THE GEOMETRY OF K -SPREADS OF HIGHER ORDER

By

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Introduction

The geometry of paths and that of K -spreads under the group of transformations as well as various groups of parameters were discussed in detail by J. DOUGLAS ([4], [5])⁽¹⁾ and by E. BORTOLLOTTI ([1]). The paths and the K -spreads are concerned with a system of differential equations of the second order. In a similar manner the case of higher order has been treated by D. D. KOSAMBI ([9]), A. KAWAGUCHI-H. HOMBU ([8]), and others.

If we are based on another group of transformations instead of the group of transformations of coordinates $x^\lambda = x^\lambda(x^i)$, we shall have another kind of geometry of paths or K -spreads different from the preceding. From this point of view we notice the paper of E. CARTAN ([2]), where the general geometry of paths of the second order was treated under the group of so-called rheonomic transformations $x^\lambda = x^\lambda(x^i, t)$, $\bar{t} = t$. Professor H. HOMBU has recently considered the case of higher order under the same group, using the CARTAN's geometric method for $m=3$ and also the method of [8] for $m \geq 3$ ([6]).

In the present paper we attempt to develop the generalized rheonomic geometry of K -spreads given by a system of partial differential equations of the m -th order

$$(1) \quad p_{a(m)}^i + H_{a(m)}^i(x^j, u^b; p_{b(1)}^j, \dots, p_{b(m-1)}^j) = 0$$

$$(i = 1, 2, \dots, n; \quad a = 1, 2, \dots, K; \quad K \leq n).$$

(1) Numbers in brackets refer to the bibliography at the end of the paper.

The generalized rheonomic geometry stands for the geometry under the group of generalized rheonomic transformations. In the $(n+K)$ -dimensional manifold of the coordinates x^i and the parameters u^a , each of the parameters is considered to have an absolute significance.

In Chapter 1, we shall survey the law of the extended transformation of K -dimensional surface elements of higher order. By virtue of the generalization of CRAIG's formulae we can define the extensors in the same manner as usual. It is to be noted that the differential of the surface element $dp_{a_{(r)}}^i$ is not an extensor, but the quantities $\delta p_{a_{(r)}}^i \equiv dp_{a_{(r)}}^i - p_{a_{(r)}b}^i du^b$ constitute the components of an extensor. We establish some theorems on affinors which we need in Chapter 2. Most of these theorems are analogous to those of A. KAWAGUCHI-H. HOMBU ([8]), but owing to the notion of extensors their proofs become simpler than in the cited paper.

Chapter 2 is the main part of the present paper. We shall establish the geometry of the system of partial differential equations (1). But, as the integrability of the differential equations is not assumed, it will be natural to consider the K -spreads to be non-holonomic of higher order. The covariant differentials of a vector and of the surface element and their relations are given there, using the theorems obtained in Chapter 1. In the same way as usual we can define the covariant derivatives of a vector and calculate curvature tensors and torsions. The rheonomic geometry of paths of higher order may be deduced from our geometry as a special case ($K=1$).

In Chapter 3, we shall treat various vectors, extensors and other quantities which will be useful in the general theory of the manifold of surface elements. Finally, we define new geometric objects in this manifold, which are called *pseudo-extensors*, and state some related theorems.

Chapter 1. Manifold of K -dimensional surface elements

§ 1. **Extended transformation of K -elements.** In an n -dimensional manifold X_n referred to a coordinate system x^i , ($i=1, 2, \dots, n$), let us consider a K -dimensional surface $x^i = x^i(u^a)$, where the u^a 's ($a=1, 2, \dots, K$; $K \leq n$) are parameters. The quantities

$$(1.1) \quad x^i = x^i(u^a), \quad p_{a(r)}^i \equiv p_{a_1 a_2 \dots a_r}^i = \frac{\partial^r x^i}{\partial u^{a_1} \partial u^{a_2} \dots \partial u^{a_r}} \quad (r=1, 2, \dots, M),$$

defined at each point of the surface, may be seen to determine the K -dimensional surface element of the M -th order, which we shall call simply K -element of the M -th order. As already mentioned in the introduction, we shall be concerned with the transformations of coordinates of the form

$$(1.2) \quad x^\lambda = x^\lambda(x^i, u^a), \quad \bar{u}^a = u^a \quad (\lambda = 1, 2, \dots, n).$$

By such a transformation the K -element is subjected to the extended transformation:

$$(1.3) \quad \left\{ \begin{array}{l} x^\lambda = x^\lambda(x^j, u^a), \quad p_{a_{(1)}}^\lambda = \frac{\partial x^\lambda}{\partial x^i} p_{a_{(1)}}^i + \frac{\partial x^\lambda}{\partial u^{a_1}}, \\ \dots\dots\dots \\ p_{a_{(M)}}^\lambda = \frac{\partial x^\lambda}{\partial x^i} p_{a_{(M)}}^i + R_{a_{(M)}}^\lambda(x^j, u^b; p_{b_{(1)}}^j, \dots, p_{b_{(M-1)}}^j). \end{array} \right.$$

When a set of quantities $x^i, p_{(r)}^i$ ($r=1, 2, \dots, M$), defined in every coordinate system and independent of any surface, is transformed by (1.2) in the same manner as (1.3), we call it generally the *K-element*. The manifold of all the *K*-elements of the *M*-th order in X_n will be denoted by $F_n^{(M)}$.

Since by (1.3)

$$\begin{aligned} \frac{\partial}{\partial x^i} p_{a_{(s)}b}^\lambda &= \frac{\partial}{\partial x^i} \left(\frac{\partial p_{a_{(s)}}^\lambda}{\partial u^b} + \sum_{s=0}^r \frac{\partial p_{a_{(s)}}^\lambda}{\partial p_{c_{(s)}}^j} p_{c_{(s)}b}^j \right) \\ &= \frac{\partial}{\partial u^b} \left(\frac{\partial p_{a_{(s)}}^\lambda}{\partial x^i} \right) + \sum_{s=0}^r \frac{\partial}{\partial p_{c_{(s)}}^j} \left(\frac{\partial p_{a_{(s)}}^\lambda}{\partial x^i} \right) p_{c_{(s)}b}^j = \left(\frac{\partial p_{a_{(s)}}^\lambda}{\partial x^i} \right)_{/b}, \end{aligned}$$

we have in general

$$(1.4) \quad \left(\frac{\partial p_{a(r)}^\lambda}{\partial x^i} \right)_{b(s)} = \frac{\partial p_{a(r)b(s)}^\lambda}{\partial x^i},$$

where the following notations are adopted:

$$A_{/b} = \frac{\partial A}{\partial u^b} + \sum_{c \geq 0} \frac{\partial A}{\partial p_{c(s)}^i} p_{(s)b}^i, \quad A_{/b(s)} = A_{/b_1/b_2 \dots /b_s}.$$

Noticing that $p_{b_{(t)}}^i$ is symmetric with respect to $b_1, b_2, \dots, b_t$, we

use, instead of usual partial derivative $\frac{\partial f}{\partial p_{b_{(t)}}^i}$, the operator

$$f_{;i}^{(b_{(t)})} \equiv \frac{\bar{\partial} f}{\partial p_{b_{(t)}}^i} = \frac{\nu_1! \nu_2! \dots \nu_s!}{t!} \frac{\partial f}{\partial p_{b_{(t)}}^i}$$

when the $b_{(t)}$'s consist of $\nu_1, \nu_2, \dots, \nu_s$ same indices ($\nu_1 + \nu_2 + \dots + \nu_s = t$). To make typography easy we shall omit this bar, however. The differential of a function f of u^a and the K -element is then written in the form

$$df = f_{;a} du^a + \sum_{r=0}^M f_{;i}^{(a_{(r)})} dp_{a_{(r)}}^i$$

and the relation

$$\frac{\partial p_{a_{(r)}}^i}{\partial p_{b_{(r)}}^j} = \delta_j^i \delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \dots \delta_{a_r}^{b_r} = \delta_j^i \delta_{(a_{(r)})}^{b_{(r)}}$$

holds good.

As in the case of ordinary transformations of coordinates the following useful formulae can be obtained:

$$(1.5a) \quad \frac{\partial p_{a_{(r)}}^\lambda}{\partial p_{b_{(t)}}^k} = \binom{r}{t} \delta_{(a_{(t)}}^{b_{(t)}} \frac{\partial p_{a_{(r-t)}}^\lambda}{\partial x^k} = \binom{r}{t} \delta_{(a_{(t)}}^{b_{(t)}} \left(\frac{\partial x^\lambda}{\partial x^k} \right)_{|a_{(r-t)}} \quad (r \geq t).$$

Proof. By definition we have

$$p_{a_{(r+1)}}^\lambda = (p_{a_{(r)}}^\lambda)_{|a_{r+1}} = \sum_{s=0}^r \frac{\partial p_{a_{(r)}}^\lambda}{\partial p_{c_{(s)}}^l} p_{c_{(s)} a_{r+1}}^l + \frac{\partial p_{a_{(r)}}^\lambda}{\partial u^{a_{r+1}}},$$

so we establish (1.5a) by induction with respect to r :

$$\begin{aligned} \frac{\partial p_{a_{(r+1)}}^\lambda}{\partial p_{b_{(t)}}^k} &= \frac{\partial p_{a_{(r)}}^\lambda}{\partial p_{c_{(t-1)}}^k} \delta_{(c_{(t-1)} a_{r+1})}^{b_{(t)}} + \sum_{s=0}^r \frac{\partial}{\partial p_{c_{(s)}}^l} \left(\frac{\partial p_{a_{(r)}}^\lambda}{\partial p_{b_{(t)}}^k} \right) p_{c_{(s)} a_{r+1}}^l + \frac{\partial}{\partial u^{a_{r+1}}} \left(\frac{\partial p_{a_{(r)}}^\lambda}{\partial p_{b_{(t)}}^k} \right) \\ &= \binom{r}{t-1} \delta_{(a_{(t)}}^{b_{(t)}} \frac{\partial p_{a_{(r-t+1)}}^\lambda}{\partial x^k} + \binom{r}{t} \delta_{(a_{(t)}}^{b_{(t)}} \left(\frac{\partial p_{a_{(r-t)}}^\lambda}{\partial x^k} \right)_{|a_{r+1}} = \binom{r+1}{t} \delta_{(a_{(t)}}^{b_{(t)}} \frac{\partial p_{a_{(r-t+1)}}^\lambda}{\partial x^k}. \end{aligned}$$

From (1.5a) we obtain immediately the relation

$$(1.5b) \quad \frac{\partial p_{a_{(r)}}^\lambda}{\partial p_{b_{(t)}}^k} = \frac{r!(t-s)!}{t!(r-s)!} \delta_{(a_{(s)}}^{b_{(s)}} \frac{\partial p_{a_{(r-s)}}^\lambda}{\partial p_{b_{(t-s)}}^k}.$$

These formulae (1.4), (1.5a), and (1.5b) make clear the laws of transformation of the surface elements. By virtue of the formulae, almost all the known theorems can be obtained in our rheonomic geometry by slight modifications.

§ 2. **Extensors.** Since by (1.3)

$$dp_{c(t)}^\lambda = \sum_{r=0}^t \frac{\partial p_{c(t)}^\lambda}{\partial p_{a(r)}^i} dp_{a(r)}^i + \frac{\partial p_{c(t)}^\lambda}{\partial u^b} du^b,$$

the quantities

$$\delta p_{a(r)}^i \equiv dp_{a(r)}^i - p_{a(r)b}^i du^b \quad (r = 0, 1, \dots, M)$$

are transformed according to

$$(1.6) \quad \delta p_{c(t)}^\lambda = \sum_{r=0}^t \frac{\partial p_{c(t)}^\lambda}{\partial p_{a(r)}^i} \delta p_{a(r)}^i.$$

When a set of the quantities $V_{a(r)}^i$ or $W_i^{a(r)}$ with the indices $i, a_1, a_2, \dots, a_r$ is subjected to the transformation

$$(1.7) \quad (a) \quad V_{b(t)}^\lambda = \sum_{r=0}^t \frac{\partial p_{b(t)}^\lambda}{\partial p_{a(r)}^i} V_{a(r)}^i, \quad (b) \quad W_{\lambda(t)}^b = \sum_{r=t}^M \frac{\partial p_{a(r)}^i}{\partial p_{b(t)}^\lambda} W_i^{a(r)},$$

$V_{a(r)}^i$ and $W_i^{a(r)}$ are then said to be the *components of extensors*, respectively, *contravariant* and *covariant of the range* $(0, M)$. $\delta p_{a(r)}^i$ are therefore the components of a contravariant extensor, and $F_{;i}^{(a(r))}$ a covariant extensor, F being a scalar. We shall derive in the following some useful theorems concerning the extensors.

THEOREM 1. If $W_i^{a(r)}$ is a covariant extensor of the range $(0, M)$, then $\binom{r}{A} W_i^{c(A)a(r-A)}$ ($0 \leq A \leq r \leq M$) for fixed $c_{(A)}$ constitute the components of a covariant extensor of the range $(0, M-A)$. ([7] A. KAWAGUCHI)⁽²⁾.

By (1.5b) the right-hand member of (1.7b) may be written in the form

$$\sum_{r=t}^M \frac{r!(t-A)!}{t!(r-A)!} \frac{\partial p_{a(r-A)}^i}{\partial p_{b(t-A)}^\lambda} W_i^{b(A)a(r-A)}$$

hence the proposition can be readily proved.

THEOREM 2. If $T_{a(r)j}^{i \quad b(s)}$ ($r \leq R \leq M, 0 \leq S \leq s$) are the components of an extensor with respect to $i, a_{(r)}$ and $j, b_{(s)}$ of the range $(0, R)$ and (S, M) respectively, then $T_{a(r)j}^{i \quad b(s)(c(t))}$ ($\text{Max}(R, M-S) < t$)

(2) The theorem is an analogy or a generalization of a theorem in the cited paper.

constitute the components of an extensor with respect to $i, a_{(r)}$; $j, b_{(s)}$ and $k, c_{(t)}$, of the range $(0, R)$, (S, M) and $(\text{Max } (R, M-S) + 1, M)$ respectively. ([3] H. V. CRAIG).

This follows at once from the definition of extensors, using the restriction on r, s , and t .

THEOREM 3. If $T_{a_{(r)}j}^{i b_{(s)}}$ are the components of an extensor of the same range $(0, M)$, then

$$\sum_{r=A}^M \binom{r}{A} T_{a_{(r-A)}i}^{i a_{(r-A)} b_{(A)}}$$

is a scalar ($A=0, 1, \dots, \text{or } M$). ([3] H. V. CRAIG).

By virtue of Theorem 1,

$$\binom{s}{A} T_{a_{(r-A)}j}^{i c_{(A)} b_{(s-A)}} \quad (A \leq s)$$

constitute the components of an extensor, contravariant with respect to $i, a_{(r-A)}$ and covariant with respect to $j, b_{(s-A)}$ of the same range $(0, M-A)$, hence our theorem is evident on account of the fact that the transformations of covariant and contravariant extensor are contragredient.

As in the case of contractions of affinars we have

THEOREM 4. If for any contravariant extensor $V_{a_{(r)}}^i$

$$\sum_{r=A}^M \binom{r}{A} T_{k(r-A)c_{(A)}}^{b_{(r-A)} V_{b_{(r-A)}}^k$$

is a scalar (an affinor or an extensor), then $T_{k(r)}^{b_{(r)}}$ are the components of a covariant extensor. ([3] H. V. CRAIG).

THEOREM 5. If the affinor T^I depends on the K -element as well as the parameters u^a and a vector v^i on the parameters, then

$$(1.8) \quad D^{a_{(A)}}(T^I, v) = \sum_{r=A}^M \binom{r}{A} T^{I(a_{(A)} b_{(r-A)})}_{;j} v^j_{/b_{(r-A)}} \quad (A = 1, 2, \dots, M)$$

constitute the components of an affinor for fixed $a_{(A)}$. ([8] A. KAWAGUCHI-H. HOMBU).

By virtue of the hypothesis and Theorem 2 we see that $T^{I(a_{(r)})}_{;i}$ ($r \geq 1$) are the components of a covariant extensor, hence by Theorem 1 $\binom{r}{A} T^{I(a_{(A)} b_{(r-A)})}_{;j}$ ($r \geq A \geq 1$) constitute the components

of a covariant extensor. Moreover, by means of (1.5a) and generalized LEIBNIZ's formula

$$(AB)_{|a_{(r)}} = \sum_{t=0}^r \binom{r}{t} A_{|a_{(t)}} B_{|a_{(r-t)}}$$

we have

$$\begin{aligned} v^u_{|f_{(r)}} &= \left(\frac{\partial x^u}{\partial x^j} v^j \right)_{|f_{(r)}} = \sum_{t=0}^r \binom{r}{t} \left(\frac{\partial x^u}{\partial x^j} \right)_{|f_{(t-D)}} v^j_{|f_{(t)}} \\ &= \sum_{t=0}^r \binom{r}{t} \delta_{f_{(t)}}^{c_{(t)}} \left(\frac{\partial x^u}{\partial x^j} \right)_{|f_{(r-t)}} v^j_{|c_{(t)}} = \sum_{t=0}^r \frac{\partial p_{f_{(t)}}^u}{\partial p_{c_{(t)}}^j} v^j_{|c_{(t)}}, \end{aligned}$$

i.e. $v^i_{|b_{(r-A)}} (r \geq A \geq 1)$ are the components of a contravariant extensor. Hence our theorem is established by Theorem 3.

Using (1.6) we have

THEOREM 6. If T^I are the components of an affnor depending on the parameters and the K -element, then the PFAFFian forms

$$(1.9) \quad \sum_{r=A}^M \binom{r}{A} T^{I(a_{(A)} b_{(r-A)})}_{;j} \mathfrak{d} p_{b_{(r-A)}}^j \quad (A = 1, 2, \dots, M)$$

for fixed $a_{(A)}$ are transformed as the components of an affnor. ([8] A. KAWAGUCHI-H. HOMBU).

If f is a scalar, the operator

$$(1.10) \quad D^{a_{(A)}} f = \sum_{r=A}^M \binom{r}{A} f^{(a_{(A)} b_{(r-A)})}_{;j} \mathfrak{d} p_{b_{(r-A)}}^j \quad (A = 0, 1, \dots, M)$$

is invariant under the transformations (1.2). The meaning of the above theorem is nothing but this invariance.

The two theorems stated above can be generalized to a certain extent in the case T^I is an extensor. Namely, if $T^{i \quad b_{(S)}}_{a_{(r)} j}$ is an extensor of the range $(0, R)$ and (S, M) respectively, then the quantities

$$(1.8a) \quad \sum_{t=A}^M \binom{t}{A} T^{i \quad b_{(S)}(c_{(A)} d_{(t-A)})}_{a_{(r)} j \quad ;k} v^k_{|d_{(t-A)}}$$

and the PFAFFian forms

$$(1.9a) \quad \sum_{t=A}^M \binom{t}{A} T^{i \quad b_{(S)}(c_{(A)} d_{(t-A)})}_{a_{(r)} j \quad ;k} \mathfrak{d} p_{d_{(t-A)}}^k,$$

$A \geq \text{Max}(R+1, M-S+1)$, constitute both the components of an extensor for fixed $c_{(A)}$.

§ 3. **PFAFFian forms with the vector character.** We have considered in Theorem 6 the PFAFFian forms which constitute the components of an affinor. When a PFAFFian form of the u^a 's and the K -element is expressed by $\delta p_{a(r)}^i$ and du^a , the coefficients of du^a are scalars. On this account we merely consider the PFAFFian forms

$$(1.11) \quad \sum_{r=0}^M \bar{P}_{i(r)}^r \delta p_{a(r)}^i$$

which are supposed to have an affinor character with respect to I . By virtue of Theorem 4, $\bar{P}_{i(r)}^r$ ($0 \leq r \leq M$) constitute the components of a covariant extensor with respect to the indices i , $a(r)$ and I respectively.

Now we have the following theorem:

THEOREM 7. If v^i is an arbitrary vector, then the differential forms

$$(1.12) \quad \bar{P}_{i(M)}^{ib(M)} \delta v^i + \sum_{r=0}^M \bar{P}_{i(r)}^{ia(r)(b(M))} v^j \delta p_{a(r)}^i \quad (b_{(M)} : \text{fixed})$$

or more generally

$$(1.13) \quad \sum_{t=0}^A \binom{M-A+t}{t} \left[\bar{P}^{M-A+t}{}_{ib(M-A)^c(t)} \delta v^i|_{c(t)} + \sum_{r=0}^M \bar{P}_{i(r)}^{ia(r)(b(M-A)^c(t))} v^j|_{c(t)} \delta p_{a(r)}^i \right] \\ (A = 0, 1, \dots, M; \quad b_{(M-A)} : \text{fixed})$$

have an affinor character with respect to I , putting

$$\delta v^i|_{c(t)} \equiv dv^i|_{c(t)} - v^j|_{c(t)a} du^a \quad (t = 0, 1, \dots, M).$$

([8] A. KAWAGUCHI-H. HOMBU).

Proof. Without loss of generality we can drop the index I throughout the proof. By virtue of the extensor character of $\bar{P}_{i(r)}^r$ and (1.5b) we have

$$\begin{aligned} & \sum_{t=0}^A \binom{M-A+t}{t} \bar{P}^{M-A+t}{}_{b(M-A)^c(t)} \delta v^i|_{c(t)} \\ &= \sum_{s=0}^A \binom{M-A+s}{s} \bar{P}^{M-A+s}{}_{b(M-A)^c(s)} \sum_{t=0}^s \frac{\partial \bar{p}_{c(s)}^\lambda}{\partial p_{c(t)}^i} \delta v^i|_{c(t)}, \\ & \sum_{t=0}^A \sum_{r=0}^M \binom{M-A+t}{t} \bar{P}_{i(r)}^{ia(r)(b(M-A)^c(t))} v^j|_{c(t)} \delta p_{a(r)}^i \end{aligned}$$

$$\begin{aligned}
&= \sum_{s=0}^A \binom{M-A+s}{s} P^{M-A+s}_{b(M-A)^{e(s)}} \sum_{t=0}^s \sum_{r=0}^s \frac{\partial^2 p_{e(s)}^\lambda}{\partial p_{a(t)}^i \partial p_{c(r)}^j} v^j_{|c(r)} \mathfrak{d} p_{a(t)}^i \\
&\quad + \sum_{s=0}^A \binom{M-A+s}{s} \sum_{r=0}^M P^{r}_{\lambda(r)} (b(M-A)^{e(s)}) v^u_{|c(s)} \mathfrak{d} p_{a(t)}^\lambda.
\end{aligned}$$

On the other hand we have the law of transformation of $\mathfrak{d} v^i_{|c(t)}$

$$(1.14) \quad \sum_{t=0}^s \left[\frac{\partial p_{e(s)}^\lambda}{\partial p_{c(t)}^i} \mathfrak{d} v^i_{|c(t)} + \sum_{r=0}^s \frac{\partial^2 p_{e(s)}^\lambda}{\partial p_{a(t)}^i \partial p_{c(r)}^j} v^j_{|c(r)} \mathfrak{d} p_{a(t)}^i \right] = \mathfrak{d} v^\lambda_{|c(s)}.$$

From these our theorem follows at once.

Combining Theorems 1 and 3 we obtain

THEOREM 8. The PFAFFIAN system of the K -element

$$\sum_{r=A}^M \binom{r}{A} P^{r}_{ib(A)^{a(r-A)}} \mathfrak{d} p_{a(r-A)}^i \quad (A = 0, 1, \dots, M)$$

has an affinor property for fixed $b_{(A)}$. ([8] A. KAWAGUCHI-H. HOMBU).

These theorems will later show us the relations between one of the PFAFFIAN systems of the K -element and the covariant differential of a vector in the theory of K -spreads.

§ 4. Covariant derivatives. The covariant derivatives of a vector v^i which we shall introduce in the manifold $F_n^{*(m-1)}$ have the form:

$$(1.15) \quad \begin{cases} \nabla_j^{a(M)} v^i = v^i_{;j}^{(a(M))} \\ \nabla_j^{a(q)} v^i = v^i_{;j}^{(a(q))} - \sum_{r=q+1}^M V^{ka(q)}_{b(r)j} v^i_{;k}^{(b(r))} + I^{*ia(q)}_{kj} v^k, \end{cases}$$

($q=0, 1, \dots, M-1$). If we define the operator

$$\bar{\nabla}_j^{a(q)} f = f^{(a(q))}_{;j} - \sum_{r=q+1}^M V^{ka(q)}_{b(r)j} f^{(b(r))}_{;k} \quad (q = 0, 1, \dots, M-1)$$

then the derivatives $\bar{\nabla}_j^{a(q)} v^i$ ($q = 1, 2, \dots, M-1$) are also covariant for fixed $a_{(q)}$ because $I^{*ia(q)}_{kj}$ ($q=1, 2, \dots, M-1$) constitute the components of an affinor.

From the enounced property of $\nabla_j^{a(q)} v^i$ ($q = 0, 1, \dots, M-1$) we have the law of transformation of $V^{ka(q)}_{b(r)j}$ and I^{*ij}_{kj} :

$$(1.16a) \quad \frac{\partial x^u}{\partial x^j} V^{va(q)}_{c(t)u} = \sum_{r=q+1}^t \binom{t}{r} \frac{\partial p_{c(t-r)}^v}{\partial x^k} V^{ka(q)}_{c(r)j} - \binom{t}{q} \delta^{a(q)}_{c(q)} \frac{\partial p_{c(t-q)}^v}{\partial x^j},$$

$$(1.16b) \quad \left\{ \begin{array}{l} \frac{\partial x^i}{\partial x^\lambda} \Gamma_{\nu\mu}^{*\lambda} = \Gamma_{kj}^{*i} \frac{\partial x^j}{\partial x^\mu} \frac{\partial x^k}{\partial x^\nu} + \frac{\partial^2 x^i}{\partial x^\nu \partial x^\mu}, \\ \frac{\partial x^\nu}{\partial x^i} V_{c(t)}^\lambda{}_\nu = \sum_{r=1}^t \binom{t}{r} \frac{\partial p_{c(t-r)}^\lambda}{\partial x^k} V_{c(r)}^k{}_{(j)} \quad (t=1, 2, \dots, M). \end{array} \right.$$

From (1.16a) we can now derive

$$(1.17) \quad \frac{\partial x^\nu}{\partial x^i} V_{c(t+\varepsilon)}^\lambda{}_\nu = \sum_{r=q+1}^t \binom{t+\varepsilon}{r+\varepsilon} \frac{\partial p_{c(t-r)}^\lambda}{\partial x^k} V_{c(r+\varepsilon)}^k{}_{(j)} - \binom{t+\varepsilon}{q+\varepsilon} \delta_{(c(q+\varepsilon))}^{a(q+\varepsilon)} \frac{\partial p_{c(t-q)}^\lambda}{\partial x^i}$$

$(q+1 \leq t \leq M-1, M-q \geq \varepsilon)$, where we put

$$(1.18) \quad V_{c(t+\varepsilon)}^{ka(q+\varepsilon)}{}^j = \frac{q!(t+\varepsilon)!}{t!(q+\varepsilon)!} \delta_{(c(\varepsilon))}^{a(\varepsilon)} V_{c(t)}^{ka(q)}{}^j.$$

Therefore we get

THEOREM 9. If the covariant derivative $\nabla_j^{a(q)} v^i$ ($0 \leq q < M$) is defined by (1.15), then the derivatives $\nabla_j^{a(q+\varepsilon)} v^i$ ($0 < \varepsilon \leq M-q$) given by (1.18) are also covariant. ([8] A. KAWAGUCHI-H. HOMBU).

Chapter 2. Connections defined in the space of K-spreads

§ 5. K-spreads. In this chapter we shall give the basis of the geometry of the partial differential equations

$$(2.1) \quad p_{c(m)}^i + H_{c(m)}^i(x^j, u^b; p_{c(1)}^j, \dots, p_{c(m-1)}^j) = 0$$

under the rheonomic transformations in X_n . Since no assumptions of existence of any solution of the differential equations are made, it is geometrically natural to consider that *to any u^a and any K-element of the $(m-1)$ -th order there corresponds a K-element of the m -th order*. The figures of the K-elements defined by (2.1), which we call K-spreads ([5] J. DOUGLAS), are in general *nonholonomic*. For the sake of convenience we denote by $F_n^{*(m-1)}$ the product of the u^a -manifold and $F_n^{(m-1)}$.

On the other hand the condition of complete integrability of (2.1) is given by

$$(2.2) \quad W_{c(m-1),ab}^i = H_{c(m-1),a/b}^i - H_{c(m-1),b/a}^i = 0,$$

using the notation

$$f_{|a} = f_{;a} + \sum_{r=0}^{m-2} f_{;i}^{(b(r))} p_{b(r),a}^i - f_{;i}^{(b(m-1))} H_{b(m-1),a}^i.$$

We shall see in §10 that the left-hand member of (2.2) represents a vector.

§6. **Covariant differential of a vector.** From (1.3) we have

$$(2.3) \quad \frac{\partial x^\lambda}{\partial x^i} H_{a(m)}^i = H_{a(m)}^\lambda + R_{a(m)}^\lambda(x^j, u^b; p_{b_1}^j, \dots, p_{b_{(m-1)}}^j)$$

and the relations

$$\frac{\partial R_{a(m)}^\lambda}{\partial p_{b_{(m-1)}}^j} = \frac{\partial p_{a(m)}^\lambda}{\partial p_{b_{(m-1)}}^j} = m \delta_{(a(m-1)}^{b_{(m-1)}} \frac{\partial p_{a(m)}^\lambda}{\partial x^j}.$$

Hence, differentiating (2.3) and using these relations, we get at once the law of transformation

$$(2.4) \quad \frac{\partial x^\lambda}{\partial x^i} G_{aj}^i = \frac{\partial x^\mu}{\partial x^j} G_{a\mu}^\lambda + \left(\frac{\partial x^\lambda}{\partial x^j} \right)_{|a}$$

where the quantities G_{aj}^i are defined by

$$(2.5) \quad G_{aj}^i = \frac{K!(m-1)!}{(K+m-1)!} H_{b_{(m-1)}a}^i (b_{(m-1)}^j).$$

Therefore we can define the covariant derivative of a vector along the K -spreads by

$$(2.6) \quad \nabla_a v^i = v^i_{|a} + G_{aj}^i v^j.$$

If we apply Theorem 5 to the vector $T_{c(m)}^i = p_{c(m)}^i + H_{c(m)}^i$ and contract with respect to some indices, we can get the derivatives (2.6) again.

Noticing that the operator (1.10) is invariant under the transformation, we have from (2.4)

$$\frac{\partial x^\lambda}{\partial x^i} \sum_{r=0}^{m-2} (r+1) G_{aj}^i ({}^{bc}_{(r)}; k) \delta p_{c(r)}^k = \frac{\partial x^\mu}{\partial x^j} \sum_{r=0}^{m-2} (r+1) G_{a\mu}^\lambda ({}^{bc}_{(r)}; j) \delta p_{c(r)}^\mu + \delta_a^b \frac{\partial^2 x^\lambda}{\partial x^j \partial x^i} \delta x^i.$$

By contraction with respect to a and b this reduces to

$$\frac{\partial x^\lambda}{\partial x^i} \sum_{r=0}^{m-2} I_{jk}^{ib(r)} \delta p_{b(r)}^k = \frac{\partial x^\mu}{\partial x^j} \sum_{r=0}^{m-2} I_{jk}^{i\lambda b(r)} \delta p_{b(r)}^\mu + \frac{\partial^2 x^\lambda}{\partial x^j \partial x^i} \delta x^i$$

where we put

$$(2.7) \quad I_{jk}^{ia(r)} = \frac{r+1}{K} G_{bj}^i ({}^{ba}_{(r)}; k) = \frac{(r+1)(m-1)!}{K(K+1)\dots(K+m-1)} H_{c_{(m-1)}b}^i ({}^{c_{(m-1)}}_{b}; j) ({}^{b_{a(r)}}_{a}; k) \\ (m-2 \geq r \geq 0).$$

From this and (1.14) we find that the differential forms

$$(2.8) \quad \delta v^i + \omega_j^i v^j \equiv \delta v^i + \sum_{r=0}^{m-2} \Gamma_{jk(r)}^{ia(r)} \delta p_{a(r)}^k v^j$$

have the vector character with respect to i . By means of (2.6) and (2.8) the covariant differential of a vector in $F_n^{*(m-1)}$ can be defined by

$$(2.9) \quad Dv^i = dv^i + G_{aj}^i v^j du^a + \sum_{r=0}^{m-2} \Gamma_{jk(r)}^{ia(r)} v^j \delta p_{a(r)}^k.$$

Making use of the covariant differential, we are able to introduce in the manifold $F_n^{*(m-1)}$ the parallel displacement of a vector v^i : Two vectors v^i and $v^i + dv^i$ at consecutive elements in $F_n^{*(m-1)}$ ($u^a, x^j, p_{a(r)}^j$) and ($u^a + du^a, x^j + dx^j, p_{a(r)}^j + dp_{a(r)}^j$) are said to be parallel when and only when $Dv^i = 0$, thus

$$dv^i = -G_{aj}^i v^j du^a - \sum_{r=0}^{m-2} \Gamma_{jk(r)}^{ia(r)} v^j \delta p_{a(r)}^k.$$

§ 7. **Covariant differentials of the K-element.** In order to define the covariant derivatives of a vector from its covariant differential in the manifold where the parallel displacement was introduced, it is commode to derive systems of the PFAFFIAN forms of the u^a 's and the K -element with the vector character. For this purpose let us apply Theorem 6 to the vector $T_{c(m)}^i = p_{c(m)}^i + H_{c(m)}^i$. By contraction with respect to $a_1 = c_1, a_2 = c_2, \dots, a_A = c_A$ we have

$$\left(\begin{matrix} K+m-1 \\ A \end{matrix} \right) \delta p_{a(m-A)}^i + \sum_{r=0}^{m-A-1} \left(\begin{matrix} A+r \\ r \end{matrix} \right) H_{a(m-A)c(A)}^i \binom{c(A)b(r)}{j} \delta p_{b(r)}^j.$$

Hence the PFAFFIAN forms

$$(2.10) \quad \delta p_{a(q)}^i = \delta p_{a(q)}^i + \sum_{r=0}^{q-1} Q_{a(q)j}^{ib(r)} \delta p_{b(r)}^j \quad (q = 0, 1, \dots, m-1)$$

possess the required property, where we put

$$(2.11) \quad Q_{a(q)j}^{ib(r)} = \frac{(K+q-1)! (m-q+r)!}{r! (K+m-1)!} H_{a(q)c(m-q)}^i \binom{c(m-q)b(r)}{j}$$

($r=0, 1, \dots, q-1$; $q=1, 2, \dots, m-1$) and

$$\delta p_{a(m-1)}^i = dp_{a(m-1)}^i + H_{a(m-1)b}^i du^b.$$

We call $\delta p_{a(q)}^i$ the q -th covariant differential of the K -element.

Now let us express the differential $\delta p_{a(q)}^i$ in terms of the covariant differentials $\delta p_{a(r)}^i$ in the form

$$(2.12) \quad \delta p_{a(q)}^i = \delta p_{a(q)}^i - \sum_{r=0}^{q-1} R_{a(q)j}^{i b(r)} \delta p_{b(r)}^j \quad (q = 0, 1, \dots, m-1).$$

From (2.10) and (2.12) we get the recursion formula

$$(2.13a) \quad R_{a(q)j}^{i b(q-1)} = Q_{a(q)j}^{i b(q-1)}, \quad R_{a(q)j}^{i b(r)} = Q_{a(q)j}^{i b(r)} - \sum_{s=r+1}^{q-1} R_{a(q)k}^{i c(s)} Q_{c(s)j}^{b(r)},$$

or

$$(2.13b) \quad R_{a(q)j}^{i b(q-1)} = Q_{a(q)j}^{i b(q-1)}, \quad R_{a(q)j}^{i b(r)} = Q_{a(q)j}^{i b(r)} - \sum_{s=r+1}^{q-1} R_{c(s)j}^{k b(r)} Q_{a(q)k}^{i c(s)},$$

($r=0, 1, \dots, q-2$; $q=1, 2, \dots, m-1$). Thus, if we write ω_j^i defined by (2.8) in the form

$$(2.14) \quad \omega_j^i = \sum_{q=0}^{m-2} \Gamma_{jk}^{ia(q)} \delta p_{a(q)}^k = \sum_{q=0}^{m-2} \Gamma_{jk}^{*ia(q)} \delta p_{a(q)}^k,$$

we have by virtue of (2.12) the following expressions of $\Gamma_{jk}^{*ia(q)}$:

$$(2.15) \quad \begin{cases} \Gamma_{jk}^{*ia(m-2)} = \Gamma_{jk}^{ia(m-2)}, \\ \Gamma_{jk}^{*ia(q)} = \Gamma_{jk}^{ia(q)} - \sum_{r=q+1}^{m-2} \Gamma_{jl}^{ib(r)} R_{b(r)k}^{la(q)} \quad (q=0, 1, \dots, m-3). \end{cases}$$

§ 8. Relations between the covariant differentials of the K-element and of a vector in $F_n^{*(m-1)}$. In the following steps we shall point out the intimate relations which lie between two kinds of the covariant differentials obtained above and moreover show that other covariant differentials different from Dv^i can be introduced.

(i). Comparing $\delta p_{c(m-1)}^i$ with (1.11), we have

$$\begin{aligned} \tilde{P}_{c(m-1)j}^{i b(r)} &= Q_{c(m-1)j}^{i b(r)} & \text{for } r &= 0, 1, \dots, m-2, \\ &= \delta_j^i \delta_c^b & \text{for } r &= m-1. \end{aligned}$$

($M=m-1$). Then the differential forms (1.12) lead us to the following

$$\left(\frac{K+m-2}{K-1} \right) \delta v^i + \sum_{r=0}^{m-2} Q_{c(m-1)j}^{i a(r)(c(m-1))} v^k \delta p_{a(r)}^j$$

which are evidently equivalent to (2.8) multiplied by $\left(\frac{K+m-2}{K-1} \right)$.

(ii) In § 6 we have got by means of Theorem 5 the covariant

derivatives along the K -spreads $\nabla_a v^i$. In like manner we can derive other covariant derivatives $\nabla_{a_{(t)}} v^i$ along the K -spreads ($t=1, 2, \dots, m-1$), which depend on the derivatives of higher order of the vector. Applying Theorem 5 for $M=m$ and $A=m-t$ ($t=1, 2, \dots, m-1$) to the vector $T_{a_{(m)}}^i = p_{a_{(m)}}^i + H_{a_{(m)}}^i$ we get namely

$$(2.16) \quad \nabla_{a_{(t)}} v^i = v^i|_{a_{(t)}} + \sum_{r=0}^{t-1} Q_{a_{(t)}}^i{}^{b_{(r)}}{}_j v^j|_{b_{(r)}} \quad (t=1, 2, \dots, m-1).$$

These derivatives will be useful to define a covariant differential different from Dv^i .

(iii) Covariant differential $\bar{D}v^i$. For this time the coefficients of $\partial p_{a_{(q)}}^i$ are

$$\begin{aligned} \bar{P}_{a_{(q)}}^i{}^{b_{(r)}}{}_j &= \partial_j^i \delta_a^b && \text{for } r=q, \\ &= Q_{a_{(q)}}^i{}^{b_{(r)}}{}_j && \text{for } r=0, 1, \dots, q-1, \\ &= 0 && \text{for all other } r. \end{aligned}$$

From the differential forms (1.13) for $M=m-1$, $A=m-q-1$ we see by contraction that the differential forms

$$\delta v^i + \sum_{t=0}^{m-q-1} \sum_{r=0}^{q-1} I_{k_j}^{i c_{(t)} a_{(r)}} v^k|_{c_{(t)}} \delta p_{a_{(r)}}^j$$

are transformed as a contravariant vector, where

$$(2.17) \quad I_{k_j}^{i c_{(t)} a_{(r)}} = \left(\frac{K+q-1}{K-1} \right)^{-1} \binom{q+t}{t} Q_{b_{(q)}}^i{}^{a_{(r)}}{}_{k_j}{}^{b_{(q)} c_{(t)}}{}_j$$

($r=0, 1, \dots, q-1$; $t=0, 1, \dots, m-q-1$; $q=1, 2, \dots, m-1$). From (2.6) and the above differential forms we obtain a covariant differential

$$(2.18) \quad \bar{D}v^i = dv^i + G_{aj}^i v^j du^a + \sum_{t=0}^{m-q-1} \sum_{r=0}^{q-1} I_{k_j}^{i c_{(t)} a_{(r)}} v^k|_{c_{(t)}} \delta p_{a_{(r)}}^j$$

($q=1, 2, \dots, m-1$). The last covariant differential depends on the derivatives of v^i up to the $(m-q-1)$ -th order, but we can remove them by means of (2.16), as follows. Let us substitute

$$- \sum_{r=0}^{m-q-2} Q_{c_{(m-q-1)}}^i{}^{b_{(r)}}{}_j v^j|_{b_{(r)}} \text{ in place of } v^i|_{c_{(m-q-1)}} \text{ in (2.18).}$$

It is then evident that this substitution preserves the vector character of the differential forms and lowers the highest order of the derivatives contained. Hence, continuing this process step by

step we can arrive at the $m-1$ different covariant differentials which depend no more on the derivatives of v^i .

§9. Covariant differentiations in $F_n^{*(m-1)}$. Now we determine the covariant derivatives of a vector field v^i in $F_n^{*(m-1)}$, making use of the covariant differentials Dv^i and $\partial p_{a(r)}^i$. The covariant derivatives $\nabla_a v^i$, $\nabla_{k^{(q)}}^i v^i$ ($q=0, 1, \dots, m-1$) are defined in the usual manner:

$$(2.19) \quad Dv^i = \nabla_a v^i du^a + \sum_{q=0}^{m-1} \nabla_{k^{(q)}}^i v^i \partial p_{a(r)}^k.$$

Obviously $\nabla_a v^i$ are the components of a contravariant vector and $\nabla_{k^{(q)}}^i v^i$ for fixed $a_{(q)}$ are the components of an affinor. From (2.9) and (2.10) we have by virtue of (2.12) and (2.15)

$$(2.20) \quad \left\{ \begin{array}{l} \nabla_a v^i = v^i_{|a} + G_{aj}^i v^j, \\ \nabla_{k^{(m-1)}}^i v^i = v^i(a_{(m-1)}^i), \\ \nabla_{k^{(q)}}^i v^i = v^i(a_{(q)}^i) + I_{jk^{(q)}}^{ia} v^j - \sum_{r=q+1}^{m-1} Q_{b(r)}^j \nabla_j^{a_{(q)}} v^i \\ \quad = v^i(a_{(q)}^i) - \sum_{r=q+1}^{m-1} R_{b(r)}^i \nabla_j^{a_{(q)}} v^i + I_{jk^{(q)}}^{ia} v^j \end{array} \right.$$

($q=0, 1, \dots, m-2$). Let us denote by $\bar{\nabla}_{k^{(q)}}^i v^i$ the expression obtained by omitting the last term in $\nabla_{k^{(q)}}^i v^i$. Since $I_{jk^{(q)}}^{ia}$ for $q \geq 1$ are components of an affinor, the derivative $\bar{\nabla}_{k^{(q)}}^i v^i$ is also covariant for $q \geq 1$.

As a special case of K -spreads we can treat the system of paths of higher order which is given by the differential equations

$$(2.21) \quad x^{(m)i} + H^i(x, t; x^{(1)}, \dots, x^{(m-1)}) = 0.$$

The important formulae which have been obtained in the above will be specialized, as follows ($K=1$):

$$(2.6') \quad \nabla v^i = v^{(1)i} + G_j^i v^j, \quad G_j^i = \frac{1}{m} H^i_{(m-1)j},$$

$$(2.9') \quad Dv^i = dv^i + G_j^i v^j dt + \sum_{r=0}^{m-2} I_{jk^{(r)}}^{i(r)} v^j dx^{(r)k},$$

$$(2.7') \quad I_{jk^{(r)}}^{i(r)} = \frac{r+1}{m} H^i_{(m-1)j; (r+1)k} \quad (m-2 \geq r \geq 0),$$

$$(2.10') \quad \partial x^{(q)i} = dx^{(q)i} + \sum_{r=0}^{q-1} Q_{k^{(r)}}^{i(r)} dx^{(r)j} \quad (q=1, 2, \dots, m-1),$$

$$(2.11') \quad Q_{(q)j}^{i(r)} = \frac{q!(m-q+r)!}{r!m!} H_{(m-q+r)j}^i$$

$$(r = 0, 1, \dots, q-1; q = 1, 2, \dots, m-1),$$

$$(2.13') \quad R_{(q)j}^{i(q-1)} = Q_{(q)j}^{i(q-1)}, R_{(q)j}^{i(r)} = Q_{(q)j}^{i(r)} - \sum_{s=r+1}^{q-1} R_{(q)k}^{i(s)} Q_{(s)j}^{k(r)} \quad (r = 0, 1, \dots, q-2),$$

$$(2.15') \quad \begin{cases} I_{jk}^{*i(m-2)} = I_{jk}^{i(m-2)}, \\ I_{jk}^{*i(q)} = - \sum_{r=q+1}^{m-1} I_{jl}^{i(r)} R_{(r)k}^{l(q)} + I_{jk}^{i(q)} \quad (q = 0, 1, \dots, m-3), \end{cases}$$

$$(2.20') \quad \begin{cases} \nabla v^i = v^{(1)i} + G_j^i v^j, \\ \nabla_{(q)k} v^i = v_{(q)k}^i + I_{jk}^{i(q)} v^j - \sum_{r=q+1}^{m-1} Q_{(r)k}^{j(q)} \nabla_{(r)j} v^i \\ = v_{(q)k}^i - \sum_{r=q+1}^{m-1} R_{(r)k}^{i(q)} v_{(r)l}^l + I_{jk}^{*i(q)} v^j \quad (q = 0, 1, \dots, m-1). \end{cases}$$

§ 10. **Curvatures and torsions.** In this paragraph we shall give the curvature tensors and the torsions of our connections which can be calculated in the usual way. If we put

$$(2.22a) \quad 2[\nabla_a \nabla_b] v^i = -R_{jab}^i v^j + W_{c(m-1)ab}^j \nabla_j^{c(m-1)} v^i,$$

$$(2.22b) \quad 2[\nabla_k^{a(q)} \nabla_b] v^i = -R_{jk_b}^{ia(q)} v^j + \sum_{r=q+1}^{m-1} S_{c(r)k}^j \nabla_{(r)k}^{a(q)} \nabla_j^{c(r)} v^i \quad (0 \leq q \leq m-1),$$

$$(2.22c) \quad 2[\nabla_k^{a(q)} \nabla_l^{b(r)}] v^i = -R_{jk_l}^{ia(q)b(r)} v^j + \sum_{s=q}^{m-1} S_{c(s)k}^j \nabla_{(s)k}^{a(q)} \nabla_j^{c(s)} \nabla_l^{b(r)} v^i \quad (0 \leq q \leq r \leq m-1),$$

then the curvatures R_{jab}^i , $R_{jk_b}^{ia(q)}$, $R_{jk_l}^{ia(q)b(r)}$ and the torsions $W_{c(m-1)ab}^i$,

$S_{c(r)k}^j \nabla_{(r)k}^{a(q)}$, $S_{c(s)k}^j \nabla_{(s)k}^{a(q)b(r)}$ are given by the following expressions:

$$(2.23a) \quad R_{jab}^i = G_{aj|b}^i - G_{bj|a}^i + G_{aj}^h G_{bh}^i - G_{bj}^h G_{ah}^i,$$

$$(2.23b) \quad R_{jk_b}^{ia(q)} = I_{jk_b}^{*ia(q)} + \sum_{s=q+1}^{m-1} R_{c(s)k_b}^h \nabla_j^{i c(s)} \nabla_{(s)k}^{a(q)} - \nabla_k^{a(q)} G_{bj}^i \\ + I_{jk}^{i(a(q-1))} \delta_{b^q}^a - \sum_{s=q+1}^{m-1} R_{c(s)k}^h \nabla_j^{i c(s)} \nabla_{(s)k}^{a(q)} \quad (0 \leq q \leq m-1),$$

$$(2.23c) \quad R_{jk_l}^{ia(q)b(r)} = \nabla_l^{b(r)} I_{jk}^{*ia(q)} - \nabla_k^{a(q)} I_{jl}^{*ib(r)} + I_{hl}^{*ib(r)} \nabla_j^{*ia(q)} - I_{hk}^{*ia(q)} \nabla_j^{*ib(r)} \\ + \sum_{s=q+1}^{m-1} I_{jh}^{i c(s)} \nabla_l^{b(r)} R_{c(s)k}^h - \sum_{s=r+1}^{m-1} I_{jh}^{i c(s)} \nabla_k^{a(q)} R_{c(s)l}^{b(r)} \quad (0 \leq q \leq r \leq m-1),$$

$$(2.24a) \quad W_{c_{(m-1)}ab}^j = H_{c_{(m-1)}a/b}^j - H_{c_{(m-1)}b/a}^j,$$

$$(2.24b) \quad S_{c_{(q-1)}k/b}^j = \partial_b^{a_{(q)}} \partial_k^j \quad (1 \leq q \leq m-1),$$

$$(2.24c) \quad S_{c_{(m-1)}k/b}^j = R_{c_{(m-1)}k/b}^j + \sum_{s=q+1}^{m-1} Q_{c_{(m-1)}h}^j R_{e_{(s)}k/b}^h a_{(q)} - \bar{\nabla}_{c_{(m-1)}k}^{a_{(q)}} H_{c_{(m-1)}b}^j \\ + Q_{c_{(m-1)}k/b}^j \partial_b^{a_{(q)}} - \sum_{s=q+1}^{m-1} R_{e_{(s)}k/b}^h a_{(q)} Q_{c_{(m-1)}h}^j \partial_b^{e_{(s-1)}} \partial_b^{e_{(s)}} \\ (0 \leq q < m-1),$$

$$(2.24d) \quad S_{c_{(r)}k/b}^j = R_{c_{(r)}k/b}^j + \sum_{s=q+1}^{r-1} Q_{c_{(r)}h}^j R_{e_{(s)}k/b}^h a_{(q)} + Q_{c_{(r)}k/b}^j \partial_b^{a_{(q-1)}} \partial_b^{a_{(q)}} \\ - \sum_{s=q+1}^r R_{e_{(s)}k/b}^h a_{(q)} Q_{c_{(r)}h}^j \partial_b^{e_{(s-1)}} \partial_b^{e_{(s)}} - R_{c_{(r)}k/b}^j \partial_b^{a_{(q)}} \\ (0 \leq q < r < m-1),$$

$$(2.24e) \quad S_{c_{(q)}k/b}^j = Q_{c_{(q)}k/b}^j \partial_b^{a_{(q-1)}} \partial_b^{a_{(q)}} - R_{bc_{(q)}k/b}^j a_{(q)} + \partial_{c_{(q)}}^{a_{(q)}} G_{bk}^j \quad (0 \leq q \leq m-1),$$

$$(2.24f) \quad S_{c_{(q)}k}^j a_{(q)} b_{(q)} = \partial_{c_{(q)}}^{a_{(q)}} \Gamma_{l/k}^{*j} b_{(q)} - \partial_{c_{(q)}}^{b_{(q)}} \Gamma_{l/k}^{*j} a_{(q)} \quad (0 \leq q \leq m-1),$$

$$(2.24g) \quad S_{c_{(q)}k}^j a_{(q)} b_{(r)} = \partial_{c_{(q)}}^{a_{(q)}} \Gamma_{l/k}^{*j} b_{(r)} \quad (0 \leq q < r \leq m-1),$$

$$(2.24h) \quad S_{c_{(s)}k}^j a_{(q)} b_{(r)} = \bar{\nabla}_{l(r)}^{b_{(r)}} R_{c_{(s)}k}^j a_{(q)} + \sum_{t=q+1}^{s-1} Q_{c_{(s)}h}^j \bar{\nabla}_{l(r)}^{b_{(r)}} R_{e_{(t)}k}^h a_{(q)} \\ (0 \leq q < s < r \leq m-1),$$

$$(2.24i) \quad S_{c_{(r)}k}^j a_{(q)} b_{(l)} = -\partial_{c_{(r)}}^{b_{(r)}} \Gamma_{l/k}^{*j} a_{(q)} + \bar{\nabla}_{l(r)}^{b_{(r)}} R_{c_{(r)}k}^j a_{(q)} + \sum_{t=q+1}^{r-1} Q_{c_{(r)}h}^j \bar{\nabla}_{l(r)}^{b_{(r)}} R_{e_{(t)}k}^h a_{(q)} \\ (0 \leq q < r \leq m-1),$$

$$(2.24j) \quad S_{c_{(s)}k}^j a_{(q)} b_{(l)} = \bar{\nabla}_{l(r)}^{b_{(r)}} R_{c_{(s)}k}^j a_{(q)} - \bar{\nabla}_{k(q)}^{a_{(q)}} R_{c_{(s)}l}^j b_{(r)} + \sum_{t=q+1}^{s-1} Q_{c_{(s)}h}^j \bar{\nabla}_{l(r)}^{b_{(r)}} R_{e_{(t)}k}^h a_{(q)} \\ - \sum_{t=r+1}^{s-1} Q_{c_{(s)}h}^j \bar{\nabla}_{k(q)}^{a_{(q)}} R_{e_{(t)}l}^h b_{(r)} \quad (0 \leq q \leq r < s \leq m-1).$$

It is remarkable that the condition of complete integrability (2.2) of the partial differential equations (2.1) is expressed by equating the torsion represented by (2.24a) to zero.

By way of (2.22) we are able to get the identities between curvatures and torsions, which are, however, omitted here.

Chapter 3. Theory of various quantities in $F_n^{*(M)}$

In this chapter we shall treat various vectors, extensors and others, which will be useful in the general theory of the geometries in $F_n^{*(M)}$. We begin with the study of ordinary extensors

and then observe that some theorems concerning contravariant extensors necessarily lead to the consideration of another wider class of geometric objects. Our study will end in the definition of this new objects and the formulation of related theorems.

§ 11. EULER's vector.

THEOREM 10. If $W_i^{a(r)}$ are the components of a covariant extensor in $F_n^{*(M)}$, then

$$(3.1) \quad \mathfrak{W}_i = \sum_{r=0}^M (-1)^r W_i^{a(r)}|_{a(r)}$$

are the components of a covariant vector. ([8] A. KAWAGUCHI-H. HOMBU).

Proof. Making use of LEIBNIZ's formula and (1.5a) we have

$$\mathfrak{W}_i = \sum_{r=0}^M (-1)^r \left(\sum_{t=r}^M W_{\lambda(t)}^{\zeta(t)} \frac{\partial p_{\zeta(t)}^{\lambda}}{\partial p_{a(r)}^i} \right) |_{a(r)} = \sum_{r=0}^M \sum_{t=r}^M (-1)^r \binom{t}{r} \left(W_{\lambda(t)}^{\zeta(t)} \left(\frac{\partial x^{\lambda}}{\partial x^i} \right) |_{c_{(t-r)}} \right) |_{c_{(r)}}.$$

Interchanging the order of summation and using

$$AB|_{a(r)} = \sum_{t=0}^r (-1)^t \binom{r}{t} \left(A|_{c_{(t)}} B \right) |_{a_{(r-t)}},$$

we see that the right-hand member is equivalent to $\frac{\partial x^{\lambda}}{\partial x^i} \mathfrak{W}_{\lambda}$. Hence the theorem is evident.

On account of this theorem and Theorem 1 we may conclude that

$$(3.2) \quad \mathfrak{W}_i^{a(A)} = \sum_{r=A}^M (-1)^r \binom{r}{A} \left(W_i^{a(A)b(r-A)} \right) |_{b(r-A)} \quad (A=0, 1, \dots, M)$$

are also components of a covariant vector for fixed $a(A)$.

If we put in particular

$$W_i^{a(r)} = F_{;i}^{(a(r))},$$

F being an arbitrary scalar, then $\mathfrak{W}_i$, $\mathfrak{W}_i^{a(A)}$, ($A=0, 1, \dots$, and M) are nothing but the well-known EULER-SYNGE's vector:

$$(3.1a) \quad E_i = \sum_{r=0}^M (-1)^r \left(F_{;i}^{(a(r))} \right) |_{a(r)},$$

$$(3.2a) \quad E_i^{a(A)} = \sum_{r=A}^M (-1)^r \binom{r}{A} \left(F_{;i}^{(a(A)b(r-A))} \right) |_{b(r-A)} \quad (A=0, 1, \dots, M).$$

THEOREM 11. If the derivative of a vector v^i in the manifold of K -elements

$$(3.3) \quad \Delta_b v^i = v^i_{/b} + \gamma^i_{bj} v^j$$

is covariant and the PFAFFIAN forms

$$(3.4) \quad P^i(d) = \sum_{r=0}^M P^{ia_{(r)}}_k \mathfrak{d} p^k_{a_{(r)}}$$

have the vector character with respect to i , then the PFAFFIAN forms

$$(3.5) \quad \Delta_b P^i(d) = \sum_{r=0}^M P^{ia_{(r)}}_k \mathfrak{d} p^k_{a_{(r)}b} + \sum_{r=0}^M [(P^{ia_{(r)}})_{/b} + \gamma^i_{bj} P^{ja_{(r)}}_k] \mathfrak{d} p^k_{a_{(r)}}$$

are transformed as components of a vector. ([8] A. KAWAGUCHI-H. HOMBU).

Proof. By hypothesis, we see that $P^{ia_{(r)}}_k$ ($0 \leq r \leq M$) are components of an extensor on account of Theorem 4, and the law of transformation of γ^i_{bj} is given by

$$\frac{\partial x^i}{\partial x^\lambda} \gamma^\lambda_{b\mu} = \frac{\partial x^j}{\partial x^\mu} \gamma^i_{bj} + \left(\frac{\partial x^i}{\partial x^\mu} \right)_{/b}.$$

Hence, the first and second terms of the right-hand member of (3.5) are transformed respectively according to

$$\frac{\partial x^i}{\partial x^\lambda} \sum_{t=0}^u \sum_{r=0}^t P^{\lambda c_{(t)}}_{\nu} \frac{\partial p^{\nu}_{c_{(t)}}}{\partial p^k_{a_{(r)}}} \mathfrak{d} p^k_{a_{(r)}b}$$

and

$$\frac{\partial x^i}{\partial x^\lambda} \sum_{r=0}^M \left[(P^{ia_{(r)}})_{/b} + \gamma^\lambda_{b\mu} P^{\mu a_{(r)}}_{\nu} \right] \mathfrak{d} p^{\nu}_{a_{(r)}} + \frac{\partial x^i}{\partial x^\lambda} \sum_{t=0}^M \sum_{r=0}^t P^{\lambda c_{(t)}}_{\nu} \left(\frac{\partial p^{\nu}_{c_{(t)}}}{\partial p^k_{a_{(r)}}} \right)_{/b} \mathfrak{d} p^k_{a_{(r)}}.$$

On the other hand, making use of the extensor character of $\mathfrak{d} p^k_{a_{(r)}}$, we find that the sum of the former expression and the second term of the latter expression is equal to

$$\frac{\partial x^i}{\partial x^\lambda} \sum_{t=0}^M P^{\lambda c_{(t)}}_{\nu} \mathfrak{d} p^{\nu}_{c_{(t)}b}.$$

Thus our theorem is proved.

This theorem shows us that the covariant derivatives of PFAFFIAN forms with the vector character can be defined in the same manner as the case of a vector.

THEOREM 12. If $W_{i(r)}^{a_{(r)}}$ are the components of a covariant extensor in $F_n^{*(M)}$, then

$$(3.6) \quad Z_i^{a_{(t)}} = \frac{1}{t} \sum_{r=t}^M (-1)^{r-t} \left(W_{i(r)}^{a_{(t)} b_{(r-t)}} \right)_{|b_{(r-t)}} \quad (t = 1, 2, \dots, M)$$

constitute the components of a covariant extensor. ([3] H. V. CRAIG).

Proof. Since $W_{i(r)}^{a_{(r)}}$ is a covariant extensor, we have

$$Z_i^{a_{(t)}} = \frac{1}{t} \sum_{r=t}^M \sum_{q=r}^M \sum_{s=0}^{r-t} (-1)^{r-t} \binom{r-t}{s} \binom{q}{r} \left(\frac{\partial x^\lambda}{\partial x^i} \right)_{|b_{(q-t-s)}} \left(W_{\lambda}^{a_{(t)} b_{(q-t)}} \right)_{|b_{(s)}}$$

using LEIBNIZ'S formula and (1.5a). Putting $q-s=v$ and interchanging the order of summation, we see that the right-hand member turns to

$$\frac{1}{t} \sum_{q=t}^M \sum_{v=t}^q (-1)^{v+q} \binom{v-1}{v-t} \left(\frac{\partial x^\lambda}{\partial x^i} \right)_{|b_{(v-t)}} \left(W_{\lambda}^{a_{(t)} b_{(q-t)}} \right)_{|b_{(q-v)}}$$

by virtue of the identity

$$\sum_{r=q+t-v}^q (-1)^{r-t} \binom{r-t}{q-r} \binom{q}{r} = (-1)^{v+q} \binom{v-1}{v-t}.$$

Interchanging once more the order of summation we have

$$Z_i^{a_{(t)}} = \sum_{v=t}^M \frac{\partial p_{b_{(v)}}^\lambda}{\partial p_{a_{(t)}}^i} Z_{\lambda}^{b_{(v)}};$$

our theorem is established.

As Professor A. KAWAGUCHI has already shown, the SYNGE'S vector (3.2) can be generalized as follows:

THEOREM 13. If $W_{i(r)}^{a_{(r)}}$ are the components of a covariant extensor in $F_n^{*(M)}$, then

$$(3.7) \quad \mathfrak{E} W_{i(t)}^{a_{(t)}} = \sum_{r=0}^H (-1)^r \binom{t+r}{r} \binom{M-t-r}{M-t-H} \left(W_{i(t)}^{a_{(t)} b_{(r)}} \right)_{|b_{(r)}}$$

($H=0, 1, \dots, M$; $t=0, 1, \dots, M-H$) are the components of a covariant extensor for each value of H . ([7] A. KAWAGUCHI).

Proof. In like manner as in Theorem 12, the calculation gives

$$\begin{aligned} \mathfrak{E} W_{i(t)}^{a_{(t)}} &= \sum_{r=0}^H (-1)^r \binom{t+r}{r} \binom{M-t-r}{M-t-H} \sum_{q=0}^r \binom{r}{q} \\ &\quad \times \sum_{s=t+r}^M \binom{s}{t+r} \left(\frac{\partial x^\lambda}{\partial x^i} \right)_{|c_{(s-t-q)}} \left(W_{\lambda}^{a_{(t)} b_{(q)} c_{(s-t-q)}} \right)_{|b_{(q)}}. \end{aligned}$$

By the substitution $s=v+q$ and the interchange of the order of summation we get

$$\overset{H}{\mathcal{E}} W_i^{a_{(t)}} = \sum_{v=t}^{M-H} \binom{v}{t} \binom{\partial x^\lambda}{\partial x^i} \Big|_{c_{(v-t)}} \sum_{q=0}^H (-1)^q \binom{v+q}{q} \binom{M-q-v}{M-H-v} \left(W_{\lambda}^{a_{(t)} b_{(q)} c_{(v-t)}} \right) \Big|_{b_{(q)}}$$

on account of the identity

$$\begin{aligned} & \sum_{r=q}^H \sum_{v=r-q+t}^{M-q} (-1)^r \binom{t+r}{r} \binom{M-t-r}{M-t-H} \binom{r}{q} \binom{v+q}{t+r} \\ &= \sum_{v=t}^{M-H} (-1)^q \binom{M-q-v}{M-H-v} \binom{v+q}{q} \binom{v}{t}. \end{aligned}$$

The right-hand member of the expression can be written, by virtue of (3.7) and (1.5a), in the form

$$\sum_{v=t}^{M-H} \frac{\partial p_{b_{(v)}}^\lambda}{\partial p_{a_{(t)}}^i} \overset{H}{\mathcal{E}} W_{\lambda}^{b_{(v)}},$$

which completes the proof.

§ 12. Definition of new geometric objects. In the case $K > 1$ the operator $\overset{H}{\mathcal{E}}$ in Theorem 13 can not be extended in general to contravariant extensors. The extension will be possible only by the introduction of the following geometric objects.

DEFINITION. When a set of quantities $X_{a_{(A+r)}}^i$ or $Y_i^{a_{(A+r)}}$ ($M-A \geq r \geq 0$) is subjected to the law of transformation

$$\begin{aligned} (a) \quad X_{a_{(A+r)}}^\lambda &= \sum_{s=0}^r \frac{\partial p_{a_{(r)}}^\lambda}{\partial p_{(s)}^i} X_{a_{(A)}}^i \Big|_{b_{(s)}}, \\ (b) \quad Y_i^{a_{(A+r)}} &= \sum_{s=r}^{M-A} \frac{\partial p_{b_{(s)}}^\lambda}{\partial p_{a_{(r)}}^i} Y_{\lambda}^{b_{(s)} a_{(A)}}, \end{aligned} \quad (3.8)$$

then $X_{a_{(A+r)}}^i$ and $Y_i^{a_{(A+r)}}$ ($M-A \geq r \geq 0$) are said, respectively, to be the *contravariant* and *covariant pseudo-extensor* of the class A and the range (A, M) .

$Y_i^{a_{(A+r)}}$ is a covariant pseudo-extensor of the class A and the range (A, M) when and only when for fixed $a_{(A)}$ it turns to an ordinary extensor of the range $(0, M-A)$. The pseudo-extensors of the class zero are ordinary extensors.

The transformations of covariant and contravariant pseudo-

extensor of the same class and the same range are contragredient. Indeed, we can write (3.8a) and (3.8b) in the form

$$X_{a(A+r)}^\lambda = \sum_{s=0}^r P_{a(A+r)}^\lambda{}_{b(A+s)} X_{b(A+s)}^i, \quad Y_i^{b(A+s)} = \sum_{r=s}^{M-A} P_{a(A+r)}^\lambda{}_{b(A+s)} Y_{\lambda(A+r)}^a,$$

where we put

$$P_{a(A+r)}^\lambda{}_{b(A+s)} = \binom{r}{s} \delta_{a(A+s)}^{b(A+s)} \left(\frac{\partial x^\lambda}{\partial x^i} \right)_{/a_{(r-s)}}.$$

In the following lines we shall state some basic theorems concerning pseudo-extensors.

THEOREM 14. If $X_{a(A+r)}^i$ ($M-A \geq r \geq 0$) is a contravariant pseudo-extensor of the class A and the range (A, M) and the components for all $r=0, 1, \dots, \varepsilon-1$ vanish, then $\binom{r+\varepsilon}{\varepsilon}^{-1} X_{a(A+\varepsilon+r)}^i$ is a contravariant pseudo-extensor of the class $(A+\varepsilon)$ and the range $(A+\varepsilon, M)$.

Proof. By hypothesis we have from (3.8a)

$$\begin{aligned} X_{a(A+\varepsilon+r)}^\lambda &= \sum_{s=0}^r \frac{\partial p_{a(r+\varepsilon)}^\lambda}{\partial p_{b(s+\varepsilon)}^i} X_{a(A)}^i{}_{b(s+\varepsilon)} \\ &= \binom{r+\varepsilon}{\varepsilon} \sum_{s=0}^r \binom{s+\varepsilon}{\varepsilon}^{-1} \frac{\partial p_{a(r)}^\lambda}{\partial p_{b(s)}^i} X_{a(A+\varepsilon)}^i{}_{b(s)} \quad (0 \leq r \leq M-A-\varepsilon), \end{aligned}$$

making use of (1.5b). Hence $\binom{r+\varepsilon}{\varepsilon}^{-1} X_{a(A+\varepsilon+r)}^i$ is a pseudo-extensor of the class $(A+\varepsilon)$.

Concerning the covariant pseudo-extensor we have in like manner

THEOREM 15. If $Y_i^{a(A+r)}$ ($0 \leq r \leq M-A$) is a covariant pseudo-extensor of the class A and the range (A, M) then $\binom{r+\varepsilon}{\varepsilon} Y_i^{a(A+\varepsilon+r)}$ is covariant of the class $(A+\varepsilon)$ and the range $(A+\varepsilon, M)$.

THEOREM 16. If $X_{a(B+s)}^i$ is a contravariant pseudo-extensor of the class B and the range (B, M) , then the quantities

$$(3.9) \quad \Xi_{a(A+B+r)}^i(X) = \sum_{s=0}^A (-1)^{A-s} \binom{A}{s} X_{a(B+s+r)}^i{}_{/a_{(A-s)}}$$

($A=1, 2, \dots$ or $M-B$; $r=0, 1, \dots, M-A-B$) constitute a contravariant pseudo-extensor of the class $(A+B)$ and the range $(A+B, M)$.

Proof. Making use of LEIBNIZ's formula and (1.5a), we have

$$\begin{aligned} \mathfrak{E}_{a(A+B+r)}^{\lambda}(X) &= \sum_{s=0}^A \sum_{t=0}^{r+s} \sum_{u=0}^{A-s} (-1)^{A-s} \binom{A}{s} \binom{A-s}{u} \binom{r+s}{t} \\ &\quad \times \left(\frac{\partial x^{\lambda}}{\partial x^i} \right)_{/ (a(r+s+u-t))} X_{a(B+t)/a(A-s-u)}^i. \end{aligned}$$

Putting $u+s=v$ and interchanging the order of summation we get

$$\mathfrak{E}_{a(A+B+r)}^{\lambda}(X) = \sum_{v=0}^A \sum_{t=v}^{r+v} (-1)^{A-v} \binom{A}{v} \binom{r}{t-v} \left(\frac{\partial x^{\lambda}}{\partial x^i} \right)_{/ (a(r+v-t))} X_{a(B+t)/a(A-v)}^i$$

on account of the identity

$$\begin{aligned} \sum_{s=0}^v (-1)^s \binom{v}{s} \binom{s+r}{t} &= 0 && \text{for } t < v, \\ &= (-1)^s \binom{r}{t-v} && \text{for } t \geq v. \end{aligned}$$

By the substitution $t-v=w$ it is seen that

$$\mathfrak{E}_{a(A+B+r)}^{\lambda}(X) = \sum_{w=0}^r \binom{r}{w} \left(\frac{\partial x^{\lambda}}{\partial x^i} \right)_{/ (a(r-w))} \mathfrak{E}_{a(A+B+w)}^i(X) = \sum_{w=0}^r \frac{\partial p_{a(r)}^{\lambda}}{\partial p_{b(w)}^i} \mathfrak{E}_{a(A+B)b(w)}^i(X).$$

Hence our theorem is proved.

For $B=0$, i.e. for a contravariant extensor (3.9) reduces to the operator $\mathfrak{E}$ of Professor A. KAWAGUCHI slightly modified ([7]). In this manner we may rewrite some theorems which were given there in a somewhat wrong way.

As to the covariant pseudo-extensor, Theorem 13 may be generalized as follows:

THEOREM 17. If $Y_i^{a(B+r)}$ is a covariant pseudo-extensor of the class B and the range (B, M) , then

$$(3.10) \quad \mathfrak{E}_{i^{a(B+s)}}^A(Y) = \sum_{r=0}^A (-1)^r \binom{s+r}{r} \binom{M-B-s-r}{M-B-s-A} \left(Y_i^{a(B+s)b(r)} \right)_{/ b(r)}$$

($A=1, 2, \dots$ or $M-B$; $s=0, 1, \dots, M-A-B$) is a covariant pseudo-extensor of the class B and the range $(B, M-A)$.

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