}{1.16} \leq \frac{k}{\ell} < 1,
\]

so

\[
\frac{k}{\ell} \in \left(\left(2 \cos\left(\frac{7\pi}{24}\right)\right)^{-\ell+k}, 1\right).
\]

Therefore, \(\theta'\) exists. Because the left-hand side of (5.12) is strictly increasing, \(\theta'\) is unique.

It remains to show that \(g(\theta)\) is strictly decreasing on \([7\pi/12, \theta']\) and strictly increasing on \([\theta', 2\pi/3)\). At \(\theta = 7\pi/12\),

\[
g'(\theta) = \ell \left(2 \cos\left(\frac{\theta}{2}\right)\right)^{-\ell-1} - k \left(2 \cos\left(\frac{\theta}{2}\right)\right)^{-k-1}
\]

\[
\leq \frac{1.16k}{(2 \cos(7\pi/24))^{k+3}} - k \left(2 \cos(7\pi/24)\right)^{k+1} < 0,
\]

Zeros of Eisenstein series
so \( g'(7\pi/12) < 0 \), and from the uniqueness of \( \theta' \) it follows that \( g'(\theta) < 0 \) on \([7\pi/12, \theta')\). At \( \theta = 2\pi/3 \),
\[
\ell \left( 2 \cos \left( \frac{\theta}{2} \right) \right)^{\ell-1} - k \left( 2 \cos \left( \frac{\theta}{2} \right) \right)^{k-1} = \ell - k > 0,
\]
so \( g'(2\pi/3) > 0 \), and from the uniqueness of \( \theta' \) it follows that \( g'(\theta) > 0 \) on \((\theta', 2\pi/3)\).
Therefore, on the interval \([7\pi/12, 2\pi/3)\), \( g(\theta) \) is minimized at \( \theta' \), where \( 2 \cos(\theta/2)^{k-\ell} = k/\ell \).

Hence,
\[
\left( 2 \cos \left( \frac{\theta}{2} \right) \right)^{-\ell} - \left( 2 \cos \left( \frac{\theta}{2} \right) \right)^{-k} \geq \left( 2 \cos \left( \frac{\theta'}{2} \right) \right)^{k-\ell} - 1 \left( 2 \cos \left( \frac{\theta'}{2} \right) \right)^{-k} = \left( k/\ell - 1 \right) \left( 2 \cos \left( \frac{\theta'}{2} \right) \right)^{-k} > k/\ell - 1,
\]
where the last inequality holds since \( k/\ell < 1 \) and \( 2 \cos(\theta/2) > 1 \) as \( \theta' \in [7\pi/12, 2\pi/3) \).

Since \( \ell > k \geq 114 \), using (1.1) and (1.2) and following [8, (4.25)] and Section 4.2 in [5],
\[
F_{\ell}(\theta) - F_k(\theta) = 2 \cos \left( \frac{\ell \theta}{2} \right) - 2 \cos \left( \frac{k \theta}{2} \right) + \left( 2 \cos \left( \frac{\theta}{2} \right) \right)^{-\ell} - \left( 2 \cos \left( \frac{\theta}{2} \right) \right)^{-k} + R_{\ell}(\theta) - R_k(\theta)
\]
\[
> \frac{1.75(\ell - k)\pi}{k} + \frac{k}{\ell} - 1 - 10(\sqrt{2})^{-k}
\]
\[
= \left( 1.75\pi - \frac{k}{\ell} \right) \frac{\ell - k}{k} - 10(\sqrt{2})^{-k}
\]
\[
> (1.75\pi - 1) \frac{2}{k} - 10(\sqrt{2})^{-k} > 0.
\]
By [8, Lemma 4.1], \( 0 < T_k(\theta) < 1 \) for all \( \theta \in [7\pi/12, (2/3 - 2/(3k))\pi] \), so \( F_k(\beta_{k,i+1}) > 0 \). Together with the fact that \( F_{\ell}(\theta) - F_k(\theta) > 0 \), we see that \( F_{\ell}(\theta) > F_k(\theta) \geq 0 \) on \([\beta_{k,i+1}, \beta_{k,i+1}^*] \). Therefore, \( \beta_{k,i+1}^* < \beta_{k,i+1}^* \), showing (5.8). However, we assumed that \( \beta_{k,i+1}^* \geq \beta_{k,i+1}^* \), so this is a contradiction. Thus, this subcase is not possible.

**Subcase 3.2**: \( \beta_{k,i+1} > \beta_{k,i+1}^* \) and \( \beta_{k,i+1} > \beta_{k,i+1}^* \). This corresponds to Type 3 (see Figure 5.1).

We follow Section 4.3 in [8] and Section 4.3 in [5]. Again, recall that \( \ell/k \in I_{k,i+1}, \ell,i+1 \), \( \beta_{k,i} > \beta_{k,i+1} > \beta_{k,i+1}^* \), and \( \beta_{k,i+1}^* \leq \beta_{k,i+1}^* \) by the assumptions of Lemma 5.3. We will show that
\[
\beta_{k,i+1}^* > \beta_{k,i+1}^*, \tag{5.13}
\]
which gives a contradiction, meaning that this case is in fact impossible.

Since \( k \geq 114 \), \( |\beta_{k,i+1}^* - \beta_{k,i+1}| \leq \pi/(3\ell) \) by Lemma 2.1. Thus, if \( \beta_{k,i+1} > \beta_{k,i+1} + \pi/(3\ell) \), (5.13) clearly holds because \( \beta_{k,i+1} > \beta_{k,i+1}^* \). Hence it suffices to show (5.13) when
\( \beta_{\ell,i+1} \leq \beta_{k,i+1} + \pi/(3\ell) \). Note that

\[
\beta_{k,i+1} \geq \beta_{\ell,i+1} - \frac{\pi}{3\ell} > \beta_{k,i+1} - \frac{\pi}{3k},
\]

so we will show that

\[
F_\ell(\theta) - F_k(\theta) < 0
\]

for \([\beta_{k,i+1} - \pi/(3k), \beta_{k,i+1}]\) to obtain the contradiction.

We let

\[
h(\theta) = -g(\theta) = \left(2 \cos\left(\frac{\theta}{2}\right)\right)^{-k} - \left(2 \cos\left(\frac{\theta}{2}\right)\right)^{-\ell}
\]

and retain the same \(\theta'\) as in Subcase 3.1. Note that we have that \(h(\theta)\) is strictly increasing on \([7\pi/12, \theta']\).

We will show that \(\beta_{k,i+1} < \theta'\) so that \(h(\theta)\) is minimized at \(7\pi/12\) on \([7\pi/12, \beta_{k,i+1}]\). Note from (5.7) that \([7\pi/12, \beta_{k,i+1}] \supset [\beta_{k,i+1} - \pi/(3k), \beta_{k,i+1}]\). Recall from (5.12) that \(\theta'\) is precisely the value that satisfies

\[
\left(2 \cos\left(\frac{\beta_{k,i+1}}{2}\right)\right)^{-\ell+k} = \frac{k}{\ell}
\]

and that the left-hand side of the above equation is strictly increasing. Thus, it suffices to show that

\[
\left(2 \cos\left(\frac{\beta_{k,i+1}}{2}\right)\right)^{-\ell+k} < \frac{k}{\ell}.
\]

Since

\[
\beta_{k,i+1} = \pi \left(\frac{2}{3} - \frac{6i + 2\kappa + 9}{3k}\right),
\]

\(\cos(\beta_{k,i+1}/2)\) is minimized when \(i, \kappa = 0\). Because \(k < \ell\),

\[
\left(2 \cos\left(\frac{\beta_{k,i+1}}{2}\right)\right)^{-\ell+k} - \ell - k \geq \left(2 \cos\left(\frac{1}{3} - \frac{3}{2k}\right)\right)^{-\ell+k} - 1 - \frac{\ell - k}{k}
\]

\[
> \left(2 - 3 \left(\frac{1}{3} - \frac{3}{2k}\right)^{-\ell+k} - 1 - \frac{\ell - k}{k}
\]

\[
= \left(1 + \frac{9}{2k}\right)^{-\ell+k} - 1 - \frac{\ell - k}{k}
\]

\[
\geq 1 + \frac{9(\ell - k)}{2k} - 1 - \frac{\ell - k}{k}
\]

\[
> 0,
\]

where the second inequality holds because \(2 \cos(\pi x) > 2 - 3x\) for \(x \in (0, 1/3)\) and the third holds by the Binomial theorem.

Therefore,

\[
\left(2 \cos\left(\frac{\beta_{k,i+1}}{2}\right)\right)^{-\ell+k} < \frac{k}{\ell}.
\]
This tells us that $\beta_{k,i+1} < \theta'$. Thus, on the interval $[7\pi/12, \beta_{k,i+1}]$, $h(\theta)$ is minimized at $7\pi/12$. By (1.1) and (1.2), for $k \geq 114$,

\[
T_k(\theta) - T_\ell(\theta) = h(\theta) + R_k(\theta) - R_\ell(\theta)
\]

\[
> h\left(\frac{7\pi}{12}\right) - 10(\sqrt{2})^{-k}
\]

\[
= \left(2 \cos\left(\frac{7\pi}{24}\right)\right)^{-k} - \left(2 \cos\left(\frac{7\pi}{24}\right)\right)^{-\ell} - 10(\sqrt{2})^{-k}
\]

\[
\geq \left(2 \cos\left(\frac{7\pi}{24}\right)\right)^{-k} - \left(2 \cos\left(\frac{7\pi}{24}\right)\right)^{-k-6} - 10(\sqrt{2})^{-k}
\]

\[
> 0.
\]

(5.14)

Here we have used that $\ell \geq k + 6$, since $\ell$ and $k$ are the same mod 6 in this case.

We now show that $2 \cos(\ell \theta/2) - 2 \cos(k \theta/2) < 0$ for $\theta \in [\beta_{k,i+1} - \pi/(3k), \beta_{k,i+1}]$ (as mentioned in Section 4.3 of [5] and [8, Lemma 4.5]). Similar to the notation in Section 4 of [5], let $\theta = \beta_{k,i+1} - \delta$, where $0 \leq \delta < \pi/(3k)$. Recall from (5.9) that

\[
c_i = \frac{\pi (\ell - k)(6i + 2\ell + 9)}{12k}.
\]

Then following our work in Subcase 3.1, we know that

\[
2 \cos\left(\frac{\ell \theta}{2}\right) - 2 \cos\left(\frac{k \theta}{2}\right) = (-1)^{(k - \ell)/3 - i} \cos\left(c_i + \frac{\ell - k}{4} \delta + \frac{k}{2} \delta\right) \sin\left(c_i + \frac{\ell - k}{4} \delta\right).
\]

As before, the sign of $\beta_{k,i+1}^* - \beta_{k,i+1}$ must be $(-1)^{(k - \ell)/3 - i}$. Since we are in Type 3, $\beta_{k,i+1}^* - \beta_{k,i+1} < 0$, so $(-1)^{(k - \ell)/3 - i} = -1$.

Again following the method in Subcase 3.1,

\[
c_i + \frac{\ell - k}{4} \delta \leq c_i + \frac{\ell - k}{3k} \pi
\]

\[
= \frac{(\ell - k)(6i + 2\ell + 10)\pi}{12k}
\]

\[
< \left(\frac{6i + 2\ell + 9.15}{6i + 2\ell + 8.85} - 1\right) \left(\frac{6i + 2\ell + 10\pi}{12}\right)
\]

\[
= \left(\frac{0.3}{6i + 2\ell + 8.85}\right) \left(\frac{6i + 2\ell + 10\pi}{12}\right)
\]

\[
\leq \left(\frac{0.3\pi}{12}\right) < \frac{\pi}{30}.
\]

Likewise, we have

\[
0 \leq \frac{k}{2} \delta < \frac{k}{2} \cdot \frac{\pi}{3k} = \frac{\pi}{6},
\]

so

\[
0 < c_i \leq c_i + \frac{\ell - k}{4} \delta \leq c_i + \frac{\ell - k}{4} \delta + \frac{k}{2} \delta < \frac{\pi}{2}.
\]
Thus, on the interval \( \theta \in [\beta_{k,i+1} - \pi/(3k), \beta_{k,i+1}] \), all trigonometric functions are positive, so

\[
2 \cos \left( \frac{\ell \theta}{2} \right) - 2 \cos \left( \frac{k \theta}{2} \right) = (-1)^{(k-\ell)/3-i} \cdot 4 \cos \left( c_i + \frac{\ell - k}{4} \delta + \frac{k}{2} \right) 
\times \sin \left( c_i + \frac{\ell - k}{4} \delta \right) < 0,
\]

so following the methods of [8, Lemma 4.5] and Section 4.3 in [5] and using (1.1) and (5.14),

\[
F_\ell(\theta) - F_k(\theta) = 2 \cos \left( \frac{\ell \theta}{2} \right) - 2 \cos \left( \frac{k \theta}{2} \right) + T_\ell(\theta) - T_k(\theta) < 0.
\]

We conclude that on the interval \([\beta_{k,i+1} - \pi/(3k), \beta_{k,i+1}]\), \( \beta_{k,i+1}^* < \beta_{\ell,i+1}^* \), showing (5.13). This is a contradiction, so this subcase is not possible either. Consequently, the proof of Case 3 is complete.

Since we have covered all cases, the proof of Lemma 5.3 is complete. \( \square \)

6. Proof of Theorem 1.2

We now combine all the results in previous sections to prove Theorem 1.2. We restate the theorem for convenience.

**Theorem 1.2.** Let \( \ell > k \geq 24 \) be positive even integers with \( k \neq 26 \). Then between any two zeros of \( F_k(\theta) \) on the interval \((\pi/2, 2\pi/3)\) there exists at least one zero of \( F_\ell(\theta) \).

**Proof.** We will show that for all indices \( 0 \leq i \leq n_k - 2 \), there is a zero of \( F_\ell \) strictly between \( \beta_{k,i}^* \) and \( \beta_{k,i+1}^* \).

If \( \ell/k > 1.16 \), then the theorem holds by Proposition 3.1. Moreover, if \( \ell/k \leq 1.16 \), \( k \geq 28 \), and \( i < k/52 \), then by Proposition 5.1 we are done. Thus, the remaining cases are when \( \ell/k \leq 1.16 \) and \( i \geq k/52 \), or when \( \ell/k \leq 1.16 \) and \( k = 24 \). Note that these cases are not necessarily mutually exclusive.

If \( i \geq k/52 \), then

\[
\beta_{k,i+1} < \beta_{k,i}^* = \pi \left( \frac{2}{3} - \frac{6i + 2k + 3}{3k} \right) \leq \pi \left( \frac{2}{3} - \frac{6 \cdot k/52 + 3}{3k} \right) < \pi \left( \frac{2}{3} - \frac{1}{26} \right) < 0.658\pi.
\]

Applying Lemmas 2.1 and 2.2 and letting \( k \geq 304 \) gives that

\[
\beta_{k,i+1}^* < \beta_{k,i}^* \leq \beta_{k,i} + \frac{\pi}{3k} < \beta_{k,i} + \frac{\pi}{3 \cdot \frac{1}{26}} + \frac{\pi}{3 \cdot 304} < 0.658\pi,
\]

so the theorem holds by Proposition 4.2. Thus, the only remaining case is when \( k \leq 302 \) and \( \ell/k \leq 1.16 \), which has been verified computationally and by using [8, Theorem 1.1] and [5, Theorem 1.2].

We have exhausted all cases, and the result holds for all \( i \), so the proof is complete. \( \square \)
7. Indivisibility between Eisenstein series

Besides the interest on its own, Theorem 1.2 also has an application on the indivisibility between Eisenstein series and thus on the linear independence of special values of \( L \)-functions. We first recall the definition of zero polynomials associated to the Eisenstein series \( E_k \).

**Definition 7.1.** (Zero polynomial of \( E_k \)) Let
\[
\varphi_k(x) := \prod_{i=1}^{n(k)} (x - j(z_i)),
\]
where \( z_i \) runs over non-trivial zeros of \( E_k \), and \( j(z) \) is the \( j \)-invariant function.

Theorem 1.2 can be applied to study the indivisibility between the zero polynomials of Eisenstein series and hence between Eisenstein series themselves. In the rest of this section we write \( M_k \) for the vector space of modular forms of weight \( k \) for \( \text{SL}_2(\mathbb{Z}) \) and \( S_k \) for its subspace of cuspforms.

**Proposition 7.2.** Let \( n_k = \dim(S_k) \). If \( \ell > k \geq 24, k \neq 26 \) and \( 2n_k > n_\ell + 1 \), then
\[
\varphi_k(x) \nmid \varphi_\ell(x).
\]
Consequently, \( E_k \nmid E_\ell \) under the same conditions.

**Proof.** Suppose on the contrary that \( \varphi_k(x) \mid \varphi_\ell(x) \). Thus, the set of zeros of \( \varphi_\ell \) contains all \( n_k \) zeros of \( \varphi_k \). Theorem 1.2 implies that \( \varphi_\ell \) also contains at least \( n_k - 1 \) zeros between the zeros of \( \varphi_k \). So \( \varphi_\ell \) has at least \( n_k + (n_k - 1) \) distinct zeros. Since the actual number of zeros of \( \varphi_\ell \) is given by \( n_\ell \), we get that \( n_\ell \geq 2n_k - 1 \), which is a contradiction. \( \square \)

Note that \( n_k \sim k/12 \), so Proposition 7.2 holds roughly when \( k < \ell < 2k \). Of course, if one assumes the following Conjecture 7.3 on the zero polynomials of Eisenstein series, then it is clear that \( \varphi_k \nmid \varphi_\ell \) for \( k < \ell \) (or at least for \( n_k < n_\ell \)). We want to point out that Beyerl et al [1, Theorem 1.5] also investigated the divisibility between Eisenstein series conditioned upon Conjecture 7.3.

**Conjecture 7.3.** (Cornelissen [3] and Gekeler [4]) The zero polynomials \( \varphi_k(x) \) are irreducible over \( \mathbb{Q} \).

We now relate the divisibility between Eisenstein series to the special values of \( L \)-functions associated to eigenforms. First note that \( E_k \mid E_\ell \) if and only if there exist constants \( c_g \) such that
\[
E_\ell = E_k \cdot \left( E_{\ell-k} + \sum_g c_g g \right),
\]
where the sum is over all normalized cuspidal eigenforms \( g \) of weight \( \ell - k \).

Now let \( f \) be any normalized cuspidal eigenform of weight \( \ell \). Then its Petersson inner product with \( E_\ell \) vanishes, i.e.,
\[
\langle E_\ell, f \rangle = 0.
\]
Thus, (7.1) holds if and only if
\[
\left( E_k E_{\ell-k} + \sum_g c_g E_k g, f \right) = 0 \tag{7.2}
\]
for every eigenform \( f \in S_\ell \).

On the other hand, using the Rankin–Selberg convolution method \([2, 11]\), one has that, for normalized eigenforms \( f \in S_\ell \) and \( g \in S_{\ell-k} \),
\[
\langle E_k E_{\ell-k}, f \rangle = -\frac{\ell - k}{B_{\ell-k}} (4\pi)^{1-\ell} \Gamma(\ell - 1) \frac{L(\ell - 1, f) L(k, f)}{\zeta(k)}, \tag{7.3}
\]
and
\[
\langle E_k g, f \rangle = \frac{1}{2} (4\pi)^{1-\ell} \Gamma(\ell - 1) \frac{L(\ell - 1, g \otimes f)}{\zeta(k)}, \tag{7.4}
\]
where \( L(s, f) \) is the standard \( L \)-function associated to \( f \) and \( L(s, g \otimes f) \) is the Rankin–Selberg convolution \( L \)-function associated to \( g \) and \( f \).

Let \( \{f_1, \ldots, f_{n_\ell}\} \) be the normalized eigenform basis for \( S_\ell \) and let \( \{g_0, g_1, \ldots, g_{n_{\ell-k}}\} \) be the normalized eigenform basis for \( M_{\ell-k} \) with \( g_0 = E_{\ell-k} \).

For each \( 0 \leq i \leq n_{\ell-k} \) we define a column vector \( \vec{L}(g_i, f) = [L(g_i, f_1), L(g_i, f_2), \ldots, L(g_i, f_{n_\ell})]^T \in \mathbb{C}^{n_\ell} \), where, for each \( 1 \leq j \leq n_\ell \),
\[
L(g_0, f_j) := L(\ell - 1, f_j) L(k, f_j) \quad \text{and} \quad L(g_i, f_j) := L(\ell - 1, g_i \otimes f_j), \quad i \geq 1.
\]

All combined, we obtain the following.

**Proposition 7.4.** The divisibility \( E_k \mid E_\ell \) holds true if and only if the vectors \( \vec{L}(g_i, f) \) are linearly dependent for \( i = 0, \ldots, n_{\ell-k} \).

**Proof.** First, suppose \( E_k \mid E_\ell \). By (7.2), (7.3), and (7.4), there exist not all zero constants \( c_i \) such that
\[
\sum_{i=0}^{n_\ell} c_i \vec{L}(g_i, f) = \vec{0},
\]
with \( c_0 = 1 \) in fact. So these vectors are linearly dependent.

Conversely, suppose there is a dependence relation like the above for not all zero \( c_i \). Translating it into each \( f_j \)-component, we obtain
\[
c_0 L(\ell - 1, f_j) L(k, f_j) + c_1 L(\ell - 1, g_1 \otimes f_j) + \cdots + c_{n_{\ell-k}} L(\ell - 1, g_{n_{\ell-k}} \otimes f_j) = 0.
\]
We only need to show that \( c_0 \neq 0 \). Suppose on the contrary that \( c_0 = 0 \), then we get for each \( j \) that
\[
c_1 L(\ell - 1, g_1 \otimes f_j) + \cdots + c_{n_\ell} L(\ell - 1, g_{n_{\ell-k}} \otimes f_j) = 0,
\]
which, according to (7.2)–(7.4), becomes (after multiplying suitable factors on \( c_i \))
\[
\left( \sum_{i=1}^{n_\ell} c_i' E_k g_i, f_j \right) = 0
\]
for all \( 1 \leq j \leq n_\ell \). However, this means that \( \sum_{i=1}^{n_\ell} c_i' E_k g_i = 0 \), or \( c_i' = 0 \) for all \( i \) because \( \{E_k g_i\} \) are linearly independent. \( \square \)
In other words, Proposition 7.4 claims that $E_k \nmid E_\ell$ if and only if the $n_\ell \times (n_\ell-k+1)$ matrix

$$
\begin{bmatrix}
L(g_0, f_1) & L(g_1, f_1) & \cdots & L(g_{n_\ell-k}, f_1) \\
L(g_0, f_2) & L(g_1, f_2) & \cdots & L(g_{n_\ell-k}, f_2) \\
\vdots & \vdots & \ddots & \vdots \\
L(g_0, f_{n_\ell}) & L(g_1, f_{n_\ell}) & \cdots & L(g_{n_\ell-k}, f_{n_\ell})
\end{bmatrix}
$$

has rank $n_\ell-k+1$. A special case of Proposition 7.4 occurs when $n_\ell = n_\ell-k$. Then the vectors $\vec{L}(g_i, f)$ are trivially linearly dependent; therefore $E_k \mid E_\ell$ in this case. In fact, this can also be seen from the isomorphism $M_{\ell-k} \xrightarrow{E_k} M_\ell$ between spaces of modular forms of different weights.

As a corollary of Theorem 1.2 we obtain the following result.

**Proposition 7.5.** If $\ell > k \geq 24$ and $n_\ell < 2n_k - 1$, then the vectors $\vec{L}(g_i, f)$ for $i = 0, \ldots, n_{\ell-k}$ are linearly independent.

**Proof.** This follows immediately from Propositions 7.2 and 7.4.

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