MULTIPLE ZETA FUNCTIONS OF KANEKO–TSUMURA TYPE AND THEIR VALUES AT POSITIVE INTEGERS

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Abstract. Kaneko and Tsumura introduced a new kind of multiple zeta function \( \eta(k_1, \ldots, k_r; s_1, \ldots, s_r) \). This is an analytic function of complex variables \( s_1, \ldots, s_r \), while \( k_1, \ldots, k_r \) are nonpositive integer parameters. In this paper, we first extend this function to an analytic function \( \eta(s'_1, \ldots, s'_r; s_1, \ldots, s_r) \) of \( 2r \) complex variables. Then we investigate its special values at positive integers. In particular, we prove some linear relations among these \( \eta \)-values and the multiple zeta values \( \zeta(k_1, \ldots, k_r) \) of Euler–Zagier type.

1. Introduction

In [4], Kaneko and Tsumura introduced and studied a new kind of multiple zeta function,

\[
\eta(k_1, \ldots, k_r; s) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{\text{Li}_{k_1,\ldots,k_r}(1-e^t)}{1-e^t} t^{s-1} dt, \quad (1.1)
\]

which is a ‘twin sibling’ of the Arakawa–Kaneko multiple zeta function [1]

\[
\xi(k_1, \ldots, k_r; s) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{\text{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{e^t - 1} t^{s-1} dt. \quad (1.2)
\]

Here \( k_1, \ldots, k_r \) are integers, \( s \) is a complex variable and \( \text{Li}_{k_1,\ldots,k_r} \) denotes the multiple polylogarithm of one variable:

\[
\text{Li}_{k_1,\ldots,k_r}(z) := \sum_{0 < n_1 < \cdots < n_r} \frac{z^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}}.
\]

Among other things, when \( r = 1 \), they proved the equality

\[
\eta(k; l) = \eta(l; k) \quad (1.3)
\]

for nonpositive integers \( k \) and \( l \), and experimentally observed that the same equality holds even when \( k \) and \( l \) are positive integers.

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In [4, Section 5], Kaneko and Tsumura also considered a variant of (1.1) with \( r \) complex variables:

\[
\eta(k_1, \ldots, k_r; s_1, \ldots, s_r) := \frac{1}{\prod_{j=1}^{r} \Gamma(s_j)} \int \cdots \int_0^\infty \frac{\text{Li}_{k_1,\ldots,k_r}(1 - e^{\sum_{j=1}^r t_j}, 1 - e^{\sum_{j=2}^r t_j}, \ldots, 1 - e^{t_r})}{\prod_{j=1}^{r} (1 - e^{\sum_{j=i}^r t_j})} \times \prod_{j=1}^{r} t_j^{s_j - 1} dt_j,
\]

(1.4)

where

\[
\text{Li}_{k_1,\ldots,k_r}(z_1, \ldots, z_r) := \sum_{0 < n_1 < \cdots < n_r} \frac{z_1^{n_1} z_2^{n_2 - n_1} \cdots z_r^{n_r - n_{r-1}}}{n_1^{k_1} \cdots n_r^{k_r}}
\]

(1.5)

is the multiple polylogarithm of \( r \) variables. For certain technical reasons, their consideration of the function (1.4) is limited to the case in which \( k_1, \ldots, k_r \) are nonpositive integers.

In the present paper, we extend the function (1.4) to a holomorphic function of \( 2r \) complex variables \( \eta(s'_1, \ldots, s'_r; s_1, \ldots, s_r) \), which satisfies

\[
\eta(s'_1, \ldots, s'_r; s_1, \ldots, s_r) = \eta(s_1, \ldots, s_r; s'_1, \ldots, s'_r).
\]

(1.6)

When \( r = 1 \), it also gives an extension of the function (1.1). In particular, we obtain a proof of the equality (1.3) for arbitrary complex numbers \( k \) and \( l \).

The second and the main purpose of this paper is to study the special values of the function \( \eta(s'_1, \ldots, s'_r; s_1, \ldots, s_r) \) at positive integers. We prove certain linear relations among these values and the multiple zeta values

\[
\zeta(k_1, \ldots, k_r) := \sum_{0 < n_1 < \cdots < n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}
\]

(1.7)

for positive integers \( k_1, \ldots, k_r \) with \( k_r > 1 \). For example, as special cases of our result (Theorem 4.4), we can show that

\[
\eta(1, \ldots, 1; 1, \ldots, 1) = \zeta(2, \ldots, 2)
\]

and

\[
\eta(k; l) = \sum_{0 < a_1 \leq \cdots \leq a_k = b_l \geq b_{l-1} \geq \cdots \geq b_1 \geq 0} \frac{1}{a_1 \cdots a_k b_1 \cdots b_l}
\]

(the right-hand side of the latter identity can be expressed as a finite sum of multiple zeta values).

The contents of this paper are as follows. In Section 2, we define the function \( \eta(s'_1, \ldots, s'_r; s_1, \ldots, s_r) \) and prove its analytic continuation to \( \mathbb{C}^{2r} \) by the classical contour integral method. In Section 3, basic formulas on its special values at positive integers are obtained. Some of them are used in Section 4, where we discuss relations of \( \eta \)-values with the multiple zeta values. Finally, in Appendix A, we prove a formula which expresses the values \( \eta(k_1, \ldots, k_r; l) \) of the function in (1.1), where \( k_1, \ldots, k_r \) and \( l \) are positive integers, in terms of the multiple zeta values.
2. Definition of $\eta(s'_1, \ldots, s'_r; s_1, \ldots, s_r)$

Let $r$ be a positive integer. The definition (1.5) of the multiple polylogarithm is meaningful for arbitrary complex numbers $k_1, \ldots, k_r$ and complex numbers $z_1, \ldots, z_r$ of absolute values less than 1. We begin with its analytic continuation. For a positive real number $\varepsilon$, denote by $C_\varepsilon$ the contour which goes from $+\infty$ to $\varepsilon$ along the real line, goes round counterclockwise along the circle of radius $\varepsilon$ about the origin, and then goes back to $+\infty$ along the real line, as shown.

![Contour Diagram]

**Lemma 2.1.** The multiple polylogarithm $\text{Li}_s(z)$, where $s = (s_1, \ldots, s_r)$, $z = (z_1, \ldots, z_r) \in \mathbb{C}$ and $|z_i| < 1$, has the following integral expression:

$$
\text{Li}_s(z) = \prod_{j=1}^r \frac{\Gamma(1 - s_j)}{2\pi i e^{\pi i s_j}} \int_{C_\varepsilon} \prod_{j=1}^r \frac{\zeta^{u_j s_j - 1}}{e^{u_j + \cdots + u_r} - z_j}.
$$

(2.1)

Here we assume that $\varepsilon > 0$ is sufficiently small.

By (2.1), $\text{Li}_s(z)$ is holomorphically continued to the region $(s, z) \in \mathbb{C}^r \times (\mathbb{C} \setminus \mathbb{R}_{\geq 1})^r$.

**Proof.** First we note that

$$
\prod_{j=1}^r \Gamma(s_j) \cdot \text{Li}_s(z) = \prod_{j=1}^r \Gamma(s_j) \sum_{l_1, \ldots, l_r > 0} \frac{l_1 l_2 \cdots l_r}{l_1 (l_1 + l_2)^2 \cdots (l_1 + \cdots + l_r)^r}
$$

$$
= \sum_{l_1, \ldots, l_r > 0} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^r e^{-(l_1 + \cdots + l_r)u_j} z_j^{l_j} u_j^{s_j - 1} \ du_j
$$

$$
= \int \cdots \int_0^\infty \prod_{j=1}^r z_j^{s_j - 1} u_j^{s_j - 1} \ du_j,
$$

that is,

$$
\text{Li}_s(z) = \prod_{j=1}^r \frac{1}{\Gamma(s_j)} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^r \frac{z_j^{s_j - 1} u_j^{s_j - 1}}{e^{u_j + \cdots + u_r} - z_j}.
$$

(2.2)

This gives an analytic continuation to the region

$$(s, z) \in \{s \in \mathbb{C} \mid \Re(s) > 0\}^r \times (\mathbb{C} \setminus \mathbb{R}_{\geq 1})^r.$$

Moreover, for each $z \in (\mathbb{C} \setminus \mathbb{R}_{\geq 1})^r$, there exists a neighborhood $K$ of $z$ and $\varepsilon_0 > 0$ such that $e^{u_j + \cdots + u_r} - z_j' \neq 0$ for $j = 1, \ldots, r$ whenever $(z'_1, \ldots, z'_r) \in K$, $0 < \varepsilon < \varepsilon_0$ and $u_1, \ldots, u_r \in C_\varepsilon$. If this is the case, we have

$$
\int \cdots \int_0^\infty \prod_{j=1}^r z_j^{s_j - 1} u_j^{s_j - 1} \ du_j = \prod_{j=1}^r \frac{1}{e^{2\pi i s_j} - 1} \int_{C_\varepsilon} \prod_{j=1}^r \frac{z_j^{s_j - 1} u_j^{s_j - 1}}{e^{u_j + \cdots + u_r} - z_j}.
$$

(2.3)
The formula (2.1) is deduced from (2.2) and (2.3) because
\[ \Gamma(s_j) \Gamma(1 - s_j) = \frac{\pi}{\sin \pi s_j} = \frac{2\pi i}{e^{\pi i s_j} - e^{-\pi i s_j}} = \frac{2\pi i e^{\pi i s_j}}{e^{2\pi i s_j} - 1}. \]

It is easy to see that (2.1) gives a meromorphic continuation to \( \mathbb{C}^r \times (\mathbb{C} \setminus \mathbb{R}_{\geq 1})^r \). The possible poles \( s_j = 1, 2, 3, \ldots \), which come from the factor \( \Gamma(1 - s_j) \), are removable, since we already know the holomorphy on \( \mathfrak{R}(s_j) > 0 \) from (2.2).

\[ \tag{2.4} \]

Remark 2.2. Tsumura pointed out to the author that the above lemma is a special case of Komori’s result \([5]\).

Now we define the main object of this article.

\[ \text{Definition 2.3. For } s = (s_1, \ldots, s_r), s' = (s'_1, \ldots, s'_r) \in \mathbb{C}^r \text{ with } \Re(s_j) > 0, \text{ we define } \]
\[ \eta(s'; s) := \prod_{j=1}^{r} \frac{1}{\Gamma(s_j)} \int \cdots \int_{0}^{\infty} \frac{\text{Li}_k(1 - e^{s_1 + \cdots + s_r}) \cdots (1 - e^{s_r})}{(1 - e^{s_1 + \cdots + s_r}) \cdots (1 - e^{s_r})} \prod_{j=1}^{r} t_j^{s_j - 1} dt_j. \]

Let us discuss the convergence of the integral (2.4) and its analytic continuation. For this purpose, the following estimate is useful.

\[ \text{Lemma 2.4. The function } (e^{(u+1)/2})/(e^u + e^t - 1) \text{ is bounded on } \]
\[ u, t \in D_\varepsilon := \{ z \in \mathbb{C} \mid \Re(z) \geq -\varepsilon, \ -\varepsilon \leq \Im(z) \leq \varepsilon \} \]
\[ \text{for sufficiently small } \varepsilon > 0. \]

\[ \text{Proof. Put } x = e^u - 1/2 \text{ and } y = e^t - 1/2. \text{ We bound the product } \]
\[ \left| \frac{e^{(u+1)/2}}{e^u + e^t - 1} \right| = \frac{\Re(x) + \Re(y)}{|x + y|} \cdot \frac{|x| + |y|}{\Re(x) + \Re(y)} \cdot \frac{|xy|^{1/2}}{|x| + |y|} \cdot \frac{|e^u|^{1/2}}{x} \cdot \frac{|e^t|^{1/2}}{y} \]
\[ \text{factorwise.} \]

The first and third factors are bounded by 1 and 1/2, respectively. It is also easy to see that the fourth and fifth factors are bounded; indeed, \( e^u / (e^u - 1/2) \) is a continuous function on \( D_\varepsilon \), hence is bounded on any bounded region, and tends to 1 with \( \Re(u) \to \infty \).

To bound the second factor, note that there is a constant \( \theta = \theta_\varepsilon > 0 \), depending only on \( \varepsilon \), such that \( |\arg x|, |\arg y| \leq \theta \) holds for any \( u, t \in D_\varepsilon \). If \( \varepsilon > 0 \) is sufficiently small, \( \theta \) also becomes arbitrarily small. In particular, we may assume that \( \theta < \pi/2 \). Then we have \( \Re(x) \geq |x| \cos \theta, \Re(y) \geq |y| \cos \theta \) and hence
\[ \frac{|x| + |y|}{\Re(x) + \Re(y)} \leq \frac{1}{\cos \theta}, \]
\[ \text{which gives a bound for the second factor.} \]

To examine the convergence of the integral (2.4), we substitute the expression (2.1) into it. Then we have
\[ \eta(s'; s) = \prod_{j=1}^{r} \frac{\Gamma(1 - s_j)}{2\pi i e^{\pi i s_j'} \Gamma(s_j)} \int \cdots \int_{(\mathbb{R}_{>0})^r} \int_{(C_\varepsilon)^r} \prod_{j=1}^{r} e^{u_j + \cdots + u_r} + e^{t_j + \cdots + t_r} - 1. \]
\[ \tag{2.5} \]
By Lemma 2.4 applied to $D_{re}$ in place of $D_s$, this integrand is bounded by a constant multiple of
\[
\left| \prod_{j=1}^{r} \frac{u_j^{s_j-1} t_j^{s_j-1}}{e^{(u_j+\cdots+u_r+t_1+\cdots+t_r)/2}} \right| = \prod_{j=1}^{r} \frac{u_j^{s_j-1}}{e^{u_j/2}} \cdot \prod_{j=1}^{r} \frac{t_j^{s_j-1}}{e^{t_j/2}}.
\]
Hence the absolute convergence of the integral (2.4) is reduced to those of one-variable integrals, which are elementary and well known. Note that, if $\Re(s_j), \Re(s'_j) > 0$, we may replace (2.5) by the simpler and symmetric expression
\[
\eta(s'; s) = \prod_{j=1}^{r} \frac{1}{\Gamma(s_j) \Gamma(s'_j)} \int_0^\infty \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{r} \frac{u_j^{s_j-1} t_j^{s_j-1}}{e^{u_j+\cdots+u_r+e^{t_j+\cdots+t_r} - 1}} du_j dt_j.
\]
Moreover, the same estimate shows that we can transform each integral on $t_j \in \mathbb{R}>0$ in (2.5) to the integral along the contour $C_{e}$ to obtain
\[
\eta(s'; s) = \prod_{j=1}^{r} \frac{\Gamma(1-s_j) \Gamma(1-s'_j)}{(2\pi i)^2 e^{\pi(s_j+s'_j)}} \int_{(C_{e})^{2r}} \prod_{j=1}^{r} \frac{u_j^{s_j-1} t_j^{s_j-1}}{e^{u_j+\cdots+u_r+e^{t_j+\cdots+t_r} - 1}} du_j dt_j,
\]
and that this integral is convergent for any $s, s' \in \mathbb{C}'$. Therefore, we have shown the following proposition.

**Proposition 2.5.** The function $\eta(s'; s)$ can be holomorphically continued to $\mathbb{C}' \times \mathbb{C}'$, and satisfies $\eta(s'; s) = \eta(s; s')$.

**Remark 2.6.** Recall that the Euler–Zagier multiple zeta function
\[
\xi(s_1, \ldots, s_r) = \sum_{0 < n_1 < \cdots < n_r} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}
\] has the integral expression
\[
\xi(s_1, \ldots, s_r) = \prod_{j=1}^{r} \frac{1}{\Gamma(s_j)} \int_0^\infty \cdots \int_0^\infty \frac{u_j^{s_j-1}}{e^{u_j+\cdots+u_r - 1}} du_j.
\]
This formula, together with (2.6), suggests that our function $\eta(s'; s)$ may be regarded as a ‘double multiple zeta function’. Furthermore, we may also consider a ‘multiple multiple zeta function’
\[
\eta(s_1, \ldots, s_l) = \prod_{i=1}^{l} \prod_{j=1}^{r} \frac{1}{\Gamma(s_{ij})} \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{l} t_{ij}^{s_{ij}-1}}{\sum_{i=1}^{l} e^{t_{ij}+\cdots+t_{ir} - 1}} dt_{ij}
\]
fors $s_l = (s_{i1}, \ldots, s_{ir})$. In this paper, however, we do not pursue such a generalization for $l \geq 3$.

### 3. Special values at positive integers

From now on, we study the values of $\eta$ at positive integers. In particular, we are interested in the relationship between these values and the multiple zeta values. First let us recall some basic notation on the multiple zeta values.
A finite sequence $\mathbf{k} = (k_1, \ldots, k_r)$ of positive integers is called an index. We put
\[
|\mathbf{k}| := k_1 + \cdots + k_r, \quad d(\mathbf{k}) := r,
\]
and call them the weight and the depth of $\mathbf{k}$, respectively.

An index $\mathbf{k} = (k_1, \ldots, k_r)$ is called admissible if $k_r > 1$ (or $r = 0$, that is, $\mathbf{k}$ is the empty index). When this is the case, the multiple zeta value $\zeta(\mathbf{k})$ is defined by the multiple series (1.7), and has the iterated integral expression
\[
\zeta(\mathbf{k}) = \int_{(x_{j_1}) \in \Delta(\mathbf{k})} \prod_{j=1}^{r} \frac{dx_{j_1}}{1-x_{j_1}} \frac{dx_{j_2}}{x_{j_2}} \cdots \frac{dx_{j_k}}{x_{j_k}}.
\]  

(3.1)

Here $\Delta(\mathbf{k})$ is a domain of dimension $|\mathbf{k}|$ defined by
\[
\Delta(\mathbf{k}) := \left\{ (x_{ji})_{j=1,\ldots,r, i=1,\ldots,k_j} \mid 0 < x_{11} < \cdots < x_{1k_1} < x_{21} < \cdots < x_{2k_2} < \cdots \right. \\
\left. \cdots < x_{rk_r} < 1 \right\}.
\]

Now we return to the study of $\eta$-values. We start with an integral expression for $\eta(\mathbf{k}; \mathbf{l})$ similar to (3.1), using domains of the form
\[
\nabla(\mathbf{l}):= \left\{ (x_{ji})_{j=1,\ldots,r, i=1,\ldots,k_j} \mid 1 > x_{11} > \cdots > x_{1k_1} > x_{21} > \cdots > x_{2k_2} > \cdots \right. \\
\left. \cdots > x_{rk_r} > 1 \right\}.
\]

PROPOSITION 3.1. For any indices $\mathbf{k} = (k_1, \ldots, k_r)$ and $\mathbf{l} = (l_1, \ldots, l_r)$ of depth $r > 0$, we have
\[
\eta(\mathbf{k}; \mathbf{l}) = \int_{(x_{ji}) \in \nabla(\mathbf{k})} \prod_{j=1}^{r} \left\{ \frac{dx_{j_1} dy_{j_1}}{1-x_{j_1} y_{j_1}} \frac{dx_{j_2} dy_{j_2}}{1-x_{j_2} y_{j_2}} \cdots \frac{dx_{j_k} dy_{j_k}}{1-x_{j_k} y_{j_k}} \right\}.
\]  

(3.2)

Proof. Since all variables are positive, we can use the integral (2.6):
\[
\eta(\mathbf{k}; \mathbf{l}) = \prod_{j=1}^{r} \frac{1}{\Gamma(k_j) \Gamma(l_j)} \int_0^\infty \int_0^\infty \prod_{j=1}^{r} \frac{u_j^{k_j-1} l_j^{l_j-1}}{e^{u_j} + \cdots + e^{u_r} + e^{l_j} + \cdots + l_r}.
\]

Then we make the change of variables
\[
x_j = 1 - e^{-(u_1+\cdots+u_r)}, \quad y_j = 1 - e^{-(l_1+\cdots+l_r)},
\]
which leads to
\[
\eta(\mathbf{k}; \mathbf{l}) = \prod_{j=1}^{r} \frac{1}{\Gamma(k_j) \Gamma(l_j)} \int_{1 \geq x_1 > \cdots > x_r > 0} \int_{1 \geq y_1 > \cdots > y_r > 0} \prod_{j=1}^{r} \left( \log \frac{1-x_{j+1}}{1-x_j} \right)^{k_j-1} \left( \log \frac{1-y_{j+1}}{1-y_j} \right)^{l_j-1} \frac{dx_j dy_j}{1-x_j y_j}.
\]  

(3.3)

Moreover, we have
\[
\frac{1}{\Gamma(k_j)} \left( \log \frac{1-x_{j+1}}{1-x_j} \right)^{k_j-1} = \frac{1}{(k_j-1)!} \left( \int_{x_{j+1}}^{x_j} \frac{dx}{x} \right)^{k_j-1} = \int_{x_j>x_{j+2}>\cdots>x_{j+k_j}>x_{j+1}} \frac{dx_{j+1}}{1-x_{j+1}} \cdots \frac{dx_{j+k_j}}{1-x_{j+k_j}}.
\]
and a similar formula for
\[
\frac{1}{\Gamma(l_j)} \left( \log \frac{1 - y_{j+1}}{1 - y_j} \right)^{l_j - 1}.
\]
If we substitute them into (3.3), we get the result (3.2) (with \( x_j = x_{j1} \) and \( y_j = y_{j1} \)).

**Corollary 3.2.** For indices \( k = (k_1, \ldots, k_r) \) and \( l = (l_1, \ldots, l_r) \), we have
\[
\eta(k; l) = \sum_{m_j, n_j;(s)} \prod_{i=1}^{[k]} \frac{1}{m_i + m_{i+1} + \cdots + m_{[k]}} \prod_{i=1}^{[l]} \frac{1}{n_i + n_{i+1} + \cdots + n_{[l]}},
\]
where the summation is taken over positive integers \( m_1, \ldots, m_{[k]} \) and \( n_1, \ldots, n_{[l]} \) satisfying
\[
m_1 = n_1, \quad m_{k_1+1} = n_{l_1+1}, \quad m_{k_1+k_2+1} = n_{l_1+l_2+1}, \ldots
\]
\[
\ldots, m_{k_1+\ldots+k_{r-1}+1} = n_{l_1+\ldots+l_{r-1}+1}.
\]

**Proof.** We expand all factors of the integrand of (3.2) by
\[
\frac{1}{1 - x_j y_j} = \sum_{m=1}^{\infty} x_j^{m-1} y_j^{m-1}, \quad \frac{1}{1 - x_j} = \sum_{m=1}^{\infty} x_j^{m-1}, \quad \frac{1}{1 - y_j} = \sum_{n=1}^{\infty} y_j^{n-1}.
\]
Then, with a renumbering of variables, the integral becomes
\[
\int_{1 > x_1 > \cdots > x_{[k]} > 0} \sum_{1 > y_1 > \cdots > y_{[l]} > 0} \prod_{i=1}^{[k]} x_i^{m_i-1} \prod_{i=1}^{[l]} y_i^{n_i-1} \, dy_i.
\]
By exchanging the integral and the summation, and by integrating repeatedly, we obtain the formula (3.4).

Let \( X = (X_1, \ldots, X_r) \) and \( Y = (Y_1, \ldots, Y_r) \) be \( r \)-tuples of indeterminates, and define the generating function for the values \( \eta(k; l) \) by
\[
F_r(X; Y) = \sum_{k, l \in (\mathbb{Z}_+, 0)^r} \eta(k; l) X_1^{k_1-1} \cdots X_r^{k_r-1} Y_1^{l_1-1} \cdots Y_r^{l_r-1}.
\]

**Proposition 3.3.** We have
\[
F_r(X; Y) = \int_{1 > x_1 > \cdots > x_r > 0} \prod_{j=1}^r (1 - x_j)^{X_{j-1}-x_j} (1 - y_j)^{Y_{j-1}-y_j} \frac{dx_j \, dy_j}{1 - x_j y_j},
\]
where \( X_0 = Y_0 = 0 \).

**Proof.** We substitute the integral expression (3.3) of \( \eta(k; l) \) into the definition (3.5) of \( F_r(X; Y) \), and take the summation over \( k \) and \( l \) using
\[
\sum_{k_j=1}^{\infty} \frac{1}{\Gamma(k_j)} \left( \log \frac{1 - x_{j+1}}{1 - x_j} \right)^{k_j-1} X_j^{k_j-1} = \exp \left( X_j \log \frac{1 - x_{j+1}}{1 - x_j} \right)
\]
and similarly for \( y_j \). Then we obtain
\[
F_r(X; Y) = \int_{1 > x_1 > \cdots > x_r > x_{r+1} = 0} \prod_{j=1}^r \left( \frac{1 - x_{j+1}}{1 - x_j} \right)^{X_j} \left( \frac{1 - y_{j+1}}{1 - y_j} \right)^{Y_j} \frac{dx_j \, dy_j}{1 - x_j y_j},
\]
which is equal to (3.6).
4. Relationship with multiple zeta values

In this section, we discuss the relation of the values \( \eta(k; l) \) with the multiple zeta values \( \zeta(k) \). The first thing to be noticed is that the former can be written in terms of the latter.

**Theorem 4.1.** Let \( \mathcal{Y} \) (respectively \( \mathcal{Z} \)) denote the \( \mathbb{Q} \)-linear subspaces of \( \mathbb{R} \) spanned by \( \eta(k; l) \) for all indices \( k \) and \( l \) of the same depth (respectively spanned by \( \zeta(k) \) for all admissible indices \( k \)). Then we have \( \mathcal{Y} \subset \mathcal{Z} \).

**Proof.** Let us make the change of variables \( y_{ji} \leftrightarrow y_{ji}^{-1} \) in the integral (3.2). Then we have

\[
\eta(k; l) = \int_{(x_{ji}) \in \nabla(k)} \prod_{j=1}^{r} \left\{ \frac{dx_{j1} dy_{j1}}{(y_{j1} - x_{j1}) y_{j1}} \frac{dx_{j2}}{1 - x_{j2}} \cdots \frac{dx_{jk}}{(y_{jk} - 1) y_{jk}} \right\},
\]

where \( \Delta'(l) := \{ (y_{ji})_{j=1, \ldots, r} \mid 1 < y_{11} < \cdots < y_{11} < y_{21} < \cdots < y_{21} < \cdots < y_{r1} < \cdots < y_{rl} \} \).

This is a period integral on the moduli space \( \mathcal{M}_{0,n} \) of genus-zero curves with \( n \) marked points, where \( n = |k| + |l| + 3 \). Hence we can apply the theorem of Brown [2, Theorem 1.1] to deduce that \( \eta(k; l) \) is a \( \mathbb{Q}[2\pi i] \)-linear combination of multiple zeta values. Since \( \eta(k; l) \) is a real number, in fact, this is a \( \mathbb{Q}(2\pi i)^2 \)-linear combination. Thus we have \( \eta(k; l) \in \mathcal{Z} \), since \((2\pi i)^2 = -24\zeta(2)\) and \( \mathcal{Z} \) is a \( \mathbb{Q} \)-subalgebra of \( \mathbb{R} \). □

**Remark 4.2.** In the first version of this paper, whether \( \eta(k; l) \in \mathcal{Z} \) or not was asked as an open question. Then E. Panzer, who read it on the arXiv, immediately communicated to the author that it can be shown by using Brown’s result as above. More precisely, he claimed that the inclusion \( \mathcal{Y}_w \subset \mathcal{Z}_w \) can be obtained for any integer \( w \geq 0 \), where \( \mathcal{Y}_w \) (respectively \( \mathcal{Z}_w \)) denotes the space spanned by \( \eta(k; l) \) with \( |k| + |l| = w \) (respectively spanned by \( \zeta(k) \) with \( |k| = w \)). However, the author could not confirm this statement by simply applying Brown’s result as in the above proof.

Recently, K. Ito announced that he and Sato showed the inclusion \( \mathcal{Y}_w \subset \mathcal{Z}_w \) [3, Remark 4.2].

Brown’s work [2] actually gives an algorithm to express the value \( \eta(k; l) \) as a linear combination of multiple zeta values for any given indices \( k \) and \( l \) (this was also noted by Panzer). Such an algorithm may be available also from the method of Ito and Sato. On the other hand, as far as the author knows, an explicit formula for such an expression for general indices is still unknown. See (4.4) and (4.5) below for such formulas in two (very) special cases.

Next we show a ‘sum formula’ for \( \eta \)-values. First let us introduce some notation on indices.
Definition 4.3. We define the following.

(i) For an index \( k = (k_1, \ldots, k_r) \) of weight \( k \), put

\[ J(k) := \{k_1, k_1 + k_2, \ldots, k_1 + \cdots + k_{r-1}\} \subset \{1, 2, \ldots, k-1\}. \]

(ii) We say that \( k \) is a refinement of \( k' \), and denote \( k \geq k' \), if \( |k| = |k'| \) and \( J(k) \supset J(k') \).

(iii) For a formal linear combination \( \alpha = \sum_k a_k k \) of (finitely many) admissible indices, we linearly extend the function \( \zeta \), i.e., set \( \zeta(\alpha) = \sum_k a_k \zeta(k) \). This ‘linear extension’ principle also applies to operations below.

(iv) For an index \( k \), we denote by \( k^* \) the formal sum \( \sum_{k' \leq k} k' \) of all indices \( k' \) of which \( k \) is a refinement.

(v) For indices \( k \) and \( l \), we denote by \( k \ast l \) the harmonic product of \( k \) and \( l \). It is a formal sum of indices defined inductively by

\[ \emptyset \ast k = k \ast \emptyset = k, \]

\[ (k_1, \ldots, k_r) \ast (l_1, \ldots, l_s) = \left((k_1, \ldots, k_{r-1}) \ast (l_1, \ldots, l_s), k_r\right) \]

\[ + \left((k_1, \ldots, k_r) \ast (l_1, \ldots, l_{s-1}), l_s\right) \]

\[ + \left((k_1, \ldots, k_{r-1}) \ast (l_1, \ldots, l_{s-1}), k_r + l_s\right), \]

where \( \emptyset \) denotes the unique index of depth zero.

(vi) For indices \( k = (k_1, \ldots, k_r) \) and \( l = (l_1, \ldots, l_s) \) with \( r, s > 0 \), we set

\[ k \oplus l := \left((k_1, \ldots, k_{r-1}) \ast (l_1, \ldots, l_{s-1}), k_r + l_s\right). \]

Theorem 4.4. Denote by \( I(k, r) \) the set of all indices of weight \( k \) and depth \( r \). Then we have, for any positive integers \( k, l, r \),

\[ \sum_{k \in I(k, r)} \sum_{l \in I(l, r)} \eta(k; l) = \zeta(\frac{2, \ldots, 2}{k-r}) \ast (\frac{1, \ldots, 1, 0}{k-r}) \ast (\frac{1, \ldots, 1, 0}{l-r}). \tag{4.2} \]

Proof. First note that, while

\[ \begin{align*}
\left(1, \ldots, 1, 0\right) & \quad \text{and} \quad \left(1, \ldots, 1, 0\right), \\
\left(1, \ldots, 1, 0\right) & \quad \text{and} \quad \left(1, \ldots, 1, 0\right),
\end{align*} \]

on the right-hand side contain 0, the formal operations work well and produce a formal sum of admissible indices of weight \( k + l \). Explicitly, it is given by the multiple series

\[ \sum_{m_1 > \cdots > m_r > 0} 1 = \prod_{0 \leq a_1 \leq \cdots \leq a_r \leq 1} a_k \cdots a_{k-r} b_1 \cdots b_{l-r}. \tag{4.3} \]

Let us compute the left-hand side of (4.2). Since

\[ F_r(X, \ldots, X; Y, \ldots, Y) = \sum_{k \in I(k, r), l \in I(l, r)} \eta(k; l) X^{k-r} Y^{l-r}, \]

it is sufficient to compute this generating function. By (3.6), we have

\[ F_r(X, \ldots, X; Y, \ldots, Y) = \int_{X_1 > \cdots > X_r > 0, Y_1 > \cdots > Y_r > 0} \prod_{j=1}^{r} \frac{dx_j dy_j}{l - x_j y_j}. \]
Using the expansion
\[
\frac{1}{1 - x_j y_j} = \sum_{n_j=1}^{\infty} (x_j y_j)^{n_j - 1}
\]
and integrating with respect to \(x_r, y_r, \ldots, x_2, y_2\), it can be rewritten as

\[
\sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_2^r \cdots m_r^r} \int_0^1 \int_0^1 (1 - x)^{-X} x^{m_1 - 1} dx \int_0^1 (1 - y)^{-Y} y^{m_1 - 1} dy.
\]

Then we note that

\[
\int_0^1 (1 - x)^{-X} x^{m_1 - 1} dx = B(1 - X, m) = \frac{\Gamma(1 - X) \Gamma(m)}{\Gamma(1 - X + m)}
\]

\[
= \frac{(m - 1)!}{(1 - X)(2 - X) \cdots (m - X)}
\]

\[
= \frac{1}{m} \prod_{a=1}^m \left(1 - \frac{X}{a}\right)^{-1} = \frac{1}{m} \prod_{a=1}^m \sum_{n=0}^{\infty} \frac{X^n}{a^n}
\]

\[
= \frac{1}{m} \sum_{k=0}^{\infty} X^k \sum_{0 < a_1 \leq \ldots \leq a_k \leq m} \frac{1}{a_1 \cdots a_k}
\]

to obtain

\[
F_r(X, \ldots, X; Y, \ldots, Y) = \sum_{k, l \geq 0} X^k y^l \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_2^r \cdots m_r^r} \sum_{0 < a_1 \leq \ldots \leq a_k \leq m} \frac{1}{a_1 \cdots a_k b_1 \cdots b_l}.
\]

Hence the coefficient of \(X^{k-l} Y^{l-r}\) coincides with the series (4.3), and the result follows. \(\Box\)

**Example 4.5.**

(i) When \(r = 1\), (4.2) says that

\[
\eta(k; l) = \sum_{m>0}^{\infty} \frac{1}{m^2 a_1 \cdots a_{k-1} b_1 \cdots b_{l-1}}
\]

\[
= \zeta \left( \frac{(1, \ldots, 1)}{k} \otimes \frac{(1, \ldots, 1)}{l} \right)
\]

for any \(k, l > 0\). For example,

\[
\eta(1; 1) = \zeta(2),
\]

\[
\eta(2; 1) = \zeta(1, 2) + \zeta(3),
\]

\[
\eta(3; 1) = \zeta(1, 1, 2) + \zeta(2, 2) + \zeta(1, 3) + \zeta(4),
\]

\[
\eta(2; 2) = 2\zeta(1, 1, 2) + \zeta(2, 2) + 2\zeta(1, 3) + \zeta(4).
\]

(ii) When \(r = k = l\), (4.2) says that

\[
\eta(1, \ldots, 1; 1, \ldots, 1) = \zeta(2, \ldots, 2).
\]

(4.5)
Appendix A. Values of \( \eta(k_1, \ldots, k_r; l) \) for positive integers \( k_1, \ldots, k_r, l \)

The formula (4.4) for \( \eta(k; l) \), which is a special case of (4.2), can also be generalized to another direction, namely, a formula for values \( \eta(k_1, \ldots, k_r; l) \) of the function (1.1). Such a formula was first found by M. Kaneko as a conjecture. In this appendix, we prove it.

To state the formula, we recall the notion of the Hoffman dual \( k^\vee \) of an index \( k \). This is the unique index such that \(|k^\vee| = |k| \) and \( J(k^\vee) = \{1, \ldots, |k| - 1\} \setminus J(k) \).

**Theorem A.1.** For a nonempty index \( k = (k_1, \ldots, k_r) \) and an integer \( l > 0 \), we have

\[
\eta(k; l) = (-1)^{d(k^\vee)} \sum_{k' \supseteq k^\vee} (-1)^{d(k')} \zeta \left( (k')^* \oplus \left( \frac{1, \ldots, 1}{l} \right) \right). \tag{A.1}
\]

When \( r = 1 \), the Hoffman dual of \( k \) is

\[
(k_1, \ldots, 1) \frac{k}{k},
\]

and it has no refinement other than itself. Hence we recover (4.4) from (A.1).

To prove Theorem A.1, we need some preparations.

Fix positive integers \( k \) and \( l \). In the following, indices denoted by \( k \) or \( k' \) are of weight \( k \), and sets denoted by \( J \) or \( J' \) are subsets of \( \{1, \ldots, k - 1\} \). For such a set \( J \), we put

\[
S_J := \sum_{0 < a_1 \sqcap_1 a_2 \sqcap_2 \ldots \sqcap_{k - 1} a_k = b_1 \geq \ldots \geq b_1 > 0} \frac{1}{a_1 \cdots a_k b_1 \cdots b_1},
\]

where \( \sqcap_j \) for \( j = 1, \ldots, k - 1 \) denote the relational operators

\[
\sqcap_j = \begin{cases} < & (j \in J), \\ = & (j \notin J) \end{cases}
\]

(it also depends on \( l \), which we fixed). It is easy to see, for \( k = (k_1, \ldots, k_r) \), that

\[
S_J(k) = \sum_{0 < m_1 < \ldots < m_r = b_1 \geq \ldots \geq b_1 > 0} \frac{1}{m_1 \cdots m_r b_1 \cdots b_1} = \zeta \left( k \oplus \left( \frac{1, \ldots, 1}{l} \right) \right). \tag{A.2}
\]

**Lemma A.2.** For any index \( k \) (of weight \( k \)), we have

\[
\xi(k; l) = S_J(k), \tag{A.3}
\]

\[
\zeta \left( k^* \oplus \left( \frac{1, \ldots, 1}{l} \right) \right) = \sum_{J \subset J(k)} S_J. \tag{A.4}
\]

**Proof.** In the definition (1.2) of \( \xi(k; s) \), make a change of variable \( u = 1 - e^{-t} \) and substitute \( s = l \). Then we have

\[
\xi(k; l)
= \frac{1}{(l - 1)!} \int_0^1 \frac{\text{Li}_k(u)}{u} (- \log(1 - u))^{l - 1} \, du
= \sum_{0 < m_1 < \ldots < m_r} \frac{1}{m_1 \cdots m_r} \int_0^1 u^{m_r - 1} \, du \int_{0 < v_1 < \ldots < v_{l - 1} < u} \frac{dv_1}{1 - v_1} \cdots \frac{dv_{l - 1}}{1 - v_{l - 1}}.
\]
Here we use
\[- \log(1 - u) = \int_0^u \frac{dv}{1 - v}.\]
The identity (A.3) is obtained by computing the integration in the order \(u, v_1, \ldots, v_l\).

The identity (A.4) follows from the computation
\[
\zeta(\mathbf{k}^* \oplus (1, \ldots, 1)^*; k) = \sum_{\mathbf{k}' \geq \mathbf{k}} \zeta((\mathbf{k}')^* \oplus (1, \ldots, 1)^*; k)
\]
\[
= \sum_{\mathbf{k}' \geq \mathbf{k}} \sum_{J \subseteq J(\mathbf{k})} S_J
\]
where we use (A.2) and the correspondence between indices \(\mathbf{k}' \preceq \mathbf{k}\) and sets \(J \subseteq J(\mathbf{k})\).

In addition to the above lemma, we use the following identity due to Kaneko and Tsumura [4, Proposition 3.2]:
\[
\eta(\mathbf{k}; s) = (-1)^{d(\mathbf{k}) - 1} \sum_{\mathbf{k}' \geq \mathbf{k}} \zeta(\mathbf{k}'^*; s).
\]  

\textbf{Proof of Theorem A.1.} Let us compute the sum on the right-hand side of (A.1):
\[
\sum_{\mathbf{k}' \geq \mathbf{k}^*} (-1)^{d(\mathbf{k})} \zeta((\mathbf{k}')^* \oplus (1, \ldots, 1)^*; k) = \sum_{\mathbf{k}' \geq \mathbf{k}^*} (-1)^{d(\mathbf{k}')} \sum_{J \subseteq J(\mathbf{k})} S_J
\]
\[
= \sum_{J \supset J(\mathbf{k}^*)} (-1)^{\# J + 1} \sum_{J \subseteq J} S_J
\]
\[
= \sum_{J} S_J \sum_{J \supset J(\mathbf{k}^*)} (-1)^{\# J + 1}.
\]

Here, we use (A.4) in the first step and the correspondence between indices \(\mathbf{k}' \geq \mathbf{k}^*\) and sets \(J \supset J(\mathbf{k}^*)\) in the second step (note that \(d(\mathbf{k}') = \# J(\mathbf{k}') + 1\)). The third step is just an exchange of summations.

It is easily shown that
\[
\sum_{J \supset J(\mathbf{k}^*) \cup J} (-1)^{\# J + 1} = \begin{cases} 
0 & (J(\mathbf{k}^*) \cup J \subseteq \{1, \ldots, k - 1\}), \\
(-1)^k & (J(\mathbf{k}^*) \cup J = \{1, \ldots, k - 1\}).
\end{cases}
\]

Because of the equivalence
\[
J(\mathbf{k}^*) \cup J = \{1, \ldots, k - 1\} \iff J \supset \{1, \ldots, k - 1\} \setminus J(\mathbf{k}^*) = J(\mathbf{k}),
\]
we can continue the above computation as
\[
\sum_{J} S_J \sum_{J \supset J(\mathbf{k}^*) \cup J} (-1)^{\# J + 1} = (-1)^k \sum_{J \supset J(\mathbf{k})} S_J
\]
\[
= (-1)^k \sum_{\mathbf{k}' \geq \mathbf{k}} \zeta(\mathbf{k}'^*; l)
\]
\[
= (-1)^k (-1)^{d(\mathbf{k}) - 1} \eta(\mathbf{k}; l),
\]
using (A.3) and (A.5). Thus the desired identity (A.1) is obtained if we notice that \(k - 1 = \# J(\mathbf{k}) + \# J(\mathbf{k}^*) = (d(\mathbf{k}) - 1) + (d(\mathbf{k}^*) - 1)\), i.e., \(k - (d(\mathbf{k}) - 1) = d(\mathbf{k}^*)\).  

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REFERENCES


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