ON $q$-ANALOGUES OF ZETA FUNCTIONS OF ROOT SYSTEMS

Masaki KATO

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Abstract. Komori, Matsumoto and Tsumura introduced a zeta function $\zeta_r(s, \Delta)$ associated with a root system $\Delta$. In this paper, we introduce a $q$-analogue of this zeta function, denoted by $\zeta_r(s, a, \Delta; q)$, and investigate its properties. We show that a ‘Weyl group symmetric’ linear combination of $\zeta_r(s, a, \Delta; q)$ can be written as a multiple integral over a torus involving functions $\psi_s$. For positive integers $k$, functions $\psi_k$ can be regarded as $q$-analogues of the periodic Bernoulli polynomials. When $\Delta$ is of type $A_2$ or $A_3$, the linear combinations can be expressed as the functions $\psi_k$, which are $q$-analogues of explicit expressions of Witten’s volume formula. We also introduce a two-parameter deformation of the zeta function $\zeta_r(s, \Delta)$ and study its properties.

1. Introduction

Let $g$ be a semisimple Lie algebra of rank $r$ and $s$ be a complex variable. We define the Witten zeta function by

$$\zeta_W(s, g) = \sum_{\varphi} (\dim \varphi)^{-s}, \quad (1.1)$$

where the summation on the right-hand side runs over all finite-dimensional irreducible representations $\varphi$ of $g$. When $g = sl(2)$, the zeta function $\zeta_W(s, sl(2))$ becomes the Riemann zeta function $\zeta(s)$:

$$\zeta_W(s, sl(2)) = \zeta(s) := \sum_{n=1}^{\infty} n^{-s}.$$ 

The Witten zeta function was introduced by Zagier [14]. The reason the zeta function (1.1) was named ‘Witten’ comes from the fact that Witten [13] calculated volumes of certain moduli spaces in quantum gauge theory in terms of special values of (1.1) at positive even integers.

By using Weyl’s dimension formula (for example, see [11, Section 3.8]), the Witten zeta function (1.1) can be written explicitly. Let $(\cdot, \cdot)$ be the Killing form of $g$ and $\Delta_+$ be the set of positive roots of $g$. For a root $\alpha$ of $g$, we denote the associated coroot of $\alpha$ by $\alpha^\vee$. Then Weyl’s dimension formula states the following:

$$\dim \varphi = \prod_{\alpha \in \Delta_+} \frac{\langle \alpha^\vee, \lambda + \rho \rangle}{\langle \alpha^\vee, \rho \rangle}, \quad (1.2)$$

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where $\lambda$ is the dominant integral weight corresponding to an irreducible representation $\varphi$ and $\rho$ is the Weyl vector. Let $P_+$ be the set of all dominant integral weights. Then, by (1.2), the Witten zeta function can be written as follows:

$$\zeta_W(s, g) = \left( \prod_{\lambda \in \Delta_+} \langle \alpha^\vee, \rho \rangle \right)^s \sum_{\lambda \in P_+, \kappa \in \Delta_+} \prod_{\lambda \in \Delta_+} \langle \alpha^\vee, \lambda + \rho \rangle^{-\kappa}.$$

(1.3)

Komori, Matsumoto and Tsumura [9, 10] introduced the following zeta function associated with the root system $\Delta$ of $g$, as a multivariable generalization of (1.3). For a complex vector $s = (s_\alpha)_{\alpha \in \Delta_+}$, we define the zeta function $\zeta_r(s, \Delta)$ by

$$\zeta_r(s, \Delta) := \sum_{\lambda \in P_+, \kappa \in \Delta_+} \prod_{\lambda \in \Delta_+} \langle \alpha^\vee, \lambda + \rho \rangle^{-s_\lambda}.$$

(1.4)

The series on the right-hand side converges absolutely when $\text{Re} \, s_\alpha > 1$ ($\alpha \in \Delta_+$). When $s_\alpha = s$ for all $\alpha \in \Delta_+$, the function $\zeta_r((s, \ldots, s), \Delta)$ essentially coincides with the Witten zeta function (1.1). For details of the function $\zeta_r(s, \Delta)$, see [9, 10].

In this paper, we introduce a $q$-analogue of the zeta function (1.4) and investigate its basic properties. The $q$-analogue of (1.4) is defined by the following:

$$\zeta_r(s, a, \Delta; q) := \sum_{\lambda \in P_+, \kappa \in \Delta_+} \prod_{\lambda \in \Delta_+} \frac{a_\alpha^{\langle \alpha^\vee, \lambda + \rho \rangle}}{(1 - q^{\langle \alpha^\vee, \lambda + \rho \rangle})^{s_\lambda}} (a = (a_\alpha)_{\alpha \in \Delta_+}).$$

When $\Delta = \Delta(A_1)$, the function $\zeta_1(s, q^{s-1}, \Delta(A_1); q)$ is essentially the same as a $q$-analogue of the Riemann zeta function, introduced by Kaneko, Kurokawa and Wakayama [7]. In Section 2, we establish basic properties of the function $\zeta_r(s, a, \Delta; q)$, including its analytic continuation. In Section 3, we show that a ‘Weyl group symmetric’ linear combination of functions $\zeta_r(s, a, \Delta; q)$ can be written as a multiple integral over a torus involving functions $\psi_k$. In Section 4, we investigate basic properties of functions $\psi_k$ for positive integers $k$. In particular, we show in Proposition 4.2 that the functions $\psi_k$ can be regarded as $q$-analogues of the periodic Bernoulli polynomials. In Section 5, we show that, when $\Delta = \Delta(A_1), \Delta(A_2), \Delta(A_3)$ and all components of the vector $s$ are positive integers, the linear combination introduced in Section 4 can be written in terms of the functions $\psi_k$. When $\Delta = \Delta(A_2), \Delta(A_3)$, these expressions can be considered to be $q$-analogues of explicit expressions of Witten’s volume formula, discovered independently by Zagier, Garoufalidis and Weinstein for the $A_2$ case (see [14]) and by Gunness and Sczech [5] for the $A_3$ case. In Section 6, we introduce a $p$-deformation $\zeta_r(s, a, \beta, \Delta; p; q)$ of $\zeta_r(s, a, \Delta; q)$ and establish its basic properties. When $\Delta = \Delta(A_1)$, the function $\zeta_1(1, q e^{2 \pi \sqrt{-1} x}, 1, \Delta(A_1); p; q)$ is considered to be a generating function of the elliptic zeta values, introduced by Felder and Varchenko [4].

2. $q$-Analogues of zeta functions of root systems

In this section, we introduce a $q$-analogue of the zeta function of a root system (1.4) and investigate its basic properties.

To do this, we prepare some notation of a root system. For details of the theory of root systems, we refer to [1, 6]. Let $V$ be an $r$-dimensional real vector space with an inner product $\langle \cdot, \cdot \rangle$. We identify the dual space $V^*$ with $V$ via this inner product of $V$. Let $\Delta$ be a root
system of $V$ and

$$\alpha' := \frac{2\alpha}{(\alpha, \alpha)}$$

be the coroot of $\alpha \in \Delta$. Let $\alpha_1, \ldots, \alpha_r$ be simple roots of $\Delta$ and put $\Psi := \{\alpha_1, \ldots, \alpha_r\}$. We denote the sets of positive and negative roots of $\Delta$ by $\Delta_+$ and $\Delta_-$, respectively:

$$\Delta_+ := \{c_1\alpha_1 + \cdots + c_r\alpha_r \in \Delta \mid c_i \geq 0 \ (i = 1, \ldots, r)\},$$

$$\Delta_- := \{c_1\alpha_1 + \cdots + c_r\alpha_r \in \Delta \mid c_i \leq 0 \ (i = 1, \ldots, r)\}.$$

Let $\lambda_1, \ldots, \lambda_r$ be the fundamental weights of $\Delta$ and $P$, $P_+$ and $\rho$ be the weight lattice, the set of all dominant integral weights and the Weyl vector, respectively:

$$P := \bigoplus_{i=1}^r \mathbb{Z}\lambda_i, \quad P_+ := \bigoplus_{i=1}^r \mathbb{Z}_{\geq 0}\lambda_i, \quad \rho := \lambda_1 + \cdots + \lambda_r.$$

We are now in a position to define a $q$-analogue of the zeta function associated with the root system $\Delta$. Let $q$ be a real number satisfying $0 < q < 1$. For complex vectors $s = (s_\alpha)_{\alpha \in \Delta_+}$ and $a = (a_\alpha)_{\alpha \in \Delta_+}$, we define the function $\zeta_r(s, a, \Delta; q)$ by

$$\zeta_r(s, a, \Delta; q) := \sum_{\lambda \in P_+} \prod_{\alpha \in \Delta_+} a_\alpha^{(\alpha^\vee, \lambda + \rho)} \left(1 - q^{(\alpha^\vee, \lambda + \rho)}\right)^{s_\alpha}.$$

The series on the right-hand side of (2.1) converges absolutely for $|a_\alpha| < 1$ ($\alpha \in \Delta_+$).

When $\Delta = \Delta(A_1)$, the function $\zeta_1(s, q^{1/2}, \Delta(A_1); q)$ is a $q$-analogue of the Riemann zeta function introduced in [7], multiplied by $(1 - q)^s$. For general root systems $\Delta$, the functions $\zeta_r(s, a, \Delta; q)$ can be regarded as $q$-analogues of zeta functions (1.4). In fact, when $a_\alpha = q^{t_\alpha}$ (Re $t_\alpha > 0$) and Re $s_\alpha > 1$ for $\alpha \in \Delta_+$, we have

$$\lim_{q \to 1} (1 - q)^{|s|} \zeta_r(s, (q^{t_\alpha})_{\alpha \in \Delta_+}, \Delta; q) = \zeta_r(s, \Delta),$$

where we put

$$|s| = \sum_{\alpha \in \Delta_+} s_\alpha.$$

For $\Delta = \Delta(A_r)$, $\Delta(B_r)$, $\Delta(C_r)$, $\Delta(D_r)$, the functions $\zeta_r(s, a, \Delta; q)$ can be expressed explicitly, as follows.

**Example 2.1.** Let $\Delta = \Delta(A_r)$. Let $\{e_1, \ldots, e_{r+1}\}$ be the standard basis of $(r + 1)$-dimensional real vector space $\mathbb{R}^{r+1}$. Then we have the following:

$$V = \left\{ \sum_{i=1}^{r+1} x_i e_i \mid \sum_{i=1}^{r+1} x_i = 0 \right\},$$

$$\Delta(A_r) = \{e_i - e_j \mid 1 \leq i, j \leq r + 1, \ i \neq j\},$$

$$\Delta_+(A_r) = \{e_i - e_j \mid 1 \leq i < j \leq r + 1\},$$

$$\Psi(A_r) = \{e_1 - e_2, e_2 - e_3, \ldots, e_r - e_{r+1}\}.$$
Thus we have
\[
\zeta_r(s, a, \Delta(A_r); q) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{1 \leq i < j \leq r+1} \frac{a_{ij}^{\langle \sum_{i \leq k < j} a_k \cdot m_1 + \cdots + m_r \cdot \lambda_r \rangle}}{(1 - q^{\langle \sum_{i \leq k < j} a_k \cdot m_1 + \cdots + m_r \cdot \lambda_r \rangle}) a_{ij}}
\]
\[
= \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{1 \leq i < j \leq r+1} \frac{a_{ij}^{m_1 + \cdots + m_r - 1}}{(1 - q^{m_1 + \cdots + m_r}) a_{ij}}.
\]
where we put \(a_\alpha = a_{ij}\) and \(s_\alpha = s_{ij}\) for \(\alpha = e_i - e_j\). In particular, by putting
\[
s_{ij} = 0 \quad (i, j) \neq (1, 2), (1, 3), \ldots, (1, r+1),
\]
\[
a_{ij} = \begin{cases} q^{s_{ij}-1} & (i, j) = (1, 2), (1, 3), \ldots, (1, r+1), \\ 0 & \text{otherwise}, \end{cases}
\]
we obtain a \(q\)-analogue of the multiple zeta function (see \([15]\))
\[
\zeta_q(s_{12}, \ldots, s_{1,r+1}) := (1 - q)^{s_{12} + \cdots + s_{1,r+1}} \sum_{k_1 > \cdots > k_r > 0} q^{k_1 (s_{12} - 1) + \cdots + k_r (s_{1,r+1} - 1)} (1 - q^{k_1})^{s_{12}} \cdots (1 - q^{k_r})^{s_{1,r+1}},
\]
multiplied by \((1 - q)^{-s_{12} - \cdots - s_{1,r+1}}\).

**Example 2.2.** When \(\Delta = \Delta(B_r)\), we have the following:
\[
V = \mathbb{R}^r,
\]
\[
\Delta(B_r) = \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq r \} \cup \{ \pm e_i \mid 1 \leq i \leq r \},
\]
\[
\Delta_+(B_r) = \{ e_i \pm e_j \mid 1 \leq i < j \leq r \} \cup \{ e_i \mid 1 \leq i \leq r \},
\]
\[
\Psi(B_r) = \{ \alpha_j = e_j - e_{j+1} \mid 1 \leq j \leq r - 1 \} \cup \{ \alpha_r = e_r \}.
\]
The simple coroots are given by
\[
\alpha_j^\vee = e_j - e_{j+1} \quad (1 \leq j \leq r - 1),
\]
\[
\alpha_r^\vee = 2e_r,
\]
and the positive coroots can be written as
\[
\begin{cases}
(e_j + e_j)^\vee = \sum_{i \leq k < j} \alpha_k^\vee + 2 \sum_{j \leq k < r} \alpha_k^\vee + \alpha_r^\vee & (1 \leq i < j \leq r), \\
(e_j - e_j)^\vee = \sum_{i \leq k < j} \alpha_k^\vee + \alpha_r^\vee & (1 \leq i < j \leq r), \\
(e_i)^\vee = 2 \sum_{i \leq k < r} \alpha_k^\vee + \alpha_r^\vee & (1 \leq i \leq r).
\end{cases}
\]
Thus we have
\[
\zeta_r(s, a, \Delta(B_r); q) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{1 \leq i \leq r} \frac{a_i^{2(m_1 + \cdots + m_r - 1) + m_r}}{(1 - q^{2(m_1 + \cdots + m_r - 1) + m_r}) a_i}
\]
\[
\times \prod_{1 \leq i < j \leq r} \frac{a_{ij}^{m_1 + \cdots + m_j - 1}}{(1 - q^{m_1 + \cdots + m_j - 1}) a_{ij}}
\]
\[
\times \prod_{1 \leq i < j \leq r} \frac{a_{ij}^{m_1 + \cdots + m_j + 2(m_j + \cdots + m_r) + m_r}}{(1 - q^{m_1 + \cdots + m_j + 2(m_j + \cdots + m_r) + m_r}) a_{ij}}.
\]
Example 2.3. When $\Delta = \Delta(C_r)$, we have the following:

$$V = \mathbb{R}^r,$$

$$\Delta(C_r) = \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq r \} \cup \{ \pm 2e_i \mid 1 \leq i \leq r \},$$

$$\Delta_+(C_r) = \{ e_i \pm e_j \mid 1 \leq i < j \leq r \} \cup \{ 2e_i \mid 1 \leq i \leq r \},$$

$$\Psi(C_r) = \{ \alpha_j = e_j - e_{j+1} \mid 1 \leq j \leq r - 1 \} \cup \{ \alpha_r = 2e_r \}.$$

The simple coroots are given by

$$\alpha_j^\vee = e_j - e_{j+1} \quad (1 \leq j \leq r - 1),$$

$$\alpha_r^\vee = e_r,$$

and the positive coroots can be written as

$$\left\{ \begin{align*}
(e_i + e_j)^\vee &= \sum_{i \leq k < j} \alpha_k^\vee + 2 \sum_{j \leq k \leq r} \alpha_k^\vee \quad (1 \leq i < j \leq r), \\
(e_i - e_j)^\vee &= \sum_{i \leq k < j} \alpha_k^\vee \quad (1 \leq i < j \leq r), \\
(e_i)^\vee &= \sum_{i \leq k < r} \alpha_k^\vee \quad (1 \leq i \leq r).
\end{align*} \right.$$  

Thus we have

$$\zeta_r(s, a, \Delta(C_r); q) = \sum_{m_1 = 1}^{\infty} \cdots \sum_{m_r = 1}^{\infty} \prod_{1 \leq i \leq r} \frac{a_i^{m_1 + \cdots + m_r}}{(1 - q^{m_1 + \cdots + m_r}) \eta_i} \quad \times \prod_{1 \leq i < j \leq r} \frac{a_i^{m_1 + \cdots + m_j - 1}}{(1 - q^{m_1 + \cdots + m_j - 1}) \eta_{ij}} \quad \times \prod_{1 \leq i < j \leq r} \frac{a_i^{m_1 + \cdots + m_j - 1 + 2(m_j + \cdots + m_r) + m_r}}{(1 - q^{m_1 + \cdots + m_j - 1 + 2(m_j + \cdots + m_r)}) \eta_{ij}}.$$  

Example 2.4. When $\Delta = \Delta(D_r)$, we have the following:

$$V = \mathbb{R}^r,$$

$$\Delta(D_r) = \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq r \},$$

$$\Delta_+(D_r) = \{ e_i \pm e_j \mid 1 \leq i < j \leq r \},$$

$$\Psi(D_r) = \{ \alpha_j = e_j - e_{j+1} \mid 1 \leq j \leq r - 1 \} \cup \{ \alpha_r = e_{r-1} + e_r \}.$$

The simple coroots are given by

$$\alpha_j^\vee = e_j - e_{j+1} \quad (1 \leq j \leq r - 1),$$

$$\alpha_r^\vee = e_{r-1} + e_r,$$

and the positive coroots can be written as

$$\left\{ \begin{align*}
(e_i + e_r)^\vee &= \sum_{i \leq k \leq r - 2} \alpha_k^\vee + \alpha_r^\vee \quad (1 \leq i < r), \\
(e_i - e_r)^\vee &= \sum_{i \leq k < j} \alpha_k^\vee \quad (1 \leq i < j \leq r), \\
(e_i + e_j)^\vee &= \sum_{i \leq k < j} \alpha_k^\vee + 2 \sum_{j \leq k \leq r - 2} \alpha_k^\vee + \alpha_{r-1}^\vee + \alpha_r^\vee \quad (1 \leq i < j < r).
\end{align*} \right.$$
Thus we have
\[
\zeta_r(s, a, \Delta(D_r); q) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{1 \leq i < r} \frac{a_i^{m_i+\cdots+m_r-2+m_r}}{(1-q^{m_i+\cdots+m_r-2+m_r})^{s_i}} \times \prod_{1 \leq i < j \leq r} \left( \frac{a_{ij,-}^{m_i+\cdots+m_{j-1}}}{(1-q^{m_i+\cdots+m_{j-1}})^{s_{ij,-}}} \right) \times \prod_{1 \leq i < j \leq r} \left( \frac{a_{ij,+}^{m_i+\cdots+m_{j-1}+2(m_j+\cdots+m_{r-2})+m_{r-1}+m_r}}{(1-q^{m_i+\cdots+m_{j-1}+2(m_j+\cdots+m_{r-2})+m_{r-1}+m_r})^{s_{ij,+}}} \right).
\]

The following proposition implies that the function \( \zeta_r(s, a, \Delta; q) \) is meromorphically continued to the whole space as a function of \( s \) and \( a \).

**Proposition 2.5.** We have the following expression:
\[
\zeta_r(s, a, \Delta; q) = \sum_{r_0=0}^{\infty} \left( \prod_{\alpha \in \Delta_+} \frac{(s_\alpha + r_\alpha - 1)}{r_\alpha} \right) \prod_{i=1}^{r} \frac{\prod_{\alpha \in \Delta_+} (aq_\alpha)^{s_{\alpha, \lambda_i, \lambda_j}}}{1 - \prod_{\alpha \in \Delta_+} (aq_\alpha)^{s_{\alpha, \lambda_i, \lambda_j}}}.
\]

**Proof.** By the binomial expansion, we obtain
\[
(1 - q^{s_\alpha, \lambda_i, \lambda_j}) = \sum_{r_\alpha=0}^{\infty} \left( \frac{s_\alpha + r_\alpha - 1}{r_\alpha} \right) q^{s_\alpha, \lambda_i, \lambda_j}.
\]

Thus we have
\[
\zeta_r(s, a, \Delta; q) = \sum_{\lambda_i \in P_+} \prod_{\alpha \in \Delta_+} a^{s_\alpha, \lambda_i, \lambda_j} \left( \sum_{r_\alpha=0}^{\infty} \left( \frac{s_\alpha + r_\alpha - 1}{r_\alpha} \right) q^{s_\alpha, \lambda_i, \lambda_j} \right).
\]

**Remark 2.6.** It is obscure that it holds for a generic complex vector \( s \) that
\[
\lim_{q \to 1} (1 - q)^{|s|} \zeta_r(s, (q^a_\alpha)_{\alpha \in \Delta_+}, \Delta; q) = \zeta_r(s, \Delta).
\]

We note that, when \( r = 1 \), Kaneko, Kurokawa and Wakayama [7] showed that
\[
\lim_{q \to 1} (1 - q)^{s_1} \zeta_1(s, (q^a_\lambda)_{\lambda \in \Delta_+}, \Delta; q) = \zeta(s)
\]
for all \( s \in \mathbb{C}, s \neq 1 \). This Kaneko–Kurokawa–Wakayama result was generalized to the \( q \)-multiple zeta function (2.2) by Zhao [15].
3. Weyl group symmetry

Let $W$ be the Weyl group of a root system $\Delta$. That is, $W$ is a group generated by reflections $\sigma_\alpha$ with respect to the hyperplane orthogonal to $\alpha \in \Delta$: $W = \langle \sigma_\alpha \mid \alpha \in \Delta \rangle$. Let $B_k(\cdot)$ ($k = 0, 1, 2, \ldots$) be Bernoulli polynomials defined by
\[
\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!},
\]
and for $k := (k_\alpha)_{\alpha \in \Delta}$ with $k_\alpha \in \mathbb{Z}_{\geq 0}$, we put
\[
B_k(\Delta) := \int_0^1 \cdots \int_0^1 \left( \prod_{\alpha \in \Delta_+ \setminus \Psi} B_{k_\alpha}(\{x_\alpha\}) \right) \prod_{j=1}^r B_{k_{\alpha_j}}\left( \big\{ - \sum_{\alpha \in \Delta_+ \setminus \Psi} x_\alpha \langle \alpha^\vee, \lambda_j \rangle \big\} \right)
\times \prod_{\alpha \in \Delta_+ \setminus \Psi} dx_\alpha,
\]
where, for a real number $x$, $\{x\}$ denotes the fractional part of $x$. Komori, Matsumoto and Tsumura [9] obtained the following result.

**Theorem 3.1.** [9, III, Theorem 8] Assume that $\Delta$ is an irreducible root system. For $\nu \in V$, we denote the norm of $\nu$ by $||\nu|| := \langle \nu, \nu \rangle^{1/2}$ and put $k = (k||\alpha||)_{\alpha \in \Delta_+} \in \mathbb{Z}_{\geq 0}^{||\Delta_+||}$. Then we have
\[
\zeta_r(2k, \Delta) = (-1)^{||\Delta_+||} \left( \prod_{\alpha \in \Delta_+} \frac{(2\pi \sqrt{-1})^{2k_\alpha}}{(2k_\alpha)!} \right) B_{2k}(\Delta).
\]

Theorem 3.1 implies that
\[
\zeta_r((2k, \ldots, 2k), \Delta) \in \mathbb{Q}[\pi^{2k||\Delta_+||}]
\]
for $k \in \mathbb{Z}_{>0}$. This result is called Witten’s volume formula.

Komori, Matsumoto and Tsumura [9] deduced Theorem 3.1 from an integral representation of a sum of zeta functions (1.4) which has the Weyl group symmetry. We define the action of the Weyl group $W$ to the complex vector $s = (s_\alpha)_{\alpha \in \Delta_+}$ by
\[
ws = (s_{w^{-1}\alpha})_{\alpha \in \Delta}
\]
for $w \in W$, where we put $s_\alpha = s_{-\alpha}$ for $\alpha \in \Delta_-$. 

**Theorem 3.2.** [9, III, Theorem 6] We put
\[
S(s, \Delta) := \sum_{w \in W} \left( \prod_{\alpha \in \Delta_+ \cap \Delta_-} (-1)^{-s_\alpha} \right) \zeta_r(w^{-1}s),
\]
and assume that $\text{Re } s_\alpha > 1$ for $\alpha \in \Delta_+$. Then we have
\[
S(s, \Delta) = (-1)^{||\Delta_+||} \left( \prod_{\alpha \in \Delta_+} \frac{(2\pi \sqrt{-1})^{s_\alpha}}{\Gamma(s_\alpha + 1)} \right) \int_0^1 \cdots \int_0^1 \left( \prod_{\alpha \in \Delta_+ \setminus \Psi} L(s_\alpha, x_\alpha) \right)
\times \prod_{j=1}^r L(s_{\alpha_j}, - \sum_{\alpha \in \Delta_+ \setminus \Psi} x_\alpha \langle \alpha^\vee, \lambda_j \rangle) \prod_{\alpha \in \Delta_+ \setminus \Psi} dx_\alpha,
\]
where $\Gamma(s)$ denotes the gamma function and we put

$$L(s, x) := -\frac{\Gamma(s + 1)}{(2\pi - 1)^s} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{e^{2\pi\sqrt{-1}nx}}{n^s}.$$  

In this section, we consider a $q$-analogue of Theorem 3.2. We define the action of the Weyl group $W$ to the complex vector $a = (a_\alpha)_{\alpha \in \Delta_+}$ by

$$w^{-1}a = (a_\alpha)_{\alpha \in \Delta_+} (w \in W),$$

where we put $\alpha = q^{x-a}a_\alpha^{-1}$ for $\alpha \in \Delta_-$. We introduce the Weyl group symmetric sum $S(s, a, \Delta; q)$ defined by

$$S(s, a, \Delta; q) := \sum_{w \in W} \left( \prod_{\alpha \in \Delta_+ \cap w\Delta_-} (-1)^{s_\alpha} \right) \zeta_r(w^{-1}s, w^{-1}a, \Delta; q).$$

**Theorem 3.3.** Assume that $\Re s_\alpha > 0$, $q^{s_\alpha} \leq |a_\alpha| < 1$ for all $\alpha \in \Delta_+$. Then we have the following:

$$S(s, a, \Delta; q) = \frac{1}{(2\pi - 1)^{|\Delta_+\setminus\Psi|}} \int_{\mathbb{T}^{|\Delta_+\setminus\Psi|}} \left( \prod_{\alpha \in \Delta_+ \setminus \Psi} \psi_s(a_\alpha z_\alpha; q) \right)$$

$$\times \prod_{j=1}^r \psi_{s_j} \left( \prod_{\alpha \in \Delta_+ \setminus \Psi} \left( z_\alpha \right)^{-\langle \alpha', \lambda_j \rangle} \right) \prod_{\alpha \in \Delta_+ \setminus \Psi} \frac{d z_\alpha}{z_\alpha},$$

where $\mathbb{T}$ is the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$ and we put

$$\psi_s(a; q) := \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{a^n}{(1 - q^n)^s}.$$  

**Proof:** By definition, we have

$$S(s, a, \Delta; q) = \sum_{w \in W} \left( \prod_{\alpha \in \Delta_+ \cap w\Delta_-} (-1)^{s_\alpha} \right) \sum_{\lambda \in \mathbb{P}_+} \prod_{\alpha \in \Delta_+} \frac{a_\alpha^{\langle \alpha', \lambda + \rho \rangle}}{(1 - q^{\langle \alpha', \lambda + \rho \rangle})^{s_{wa}}}. $$

The product $\prod_{\alpha \in \Delta_+}$ can be decomposed as follows:

$$\prod_{\alpha \in \Delta_+} = \prod_{\alpha \in \Delta_+ \cap w^{-1}\Delta_+} \prod_{\alpha \in \Delta_+ \cap w^{-1}\Delta_-}.$$  

Furthermore it holds that

$$\prod_{\alpha \in \Delta_+ \cap w^{-1}\Delta_-} a_\alpha^{\langle \alpha', \lambda + \rho \rangle} = \prod_{\alpha \in \Delta_+ \cap w^{-1}\Delta_-} \frac{(q^{s_{wa}} a_\alpha^{-1})^{\langle \alpha', \lambda + \rho \rangle}}{(1 - q^{\langle \alpha', \lambda + \rho \rangle})^{s_{wa}}}.$$

$$= \prod_{\alpha \in \Delta_+ \cap w\Delta_-} (-1)^{s_\alpha} \prod_{\alpha \in \Delta_- \cap w^{-1}\Delta_+} \frac{a_\alpha^{\langle \alpha', \lambda + \rho \rangle}}{(1 - q^{\langle \alpha', \lambda + \rho \rangle})^{s_{wa}}}.$$
Thus we have

\[
S(s, a_1, \lambda; q) = \sum_{w \in W} \sum_{\lambda \in P_+} \prod_{\alpha \in w^{-1} \Delta_+} \frac{a_1^{(\alpha \vee, \lambda + \rho)}}{(1 - q^{(\alpha \vee, \lambda + \rho)})^{s_{w \alpha}}}
= \sum_{w \in W} \sum_{\lambda \in P_+} \prod_{\alpha \in \Delta_+} \frac{a_1^{(w^{-1} \alpha, \lambda + \rho)}}{(1 - q^{(w^{-1} \alpha \vee, \lambda + \rho)})^{s_{w \alpha}}}
= \sum_{w \in W} \sum_{\lambda \in P_+} \prod_{\alpha \in \Delta_+} \frac{a_1^{(\alpha, w(\lambda + \rho))}}{(1 - q^{(\alpha \vee, w(\lambda + \rho))})^{s_{w \alpha}}}.
\]

Let \( \Delta \) be the union of boundaries of all Weyl chambers. Then, for \( \lambda \in P \setminus \Delta \), there exist unique \( w \in W \) and \( \lambda' \in P_+ \) satisfying \( \lambda = w(\lambda' + \rho) \). Thus we have

\[
S(s, a_1, \lambda; q) = \sum_{\lambda \in P \setminus \Delta} \prod_{\alpha \in \Delta_+} \frac{a_1^{(\alpha, \lambda)}}{(1 - q^{(\alpha \vee, \lambda)})^{s_{\alpha}}}.
\]

Here, by observing

\[
\frac{a_1^{(\alpha \vee, \lambda)}}{(1 - q^{(\alpha \vee, \lambda)})^{s_{\alpha}}} = \frac{1}{2\pi \sqrt{-1}} \int_{\mathbb{T}} z_{\alpha}^{-\langle \alpha \vee, \lambda \rangle} \psi_{s_{\alpha}}(a_1 z_{\alpha}; q) \frac{dz_{\alpha}}{z_{\alpha}},
\]

we find that

\[
S(s, a_1, \lambda; q) = \sum_{\lambda \in P \setminus \Delta} \prod_{\alpha \in \Delta_+} \frac{a_1^{(\alpha \vee, \lambda)}}{(1 - q^{(\alpha \vee, \lambda)})^{s_{\alpha}}} \prod_{\alpha \in P \setminus \Delta} \frac{a_1^{(\alpha \vee, \lambda)}}{(1 - q^{(\alpha \vee, \lambda)})^{s_{\alpha}}}
\times \prod_{\alpha \in \Delta_+} \frac{1}{2\pi \sqrt{-1}} \int_{\mathbb{T}} z_{\alpha}^{-\langle \alpha \vee, \lambda \rangle} \psi_{s_{\alpha}}(a_1 z_{\alpha}; q) \frac{dz_{\alpha}}{z_{\alpha}}
\]

\[
= \frac{1}{(2\pi \sqrt{-1})^{[\Delta_+ \setminus \Psi]}} \sum_{\lambda \in P \setminus \Delta} \prod_{\alpha \in \Delta_+ \setminus \Psi} \frac{a_1^{(\alpha \vee, \lambda)}}{(1 - q^{(\alpha \vee, \lambda)})^{s_{\alpha}}}
\times \prod_{\alpha \in \Delta_+ \setminus \Psi} \int_{\mathbb{T}} z_{\alpha}^{-\langle \alpha \vee, \lambda \rangle} \psi_{s_{\alpha}}(a_1 z_{\alpha}; q) \frac{dz_{\alpha}}{z_{\alpha}}.
\]

We now write \( \lambda = \sum_{j=1}^{r} n_j \lambda_j \). Since

\[
\int_{\mathbb{T}} \psi_{s_{\alpha}}(a_1 z_{\alpha}; q) \frac{dz_{\alpha}}{z_{\alpha}} = 0,
\]

\[
S(s, a_1, \lambda; q) = \sum_{\lambda \in P \setminus \Delta} \prod_{\alpha \in \Delta_+} \frac{a_1^{(\alpha \vee, \lambda)}}{(1 - q^{(\alpha \vee, \lambda)})^{s_{\alpha}}} \prod_{\alpha \in P \setminus \Delta} \frac{a_1^{(\alpha \vee, \lambda)}}{(1 - q^{(\alpha \vee, \lambda)})^{s_{\alpha}}}
\times \prod_{\alpha \in \Delta_+} \int_{\mathbb{T}} z_{\alpha}^{-\langle \alpha \vee, \lambda \rangle} \psi_{s_{\alpha}}(a_1 z_{\alpha}; q) \frac{dz_{\alpha}}{z_{\alpha}} = 0.
\]
we can extend the summation range $P \setminus H_\Delta$ to the set of all $\lambda$ satisfying $n_j \neq 0$ ($1 \leq j \leq r$). Thus we obtain

$$S(s, a, \Delta; q) = \frac{1}{(2\pi \sqrt{-1})|\Delta_+ \setminus \Psi|} \sum_{n_j \neq 0} \prod_{1 \leq j \leq r} (1 - q^{n_j})^s \prod_{j=1}^{r} \frac{a_{n_j}^n}{a_{a_j}} \times \prod_{a \in \Delta_+ \setminus \Psi} \int_{\mathbb{T}} z^{-\langle a^{\vee}, n_1 \lambda_1 + \cdots + n_r \lambda_r \rangle} \psi_{s_{a}}(a_{a} z_a; q) \frac{dz_a}{z_a}$$

which completes the proof of the theorem.  

\[ \square \]

4. Properties of functions $\psi_k(a; q)$

In this section, we investigate basic properties of functions $\psi_k(a; q)$ for $k \in \mathbb{Z}_{\geq 0}$. The results established in this section will be used in the next section.

**Proposition 4.1.** Let $k \in \mathbb{Z}_{\geq 0}$. Then we have the following.

1. The function $\psi_k(a; q)$ satisfies the following $q$-difference relation:

$$\psi_k(q^a a; q) = \psi_k(a; q) - \psi_{k-1}(a; q),$$

where we put $\psi_0(a; q) = -1$.

2. The function $\psi_k(a; q)$ has the following symmetry:

$$\psi_k(q^k a^{-1}; q) = (-1)^k \psi_k(a; q).$$

3. The function $\psi_k(a; q)$ can be written as follows:

$$\psi_k(a; q) = \sum_{r=0}^\infty \binom{k + r - 1}{r} \left( \frac{q^r a}{1 - q^r a} + (-1)^k \frac{q^{r+k} a^{-1}}{1 - q^{r+k} a^{-1}} \right). \quad (4.1)$$

This expression gives the meromorphic continuation of $\psi_k(a; q)$ to the whole complex plane. The function $\psi_k(a; q)$ is holomorphic except at simple poles $a = q^{\mathbb{Z}_{\leq 0}}, q^{k+\mathbb{Z}_{\geq 0}}$.

**Proof.** The claims (1) and (2) are clear from the definitions. The claim (3) follows from the binomial expansion given by

$$\frac{1}{(1 - q^n)^k} = \sum_{r=0}^\infty \binom{k + r - 1}{r} q^{nr} \quad (n > 0).$$

By using Proposition 4.1(3) repeatedly, we have

$$\psi_k(q^a a; q) = \sum_{i=0}^{n} \binom{n}{i} (-1)^i \psi_{k-i}(a; q) \quad (4.2)$$

for $n \geq 1$, where we put $\psi_k(a; q) = 0$ for $k \in \mathbb{Z}_{\leq -1}$.
The following proposition implies that the function \( \psi_k(a; q) \) can be considered to be a \( q \)-analogue of the periodic Bernoulli polynomial \( B_k(\{x\}) \).

**Proposition 4.2.** Let \( k \in \mathbb{Z}_{>0} \). Then, for \( t, x \in \mathbb{R} \) and \( (t, x) \notin (\mathbb{Z}_{\leq 0} \cup (k + \mathbb{Z}_{\geq 0})) \times \mathbb{Z} \), we have

\[
\lim_{q \to 1} (1 - q)^k \psi_k(q^t e^{2\pi \sqrt{-1} x}; q) = - \frac{(2\pi \sqrt{-1})^k}{k!} B_k(\{x\}).
\]

**Proof.** By (4.2) and Proposition 4.1(2), it enough to show the proposition for \( 0 \leq t < k \). When \( 0 < t < k \), the proposition follows immediately from the following well-known Fourier series expansion of the periodic Bernoulli polynomial:

\[
B_k(\{x\}) = -k! \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{e^{2\pi \sqrt{-1}n}}{(2\pi \sqrt{-1})^k n^k}.
\]

We now show the proposition for \( t = 0 \) and \( x \notin \mathbb{Z} \) by induction on \( k \). By Proposition 4.1(3), we have

\[
\psi_1(e^{2\pi \sqrt{-1}x}; q) = 2\sqrt{-1} \sin(2\pi x) \sum_{r=1}^{\infty} \frac{q^r}{q^{2r} - 2 \cos(2\pi x)q^r + 1} + \frac{e^{2\pi \sqrt{-1}x}}{1 - e^{2\pi \sqrt{-1}x}}.
\]

It follows that

\[
\lim_{q \to 1} (1 - q) \psi_1(e^{2\pi \sqrt{-1}x}; q) = 2\sqrt{-1} \sin(2\pi x) \int_0^1 \frac{du}{u^2 - 2 \cos(2\pi x)u + 1}
\]

\[
= 2\pi \sqrt{-1} \int_0^\infty \frac{\sin(2\pi x)}{\cosh(2\pi t) - \cos(2\pi x)} dt
\]

\[
= -2\pi \sqrt{-1} B_1(\{x\}),
\]

where we put \( u = e^{-2\pi t} \) in the second equality. In the last equality, we used the integral representation of the Bernoulli polynomial (see [3, (21), p. 38]). Thus we find that the proposition holds for \( k = 1 \).

We next assume that the proposition is true for \( k \geq 1 \). By Proposition 4.1(1) and (2), we have

\[
\psi_{k+1}(e^{2\pi \sqrt{-1}x}; q) = (-1)^{k+1} \psi_{k+1}(q^k e^{2\pi \sqrt{-1}x}; q) + \psi_k(e^{2\pi \sqrt{-1}x}; q).
\]

Thus the induction hypothesis implies that

\[
\lim_{q \to 1} (1 - q)^{k+1} \psi_{k+1}(e^{2\pi \sqrt{-1}x}; q) = (-1)^{k+1} \left( - \frac{(2\pi \sqrt{-1})^{k+1}}{(k + 1)!} B_{k+1}(\{-x\}) \right)
\]

\[
= - \frac{(2\pi \sqrt{-1})^{k+1}}{(k + 1)!} B_{k+1}(\{x\}),
\]

which proves the proposition for \( k + 1 \). We thus finish the proof of the proposition. \(\square\)

By definition, the generating function of the Bernoulli polynomials \( B_k(x) \) is given by

\[
t e^{xt} / e^t - 1.
\]

Meanwhile the generating function of the functions \( \psi_k(a; q) \) becomes the Kronecker function.
PROPOSITION 4.3. We define the theta function \( \theta(a; q) \) by
\[
\theta(a; q) := \prod_{m=0}^{\infty} (1 - a q^m) (1 - a^{-1} q^{m+1})
\]
and the Kronecker function \( F(\alpha; a; q) \) by
\[
F(\alpha, a; q) := \frac{\theta'(1; q) \theta(q \alpha; q)}{\theta(q; q) \theta(\alpha; q)}
\]
for \( a, \alpha \in \mathbb{C} \). Then, for \( a \in \mathbb{C} \) satisfying \( q < |a| < 1 \), the Kronecker function \( F(\alpha, a; q) \) is expanded into a Laurent series around \( \alpha = 1 \), as follows:
\[
F(\alpha, a; q) = \frac{1}{\alpha - 1} + \sum_{k=0}^{\infty} (-1)^k \psi_k(q^k a; q)(\alpha - 1)^k.
\]

Proof. See [8, Proposition 2.2].

The following proposition will play an important role in the next section.

PROPOSITION 4.4. For \( k_1, k_2 \in \mathbb{Z}_{>0} \), we have the following:
\[
\psi_{k_1}(a_1; q) \psi_{k_2}(a_2; q) = \sum_{k=0}^{k_1} \left( \frac{k_1 + k_2 - k - 1}{k_2 - 1} \right) (-1)^{k_1-k} \psi_k(a_1 a_2; q) \psi_{k_1+k_2-k}(q^{k_1-k} a_2; q) + \sum_{l=0}^{k_2} \left( \frac{k_1 + k_2 - l - 1}{k_1 - 1} \right) (-1)^{k_2-l} \psi_l(a_1 a_2; q) \psi_{k_1+k_2-l}(q^{k_2-l} a_1; q) + \psi_{k_1+k_2}(a_1 a_2; q).
\]

Proof. It is known that the Kronecker function satisfies the following Fay’s identity (see [2] or [8, Theorem 2.3]):
\[
F(\alpha_1, a_1; q) F(\alpha_2, a_2; q) = F(\alpha_1, a_1 a_2; q) F(\alpha_1^{-1} a_2, a_2; q) + F(\alpha_2, a_1 a_2; q) F(\alpha_1 a_2^{-1}, a_1; q).
\]
We now expand both sides into Laurent series of \( \alpha_1 - 1 \) and \( \alpha_2 - 1 \), and then compare the coefficients of \( (\alpha_1 - 1)^{k_1-1} \) and \( (\alpha_2 - 1)^{k_2-1} \). Then, by Proposition 4.3, we obtain the proposition.

5. The cases of \( \Delta = \Delta(A_1), \Delta(A_2), \Delta(A_3) \)

In this section, we show that, when \( \Delta = \Delta(A_1), \Delta(A_2), \Delta(A_3) \) and all components of the vector \( s \) are positive integers, the ‘Weyl group symmetric’ linear combination of the functions \( \zeta_r(s, a, \Delta; q) \) introduced in Section 3 can be written in terms of the functions \( \psi_k \). By letting \( q \to 1 \) in this result, we obtain explicit expressions for Witten’s volume formulas (3.1) of \( A_1, A_2 \) and \( A_3 \) types.

Example 5.1. Let \( r = 1 \). By putting \( s = k(k \in \mathbb{Z}_{>1}) \) in Theorem 3.3, we have
\[
S(k, a, \Delta(A_1); q) = \psi_k(a; q).
\]
Thus we obtain
\[ \zeta_1(k, a, \Delta(A_1); q) + (-1)^{-k} \zeta_1(k, q^k a^{1}, \Delta(A_1); q) = \psi_k(a; q). \] (5.1)

In particular, when \( k = 2m \) (\( m \geq 1 \)) and \( a = q^m \), we have
\[ \zeta_1(2m, q^m, \Delta(A_1); q) = \frac{1}{2} \psi_{2m}(q^m; q). \]

We now put \( k = 2m(m \geq 1), a = q^l(0 < l < 2m) \), multiply both sides by \((1 - q)^{2m}\) and take the limit as \( q \to 1 \) in (5.1). Then, by Proposition 4.2, we obtain the well-known formula
\[ \zeta(2m) = (-1)^{m+1} \frac{B_{2m}(2\pi)^{2m}}{2(2m)!}, \]
which is due to Euler. Here \( B_n := B_n(0) \) denotes the \( n \)th Bernoulli number.

We next consider the case where \( \Delta = \Delta(A_2) \). Then for \( k \in \mathbb{Z}_{\geq 0}^3 \), the linear combination \( S(k, a, \Delta(A_2); q) \) can be written in terms of the functions \( \psi_k \), as follows.

**Theorem 5.2.** We have
\[
S(k, a, \Delta(A_2); q) = (-1)^{k_{12}} \sum_{l=0}^{k_{13}} \binom{k_{12} + k_{13} - l - 1}{k_{12} - 1} (-1)^{l_{12}} \psi_l(a_{12}a_{13}; q) {\psi}_{k_{12} + k_{13} + k_{23} - l}(a_{12} a_{23} q^{k_{12}}; q) + \sum_{l=0}^{k_{12}} \binom{k_{12} + k_{13} - l - 1}{k_{13} - 1} (-1)^{l_{12}} \psi_l(a_{12}a_{13}; q) {\psi}_{k_{12} + k_{13} + k_{23} - l}(a_{13} a_{23} q^{k_{12}}; q).
\]

**Proof.** By Theorem 3.3, we have
\[
S(k, a, \Delta(A_2); q) = \frac{1}{2\pi \sqrt{-1}} \int_{\mathbb{T}} \psi_{k_{12}}(a_{13} z_{13}; q) \psi_{k_{13}}(a_{12} z_{13}^{-1}; q) \times \psi_{k_{23}}(a_{23} z_{13}^{-1}; q) \frac{d z_{13}}{z_{13}}.
\]

Proposition 4.4 gives
\[
\psi_{k_{12}}(a_{13} z_{13}; q) \psi_{k_{13}}(a_{12} z_{13}^{-1}; q) = \sum_{k=0}^{k_{13}} \binom{k_{12} + k_{13} - k - 1}{k_{12} - 1} (-1)^{k_{13} - k} \psi_{k}(a_{12}a_{13}; q) {\psi}_{k_{12} + k_{13} - k}(q^{k_{13} - k} a_{12} z_{13}^{-1}; q)
\]
\+
\sum_{l=0}^{k_{12}} \binom{k_{12} + k_{13} - l - 1}{k_{13} - 1} (-1)^{l_{12}} \psi_l(a_{12}a_{13}; q) {\psi}_{k_{12} + k_{13} - l}(q^{k_{12} - l} a_{13} z_{13}; q)
\+
\psi_{k_{12} + k_{13}}(a_{12}a_{13}; q).
\]
Thus we find that

\[
S(k, a, \Delta(A_2); q) = \sum_{k=0}^{k_13} \binom{k_12 + k_13 - k - 1}{k_12 - 1} (-1)^{k_13-k} \psi_k(a_{12}a_{13}; q) \\
\times \frac{1}{2\pi \sqrt{-1}} \int_{\mathcal{T}} \psi_{k_23}(a_{23}z_{13}^{-1}; q) \psi_{k_{12}+k_{13}-k}(a_{12}^{-1}a_{12}z_{13}^{-1}; q) \frac{dz_{13}}{z_{13}^{23+1}} + \sum_{i=0}^{k_{12}} \binom{k_{12} + k_{13} - l - 1}{k_{13} - 1} (-1)^{k_{12}-l} \psi_l(a_{12}a_{13}; q) \psi_{k_{12}+k_{13}+k_{23}-k}(a_{12}^{-1}a_{23}q^{k_{12}}; q) \\
\times \frac{1}{2\pi \sqrt{-1}} \int_{\mathcal{T}} \psi_{k_23}(a_{23}z_{13}^{-1}; q) \psi_{k_{12}+k_{13}-l}(a_{12}a_{13}q^{k_{12}-l}; q) \frac{dz_{13}}{z_{13}^{23+1}},
\]

which completes the proof of the theorem. \(\square\)

When \(k_{12} = k_{13} = k_{23} = 2m\), Theorem 5.2 becomes

\[
S((2m), a, \Delta(A_2); q) = \sum_{i=0}^{2m} \binom{4m - i - 1}{2m - 1} \psi_i(a_{12}a_{13}; q) \\
\times (\psi_{6m-i}(a_{12}^{-1}a_{23}q^{2m}; q) + \psi_{6m-i}(a_{13}^{-1}a_{23}^{-1}q^{4m}; q)). \tag{5.2}
\]

In particular, by letting \(a_{12}, a_{13}, a_{23} \to q^m\) in (5.2), we obtain

\[
6\zeta_2((2m), (q^m), \Delta(A_2); q) = \sum_{i=0}^{2m} \binom{4m - i - 1}{2m - 1} \left(2\psi_{1,2m}(q^{2m}; q)\psi_{6m-i}(q^{2m}; q) + \binom{2m - 1}{i - 1} q^{m}\psi_{6m-i}(q^{5m-i}; q)\right),
\]

where we put

\[
\psi_{k,l}(a; q) := \psi_k(a; q) + (-1)^{k+1} \binom{l - 1}{k - 1} \frac{q^l a^{-1}}{1 - q q^l a^{-1}}
\]

for \(l \in \mathbb{Z}_{\geq 1}\).

Let us consider what is obtained by letting \(q \to 1\) in (5.2). We put

\[
a_{12} = q^t e^{2\pi \sqrt{-1} x_{12}}, \quad a_{13} = q^t e^{2\pi \sqrt{-1} x_{13}}, \quad a_{23} = q^t e^{2\pi \sqrt{-1} x_{23}},
\]

where \(t, x_{12}, x_{13}\) and \(x_{23}\) satisfy the following conditions:

\[
0 < t < 2m, \quad x_{12}, x_{13}, x_{23} \in \mathbb{R}, \quad x_{12} + x_{13}, x_{12} - x_{23}, x_{13} + x_{23} \notin \mathbb{Z}.
\]
We now multiply both sides of (5.2) by \((1 - q)^{6m}\) and take the limit as \(q \to 1\). Then, by Proposition 4.2, we have
\[
\sum_{w \in W} \zeta_2((2m), (x_\alpha)_{\alpha \in \Delta_+}, \Delta(A_2)) = (2\pi \sqrt{-1})^{6m} \sum_{i=0}^{2m} \frac{(4m - i - 1)}{2m - 1} \frac{B_i(\{x_{12} + x_{13}\})}{i!} \times \left( \frac{B_{6m-i}(\{x_{23} - x_{12}\})}{(6m - i)!} + \frac{B_{6m-i}(\{-x_{13} - x_{23}\})}{(6m - i)!} \right),
\]
where we put
\[
\zeta_r(s, a, \Delta) := \sum_{\lambda \in P_+} \prod_{\alpha \in \Delta_+} a_{\alpha}^{\langle \alpha, \lambda \rangle} \times (\langle \alpha, \lambda \rangle \in \text{even}).
\]
When \(a_\alpha = 1\) for all \(\alpha \in \Delta_+\), \(\zeta_r(s, a, \Delta)\) is equal to the zeta function of the root system \(\zeta_r(s, \Delta)\). By letting \(x_{12}, x_{13}, x_{23} \to 0\) in (5.3), we obtain the following result discovered independently by Zagier, Garoufalidis and Weinstein (see [14]):
\[
6\zeta_2((2m), \Delta(A_2)) = 8 \sum_{i=0}^{2m} \frac{(4m - i - 1)}{2m - 1} \zeta(i) \zeta(6m - i).
\]
This result is an explicit expression for Witten’s volume formula (3.1) of \(A_2\) type.

Finally, we consider the case where \(\Delta = \Delta(A_3)\). Proposition 4.4 yields the following theorem.

**THEOREM 5.3.** We have
\[
S((2m), a, \Delta(A_3); q) = \sum_{i=0}^{2m} \frac{(4m - i - 1)}{2m - 1} \left( A(a; q) + B(a; q) + C(a; q) + D(a; q) \right),
\]
where \(A(a; q), B(a; q), C(a; q)\) and \(D(a; q)\) are given by the following:
\[
A(a; q) := \sum_{0 \leq j \leq 2m} \frac{(2m + i - j - 1)}{i - 1} \frac{(6m + i - j - t - 1)}{2m - 1} \times \psi_j(a_{12}a_{13}a_{14}; q) \psi_t(a_{12}^{-1}a_{13}^{-1}a_{24}a_{34}q^j; q)
\]
\[
\times (\psi_{12m-j}^{-1}(a_{12}^{-1}a_{13}^{-1}a_{24}a_{34}q^{6m-i}; q) + \psi_{12m-j}(a_{12}^{-1}a_{23}^{-1}a_{34}^{-1}q^{4m-i}; q)),
\]
\[
B(a; q) := \sum_{0 \leq j \leq 2m} \frac{(2m + i - j - 1)}{i - 1} \frac{(6m + i - j - u - 1)}{4m + i - j - 1} \times \psi_j(a_{12}a_{13}a_{14}; q) \psi_u(a_{12}a_{13}a_{24}^{-1}a_{34}^{-1}q^{i-u}; q)
\]
\[
\times (\psi_{12m-j-u}(a_{12}^{-1}a_{23}^{-1}a_{34}q^{6m-u}; q) + \psi_{12m-j-u}(a_{12}^{-1}a_{23}a_{24}q^{4m-u}; q)),
\]
\[
C(a; q), D(a; q) \text{ are similarly defined.}
\]
Proof. Theorem 3.3 implies that

\[
\sum_{0 \leq v \leq 4m + i - k} \left( 2m + i - k - 1 \right) \left( 6m + i - k - v - 1 \right) \left( 2m - 1 \right) \times \psi \left( a_{12}a_{13}a_{14}; q \right) \psi \left( a_{14}^{-1}a_{24}^{-1}a_{34}^{-1}q^{k + v - i}; q \right) \\
\times \left( \psi \left( a_{12}a_{23}a_{24}q^{6m + i - k - v}; q \right) + \psi \left( a_{13}^{-1}a_{23}^{-1}a_{34}^{-1}q^{8m + i - k - v}; q \right) \right),
\]

\[
\sum_{0 \leq w \leq 2m} \left( 2m + i - k - 1 \right) \left( 6m + i - k - w - 1 \right) \left( 4m + i - k - 1 \right) \times \psi \left( a_{12}a_{13}a_{14}; q \right) \psi \left( a_{14}^{-1}a_{24}^{-1}a_{34}^{-1}q^{6m}; q \right) \\
\times \left( \psi \left( a_{12}a_{23}a_{24}q^{6m}; q \right) + \psi \left( a_{13}^{-1}a_{23}^{-1}a_{34}^{-1}q^{8m}; q \right) \right).
\]

By Theorem 5.2, we have

\[
S((2m), a, \Delta(A_3); q) = \int_{\mathbb{T}^2} S((2m), a_{12}z_{14}^{-1}, a_{13}, a_{23}z_{14}^{-1}z_{24}^{-1}), \Delta(A_2); q) \\
\times \psi_{2m}(a_{14}z_{14}; q) \psi_{2m}(a_{24}z_{24}; q) \psi_{2m}(a_{34}z_{14}^{-1}z_{24}^{-1}; q) \frac{dz_{14} dz_{24}}{z_{14} z_{24}}.
\]

where we put

\[
I_1 := \frac{1}{(2\sqrt{2} - 1)^2} \int_{\mathbb{T}^2} \psi \left( a_{12}a_{13}z_{14}^{-1}; q \right) \psi_{6m-i}(a_{12}^{-1}a_{23}z_{24}^{-1}q^{2m}; q) \\
\times \psi_{2m}(a_{14}z_{14}; q) \psi_{2m}(a_{24}z_{24}; q) \psi_{2m}(a_{34}z_{14}^{-1}z_{24}^{-1}; q) \frac{dz_{14} dz_{24}}{z_{14} z_{24}},
\]

\[
I_2 := \frac{1}{(2\sqrt{2} - 1)^2} \int_{\mathbb{T}^2} \psi \left( a_{12}a_{13}z_{14}^{-1}; q \right) \psi_{6m-i}(a_{13}a_{23}z_{14}^{-1}z_{24}^{-1}; q) \\
\times \psi_{2m}(a_{14}z_{14}; q) \psi_{2m}(a_{24}z_{24}; q) \psi_{2m}(a_{34}z_{14}^{-1}z_{24}^{-1}; q) \frac{dz_{14} dz_{24}}{z_{14} z_{24}}.
\]
Let us calculate the integral $I_1$ by using Proposition 4.4 repeatedly. Since
\[
\psi_l(a_{12}a_{13}z_{14}^{-1}; q)\psi_{2m}(a_{14}z_{14}; q)
= \sum_{k=0}^{i} \left( \frac{i+2m-k-1}{2m-1} \right) (-1)^{i-k}\psi_k(a_{12}a_{13}a_{14}; q)\psi_{i+2m-k}(q^{-k}a_{14}z_{14}; q)
+ \sum_{l=0}^{2m} \left( \frac{i+2m-l-1}{i-1} \right) (-1)^{l}\psi_l(a_{12}a_{13}a_{14}; q)\psi_{i+2m-l}(q^{2m-l}a_{12}a_{13}z_{14}^{-1}; q),
\]
we have
\[
I_1 = \sum_{k=0}^{i} \left( \frac{i+2m-k-1}{2m-1} \right) (-1)^{i-k}\psi_k(a_{12}a_{13}a_{14}; q)
\times \frac{1}{2\pi \sqrt{-1}} \int_{\mathbb{T}} \psi_{i+4m-k}(a_{14}a_{34}z_{24}^{-1}q^{-k}; q) dz_{24}
\times \psi_{6m-i}(a_{12}^{-1}a_{23}z_{24}^{-1}q^{2m}; q)\psi_{2m}(a_{24}z_{24}; q) dz_{24}
+ \sum_{l=0}^{2m} \left( \frac{i+2m-l-1}{i-1} \right) (-1)^{l}\psi_l(a_{12}a_{13}a_{14}; q)
\times \frac{1}{2\pi \sqrt{-1}} \int_{\mathbb{T}} \psi_{i+4m-l}(a_{12}a_{13}a_{34}^{-1}z_{24}q^{4m-l}; q) dz_{24}
\times \psi_{6m-i}(a_{12}^{-1}a_{23}z_{24}^{-1}q^{2m}; q)\psi_{2m}(a_{24}z_{24}; q) dz_{24}.
\]

Furthermore, we obtain
\[
\psi_{i+4m-k}(a_{14}a_{34}z_{24}^{-1}q^{-k}; q)\psi_{2m}(a_{24}z_{24}; q)
= \sum_{t=0}^{i+4m-k} \left( \frac{6m+i-k-u-1}{2m-1} \right) (-1)^{i+4m-k-t}\psi_t(a_{14}a_{24}a_{34}q^{-k}; q)
\times \psi_{6m+i-k-t}(a_{24}z_{24}q^{i+4m-k-t}; q)
+ \sum_{u=0}^{2m} \left( \frac{6m+i-k-u-1}{4m+i-k-1} \right) (-1)^{2m-u}\psi_u(a_{14}a_{24}a_{34}q^{-k}; q)
\times \psi_{6m+i-k-u}(a_{14}a_{34}q^{i+2m-k-u}; q),
\]
and
\[
\psi_{i+4m-l}(a_{12}a_{13}a_{34}^{-1}z_{24}q^{4m-l}; q)\psi_{2m}(a_{24}z_{24}; q)
= \sum_{v=0}^{4m-l+i} \left( \frac{6m+i-l-v-1}{2m-1} \right) (-1)^{i}\psi_v(a_{12}^{-1}a_{13}a_{24}a_{34}q^{i}; q)
\times \psi_{6m-l+i-v}(a_{24}z_{24}q^{4m-l+i-v}; q)
\]
\[+ \sum_{w=0}^{2m} \left( \frac{6m + i - l - w - 1}{4m + i - l - 1} \right) (-1)^{w+i-l} \psi_w(a_{12}^{-1}a_{13}^{-1}a_{24}a_{34}q^l; q)\]
\[\times \psi_{6m+i-l-w}(a_{12}^{-1}a_{13}^{-1}a_{34}^{-1}a_{24}q^{2m-w}; q)\].

Thus we find that

\[I_1 = \sum_{i=0}^{2m} \left( \frac{4m - i - 1}{2m - 1} \right) (A_0(a; q) + B_0(a; q) + C_0(a; q) + D_0(a; q)),\]

where we put

\[A_0(a; q) := \sum_{0 \leq j \leq 2m} \sum_{0 \leq t \leq 4m+i-j} \left( \frac{2m + i - j - 1}{i - 1} \right) \left( \frac{6m + i - j - t - 1}{2m - 1} \right) \times \psi_j(a_{12}a_{13}a_{14}; q) \psi_t(a_{12}^{-1}a_{13}^{-1}a_{24}a_{34}q^i; q) \psi_{12m-j-t}(a_{12}^{-1}a_{23}^{-1}a_{24}^{-1}q^{6m-i}; q),\]

\[B_0(a; q) := \sum_{0 \leq j \leq 2m} \sum_{0 \leq u \leq 2m} \left( \frac{2m + i - j - 1}{i - 1} \right) \left( \frac{6m + i - j - u - 1}{4m + i - j - 1} \right) \times \psi_j(a_{12}a_{13}a_{14}; q) \times \psi_u(a_{12}a_{13}^{-1}a_{24}^{-1}q^{i-u}; q) \times \psi_{12m-j-u}(a_{13}^{-1}a_{23}^{-1}a_{34}q^{6m-u}; q),\]

\[C_0(a; q) := \sum_{0 \leq k \leq i} \sum_{0 \leq v \leq 4m+i-k} \left( \frac{2m + i - k - 1}{i - k} \right) \left( \frac{6m + i - k - v - 1}{2m - 1} \right) \times \psi_k(a_{12}a_{13}a_{14}; q) \times \psi_v(a_{14}^{-1}a_{24}^{-1}a_{34}^{-1}q^{k+v}; q) \times \psi_{12m-k-v}(a_{12}^{-1}a_{23}a_{24}q^{6m+i-k-v}; q),\]

\[D_0(a; q) := \sum_{0 \leq k \leq i} \sum_{0 \leq w \leq 2m} \left( \frac{2m + i - k - 1}{i - k} \right) \left( \frac{6m + i - k - w - 1}{4m + i - k - 1} \right) \times \psi_k(a_{12}a_{13}a_{14}; q) \times \psi_w(a_{14}a_{24}a_{34}^{-1}q^{i-k}; q) \times \psi_{12m-k-w}(a_{12}^{-1}a_{14}^{-1}a_{23}a_{34}^{-1}q^{6m}; q).\]

Since the integral \(I_2\) can be calculated similarly, we finish the proof the theorem. \(\square\)

We now put \(a_{ij} = q^i e^{2\pi \sqrt{-1}t_{ij}} (0 < t < 2m, x_{ij} \in \mathbb{R})\) in Theorem 5.3. By setting \(x_{ij}\) appropriately, multiplying both sides by \((1 - q)^{12m}\) and letting \(q \to 1\), we obtain the following:

\[\sum_{w \in \mathcal{W}} \zeta_3((2m), (x_\alpha)_{\alpha \in \Delta_+}, \Delta(A_2)) = -(2\pi \sqrt{-1})^{12m} (A((x_\alpha)_{\alpha \in \Delta_+}) + B((x_\alpha)_{\alpha \in \Delta_+}) + C((x_\alpha)_{\alpha \in \Delta_+}) + D((x_\alpha)_{\alpha \in \Delta_+})), (5.4)\]
where we put

\[
A((x_\alpha)_{\alpha \in \Delta_+}) := \sum_{0 \leq j \leq 2m \atop 0 \leq t \leq 4m + i - j} \binom{2m + i - j - 1}{i - 1} \binom{6m + i - j - t - 1}{2m - 1} \times \frac{B_j([x_{12} + x_{13} + x_{14}]) \ B_t([-x_{12} - x_{13} + x_{24} + x_{34}])}{j! t!} \times \frac{B_{12m - j - t}([-x_{12} - x_{23} - x_{34}])}{(12m - t)!},
\]

\[
B_0((x_\alpha)_{\alpha \in \Delta_+}) := \sum_{0 \leq j \leq 2m \atop 0 \leq u \leq 2m} \binom{2m + i - j - 1}{i - 1} \binom{6m + i - j - u - 1}{4m + i - j - 1} \times \frac{B_j([x_{12} + x_{13} + x_{14}]) \ B_u([x_{12} + x_{13} - x_{24} - x_{34}])}{j! u!} \times \frac{B_{12m - j - u}([-x_{12} - x_{23} + x_{34}])}{(12m - j - u)!},
\]

\[
C_0((x_\alpha)_{\alpha \in \Delta_+}) := \sum_{0 \leq k \leq i \atop 0 \leq v \leq 4m + i - k} \binom{2m + i - k - 1}{i - k} \binom{6m + i - k - v - 1}{2m - 1} \times \frac{B_k([x_{12} + x_{13} + x_{14}]) \ B_v([-x_{14} + x_{24} + x_{34}])}{k! v!} \times \frac{B_{12m - k - v}([-x_{12} + x_{23} + x_{34}])}{(12m - k - v)!},
\]

\[
D_0((x_\alpha)_{\alpha \in \Delta_+}) := \sum_{0 \leq k \leq j \atop 0 \leq w \leq 2m} \binom{2m + i - k 1}{i - k} \binom{6m + i - k - w - 1}{4m + i - k - 1} \times \frac{B_k([x_{12} + x_{13} + x_{14}]) \ B_w([x_{12} + x_{13} + x_{14}])}{k! w!} \times \frac{B_{12m - k - w}([-x_{12} - x_{14} + x_{23} - x_{34}])}{(12m - k - w)!}.
\]

By letting \(x_{ij} \to 0\) in (5.4), we obtain the following result due to Gunnells and Sczech [5, Proposition 8.5], which can be regarded as an explicit expression for Witten’s volume formula (3.1) of \(A_3\) type:

\[
24 \zeta_3((2m), \Delta(A_3)) = 16 \sum_{i=0}^{2m} \binom{4m - i - 1}{2m - 1} (A + B + C + D),
\]

where we put

\[
A := \sum_{0 \leq j \leq 2m \atop 0 \leq t \leq 4m + i - j \atop j, i \equiv 0 \mod 2} \binom{2m + i - j - 1}{i - 1} \binom{6m + i - j - t - 1}{2m - 1} \zeta(j) \zeta(t) \zeta((12m - j - t)),
\]
By substituting the Kronecker function \( F \) where
deformation of the zeta function of 1.

**Definition 6.2.**
Let define the two-parameter deformation of the zeta function of the root system (1).

Two-parameter deformations of zeta functions of root systems

where \( t \) of the zeta function (2.1).

\[
\text{Proposition 6.1. Put } c(a) := a/(1 - a). \text{ Then, when } \text{Re} \, s_\alpha > 0, \ |a_\alpha| < 1 \text{ for all } \alpha \in \Delta_+,
\]

we have

\[
\zeta_r(s, a, \Delta; q) = \frac{1}{(2\pi)^{1/\Delta_+}} \int_{\mathbb{T}^{1/\Delta_+}} \prod_{i=1}^r \prod_{\alpha \in \Delta_+} (a_\alpha t_\alpha z_\alpha)^{\langle \alpha_i \rangle, \lambda_i} \times \prod_{\alpha \in \Delta_+} \psi_s(t_\alpha^{-1} z_\alpha^{-1}; q) \prod_{\alpha \in \Delta_+} \frac{d z_\alpha}{z_\alpha}, \tag{6.1}
\]

where \( t_\alpha (\alpha \in \Delta_+) \) are complex numbers satisfying \( 1 < |t_\alpha| < |a_\alpha^{-1}| \).

This proposition follows immediately from the series expression of \( c(a) \), given by

\[
c(a) = \sum_{n=1}^{\infty} a^n \quad (|a| < 1).
\]

By substituting the Kronecker function \( F(a, a; p) \) for the rational function \( c(a) \) in (6.1), we define the two-parameter deformation of the zeta function of the root system \( \Delta \), as follows.

**Definition 6.2.**
Let \( p \) be a complex number satisfying \( 0 < |p| < 1 \) and assume that \( \text{Re} \, s_\alpha > 0, \ |q_\alpha| < |a_\alpha| < 1 \) for all \( \alpha \in \Delta_+ \). We put \( \beta = (\beta_1, \ldots, \beta_r) \). We define the two-parameter deformation of the zeta function of \( \Delta \) by

\[
\zeta_r(s, a, \beta, \Delta; p, q) := \frac{1}{(2\pi)^{1/\Delta_+}} \int_{\mathbb{T}^{1/\Delta_+}} \prod_{i=1}^r F\left(\beta_i, \prod_{\alpha \in \Delta_+} (a_\alpha t_\alpha z_\alpha)^{\langle \alpha_i \rangle, \lambda_i}; p\right) \times \prod_{\alpha \in \Delta_+} \psi_s(t_\alpha^{-1} z_\alpha^{-1}; q) \prod_{\alpha \in \Delta_+} \frac{d z_\alpha}{z_\alpha},
\]

where \( t_\alpha \) are complex numbers satisfying the following:

\[
\max\{ |p^{1/\Delta_+} a_\alpha^{-1}, 1 \} < |t_\alpha| < |a_\alpha^{-1}|.
\]
PROPOSITION 6.3. We have

(1) \[ \lim_{p \to 0} \zeta_r(s, a, \beta, \Delta; p, q) = (-1)^r \zeta_r(s, a, \Delta; q). \]

(2) The function \( \zeta_r(s, a, \beta, \Delta; p, q) \) has the following series representation:

\[
\zeta_r(s, a, \beta, \Delta; p, q) = \sum_{\lambda \in \mathcal{P} \setminus \mathcal{H}_\Delta} \prod_{i=1}^r \frac{1}{\mathcal{P}(\alpha_i^\lambda)} \beta_i \prod_{\alpha \in \Delta_+} (1 - q^{(\alpha, \lambda)})_{\delta_{\alpha}}.
\]

(3) Let \(|p| < |\beta_i| < 1\) \((i = 1, \ldots, r)\). We also denote \( \beta_i \) by \( \beta_a(\alpha = \alpha_i) \). Then, by the following expression, the function \( \zeta_r(s, a, \beta, \Delta; p, q) \) becomes a meromorphic function of \( s \) and \( a \):

\[
\zeta_r(s, a, \beta, \Delta; p, q) = \sum_{w \in W} (-1)^{|w|} \Psi(\Delta_+, \Delta_-) \prod_{\alpha \in \Delta_+} (1 - q^{|\alpha|})_{\delta_{\alpha}} \prod_{\alpha \in \Delta_-} (1 - q^{\langle \alpha, \beta \rangle})_{\delta_{\alpha}}.
\]

Proof. The claim (1) follows from the fact that \( F(\alpha, a; p) \to -c(a) + 1/(\alpha - 1) \) as \( p \to 0 \).

The claim (2) is an immediate consequence of the following Laurent series expansion of the Kronecker function \( F(\alpha, a; p) \) (see [12]):

\[
F(\alpha, a; p) = \sum_{n \in \mathbb{Z}} \frac{a^n}{1 - p^n} \quad (|p| < |a| < 1).
\]

Let us prove the claim (3). Since there exist unique \( w \in W \) and \( \lambda' \in \mathcal{P}_+ \) satisfying \( \lambda = w(\lambda' + \rho) \) for all \( \lambda \in \mathcal{P} \setminus \mathcal{H}_\Delta \), claim (2) implies that

\[
\zeta_r(s, a, \beta, \Delta; p, q) = \sum_{w \in W} \sum_{\lambda' \in \mathcal{P}_+} \prod_{\alpha \in \Psi} \frac{1}{\mathcal{P}(\alpha^\lambda, w(\lambda + \rho))} \beta_a \prod_{\alpha \in \Delta_+} (1 - q^{(\alpha, \lambda)})_{\delta_{\alpha}}
\]

\[
= \sum_{w \in W} \sum_{\lambda' \in \mathcal{P}_+} \prod_{\alpha \in \Psi} \frac{1}{\mathcal{P}(\alpha, \lambda + \rho)} \beta_{w\alpha} \prod_{\alpha \in \Delta_+} (1 - q^{(\alpha, \lambda + \rho)})_{\delta_{\alpha}}.
\]
By decomposing the products $\prod_{\alpha \in w^{-1}\Psi}$ and $\prod_{\alpha \in w^{-1}\Delta_+}$ into
\[
\prod_{\alpha \in w^{-1}\Psi} = \prod_{\alpha \in w^{-1}\Psi \cap \Delta_+} \prod_{\alpha \in w^{-1}\Psi \cap \Delta_-},
\]
\[
\prod_{\alpha \in w^{-1}\Delta_+} = \prod_{\alpha \in w^{-1}\Delta_+ \cap \Delta_+} \prod_{\alpha \in w^{-1}\Delta_+ \cap \Delta_-}
\]
and using the binomial expansion, we obtain the claim. \qed

Example 6.4. When $\Delta = \Delta(A_1)$, by Proposition 6.3(2), $\zeta_1(s, a, 1, \Delta(A_1); p, q)$ can be expressed as follows:

$$
\zeta_1(1, a, 1, \Delta(A_1); p, q) = - \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{a^n}{(1 - p^n)(1 - q^n)}.
$$

Thus the function $\zeta_1(1, qe^{2\pi \sqrt{-1}x}, 1, \Delta(A_1); p, q)$ has the following Taylor series expansion around $x = 0$:

$$
\zeta_1(1, qe^{2\pi \sqrt{-1}x}, 1, \Delta(A_1); p, q) = - \sum_{k=1}^{\infty} \frac{(2\pi \sqrt{-1})^k x^{k-1}}{(k - 1)!} Z_k(p, q) x^{k-1},
$$

where we put

$$
Z_k(p, q) := \sum_{n=1}^{\infty} n^{k-1} \frac{q^n - (-1)^k p^n}{(1 - p^n)(1 - q^n)}
$$

for $k \in \mathbb{Z}_{>0}$. The numbers $Z_k(p, q)$ are essentially the same as the elliptic zeta values, introduced by Felder and Varchenko [4].

Let us consider an analogy of Theorem 3.3 for $\zeta_r(s, a, \beta, \Delta; p, q)$. We define the $p$-deformation of $S(s, a, \beta, \Delta; q)$ by

$$
S(s, a, \beta, \Delta; p, q) := \sum_{w \in W} \left( \prod_{\alpha \in \Delta_+ \cap w \Delta_-} (-1)^{-s_\alpha} \right) \zeta_r(w^{-1}s, w^{-1}a, \beta, \Delta; p, q).
$$

By a similar argument used to prove Theorem 3.3, we obtain

$$
S(s, a, \beta, \Delta; p, q) = \sum_{\lambda \in P \setminus H_\Delta} s(\lambda, \beta, \Delta) \prod_{\alpha \in \Delta_+} \frac{a^{(a', \lambda)}_\alpha}{(1 - q^{(a', \lambda)}_\alpha)^{s_\alpha}},
$$

where we put

$$
s(\lambda, \beta, \Delta) := \sum_{w \in W} \prod_{i=1}^{r} \frac{1}{p^{(a', w_\lambda)^{s_\alpha}} \beta_i - 1}.
$$

When $\Delta = \Delta(A_1), \Delta(A_2), \Delta(A_3), s(\lambda, (1, \ldots, 1), \Delta)$ can be calculated, as follows.

**Theorem 6.5.** We have

1. When $\Delta = \Delta(A_1)$, for $\lambda \in P \setminus H_\Delta$, we have

$$
s(\lambda, 1, \Delta(A_1)) = -1.
$$

Thus it holds that

$$
\lim_{\beta_i \to 1} S(s, a_{12}, \beta_1, \Delta(A_1); p, q) = -S(s, a_{12}, \Delta(A_1); q).
$$
(2) When $\Delta = \Delta(A_2)$, for $\lambda \in P \setminus H_{\Delta}$, we have

$$s(\lambda, (1, 1), \Delta(A_2)) = 1.$$ 

Thus it holds that

$$\lim_{\beta_1, \beta_2 \to 1} S(s, a, \beta, \Delta(A_2); p, q) = S(s, a, \Delta(A_2); q).$$

(3) When $\Delta = \Delta(A_3)$, for $\lambda \in P \setminus H_{\Delta}$, we have

$$s(\lambda, (1, 1, 1), \Delta(A_3)) = -1.$$ 

Thus it holds that

$$\lim_{\beta_1, \beta_2, \beta_3 \to 1} S(s, a, \beta, \Delta(A_3); p, q) = -S(s, a, \Delta(A_2); q).$$

Proof. When $\Delta = \Delta(A_r)$, the Weyl group $W$ becomes the symmetric group $S_{r+1}$ of degree $r + 1$. The group $W = S_{r+1}$ acts on the space

$$V = \left\{ \sum_{i=1}^{r+1} x_i e_i \mid \sum_{i=1}^{r+1} x_i = 0 \right\}$$

by permutations of indices of the vectors $e_i$. Thus, when $\Delta = \Delta(A_1)$, we have

$$s(\lambda, 1, \Delta(A_1)) = \frac{1}{p^{n_1} - 1} + \frac{1}{p^{-n_1} - 1} = -1$$

for $\lambda = n_1 \lambda_1$. Similarly, we have

$$s(\lambda, (1, 1), \Delta(A_2)) = \frac{1}{(p^{n_1} - 1)(p^{n_2} - 1)} + \frac{1}{(p^{-n_1} - 1)(p^{n_1+n_2} - 1)}$$

$$+ \frac{1}{(p^{-n_2} - 1)(p^{-n_1} - 1)} + \frac{1}{(p^{n_1+n_2} - 1)(p^{-n_2} - 1)} + \frac{1}{(p^{n_1} - 1)(p^{-n_2-n_1} - 1)}$$

$$= 1$$

for $\lambda = n_1 \lambda_1 + n_2 \lambda_2$ and

$$s(\lambda, (1, 1, 1), \Delta(A_3))$$

$$= \frac{1}{(p^{n_1} - 1)(p^{n_2} - 1)(p^{n_3} - 1)} + \frac{1}{(p^{-n_1} - 1)(p^{n_1+n_2} - 1)(p^{n_3} - 1)}$$

$$+ \frac{1}{(p^{-n_2} - 1)(p^{-n_1} - 1)(p^{n_1+n_2+n_3} - 1)} + \frac{1}{(p^{n_1+n_2+n_3} - 1)(p^{n_2} - 1)(p^{-n_1-n_2} - 1)}$$

$$+ \frac{1}{(p^{n_1+n_2} - 1)(p^{n_2} - 1)(p^{n_2+n_3} - 1)} + \frac{1}{(p^{n_1+n_2+n_3} - 1)(p^{-n_3} - 1)(p^{-n_2} - 1)}$$
\[
\frac{1}{(p^{n_1} - 1)(p^{n_2+n_3} - 1)(p^{-n_3} - 1)} + \frac{1}{(p^{n_2} - 1)(p^{-n_1-n_2} - 1)(p^{n_1+n_2+n_3} - 1)} + \frac{1}{(p^{n_1+n_2} - 1)(p^{n_3} - 1)(p^{-n_1-n_2} - 1)} \\
+ \frac{1}{(p^{-n_1-n_2} - 1)(p^{n_1} - 1)(p^{n_2+n_3} - 1)} + \frac{1}{(p^{n_2+n_3} - 1)(p^{-n_3} - 1)(p^{-n_1-n_2} - 1)} + \frac{1}{(p^{n_2} - 1)(p^{n_3} - 1)(p^{n_1+n_2} - 1)} \\
+ \frac{1}{(p^{n_1} - 1)(p^{n_2+n_3} - 1)(p^{n_2} - 1)} + \frac{1}{(p^{n_1+n_2} - 1)(p^{n_3} - 1)(p^{-n_1-n_2} - 1)} + \frac{1}{(p^{n_2} - 1)(p^{n_3} - 1)(p^{n_1+n_2} - 1)} \\
+ \frac{1}{(p^{n_1-n_2} - 1)(p^{n_1+n_2+n_3} - 1)(p^{n_3} - 1)} + \frac{1}{(p^{n_3} - 1)(p^{n_1-n_2} - 1)(p^{-n_3} - 1)} + \frac{1}{(p^{n_1} - 1)(p^{n_2} - 1)(p^{n_1+n_2} - 1)} \\
+ \frac{1}{(p^{n_1} - 1)(p^{n_2} - 1)(p^{n_3} - 1)} + \frac{1}{(p^{n_3} - 1)(p^{n_1-n_2-n_3} - 1)(p^{n_3} - 1)} + \frac{1}{(p^{n_1} - 1)(p^{n_2} - 1)(p^{n_1+n_2} - 1)} \\
= -1
\]

for \( \lambda = n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3 \). Thus we finish the proof of the theorem. \(\square\)

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Masaki Kato
Department of General Education
National Institute of Technology
Toyama College
13 Hongo-machi, Toyama city
Toyama 939-8630
Japan
(E-mail: mkato@nc-toyama.ac.jp)