STRUCTURE POLYNOMIAL OF LATIN RECTANGLES AND ITS APPLICATION TO A COMBINATORIAL PROBLEM

By

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Introduction.—Generalizing the classical *problème des rencontres* and *problème des ménages*, TOUCHARD asked for the number of permutations discordant with the k permutations $123 \dots n$, $234 \dots 1$, ..., $k \ k+1 \dots k-1$. The case $k=3$ was recently treated by RIORDAN [7], who obtained a recursive formula of the enumeration polynomial, but it seemed far from being direct, containing an auxiliary parameter not easily calculated.

The present paper deals chiefly with the case $k=3$, giving, above all, a recursive formula and asymptotic expansion, both in just the manner in which a future generalization is believed to take place.

1.—Latin extension and Latin contraction.—Let L be a $k \times n$ Latin rectangle in the integers $1, 2, \dots, n$. Suppose that we can add a new row to L , obtaining a $(k+1) \times n$ Latin rectangle L' . Then we call L' a *Latin extension* of L , and L a *Latin contraction* of L' . The number of Latin extensions of L will be denoted by $N(L)$, and that of Latin contractions of L by $N^*(L)$. These may be alternately defined as the numbers of permutations in the integers $1, 2, \dots, n$, which are discordant with, or imbedded into L . They depend in fact only on the equivalence class of Latin (or rather Hall) rectangles (Cf. [11]), and if L^* denotes a complementary Latin (or rather Hall) rectangle to L , then $N(L^*) = N^*(L)$. We cite here the following theorem of ERDÖS-KAPLANSKY [1] in the form improved by the author [10].

THEOREM 1. $N(L) = n! \sum (-)^u J(L; \pi) 1^{a_2} 2^{a_3} \dots (k-1)^{a_k} \sigma_{n-t} / (n)_t$,
where the summation is extended over all non-unitary restricted partitions

$$\pi : \begin{cases} t = 2a_2 + 3a_3 + \dots + ka_k, \\ u = a_2 + a_3 + \dots + a_k \end{cases}$$

and $J(L; \pi)$, the ultimate building block, denotes the number of ways of choosing t ele-

ments in different columns of L , in such a way that these t elements use up just u integers, and moreover, some a_2 of these integers appear each twice, some a_3 of them appear each thrice, ..., and some a_k of them appear each k times. σ_n is the truncated sum $\sum_{m=0}^n (-k)^m/m!$ for e^{-k} and $(n)_i$ is the Jordan factorial.

2.—Structure polynomial.—We define the structure polynomial $F(L; X)$ of Latin rectangle L by

$$F(L; X) = \sum J(L; \pi) x_2^{a_2} x_3^{a_3} \cdots x_k^{a_k},$$

and the reduced structure polynomial $f(L; x)$ by

$$f(L; x) = x^n F(L; -x^{-2}, -2x^{-3}, \dots, -(k-1)x^{-k}).$$

Then Theorem 1 may be stated compactly as

$$\text{THEOREM 1'}. \quad N(L) = f(L; E - k)0! = f(L; E)S_0,$$

where E is a usual shift operator and $S_n = n! \sigma_n$.

The use of these polynomials enables one to express some other functions of L in manageable fashion, for instance the number q_m of permutations having just m clashes with L is given by (Cf. [1], [2]) $q_m = (-)^m \sum_{i=0}^{n-m} (n-i)_m f^{(i)}(L; -k)$, and in particular,

$$\text{THEOREM 2}. \quad N^*(L) = (-)^n f(L; -k).$$

This result may be regarded as a duality theorem and becomes important in the sequel.

3.—A combinatorial lemma.—The following lemma is rather trivial, but it contains a typical argument that will be used repeatedly in our treatment.

LEMMA 1. *The number of ways of choosing a (>0) points from among the n given points on a circle, such that any two points selected are separated by at least s other points is $\frac{n}{a} \binom{n-sa-1}{a-1}$.*

PROOF. (Cf. [4], [9].) Any choice containing a fixed point is characterized by the sequence of numbers $v_1, v_2, \dots, v_a$, denoting the lengths of intervals formed by adjacent points selected. These numbers satisfy $\sum v_i = n$ and $v_i > s$. The number of solutions of this equation is, as is well known,

$\binom{n-sa-1}{a-1}$. Since any of the a points may be fixed at the beginning, the Lemma follows immediately.

4.—Enumeration polynomial.—Let $\{L\} = \{L^k, L^{k+1}, \dots, L^n, \dots\}$ be a sequence of Latin rectangles, L^n being a $k \times n$ Latin rectangle. Then we put $Q_n(E) = f(L^n; E-k)$, $G(x, E) = \sum Q_n(E) x^n$ and call $Q_n(E)$ an *enumeration polynomial*, and $G(x, E)$ its *generating function*. If the sequence consists of Latin rectangles corresponding to the problem of TOUCHARD, we denote the generating function by $G_k(x, E)$.

Before going further, we illustrate our notations by simple examples. The *problème des rencontres* is the case $k=1$ of TOUCHARD's problem. Here $J(L^n; \pi) = 0$ unless $t=u=0$, and $F(L^n; x) = 1$, $f(L^n; E-1) = Q_n(E) = (E-1)^n = \Delta^n$, where $\Delta = E-1$ and $N_n = N(L^n) = \Delta^n 0!$, $G_1(x, E) = \Delta x / (1 - \Delta x)$.

The *problème des ménages* corresponds to $k=2$. Here the significant partitions are necessarily of the form 2^a , and $J(L^n; 2^a)$ is precisely given by Lemma 1 with $s=1$. Hence we have, by putting $\Delta = E-2$,

$$\begin{aligned} Q_n(E) &= \Delta^n + \sum_{a=1}^{\lfloor n/2 \rfloor} (-)^a \frac{n}{a} \binom{n-a-1}{a-1} \Delta^{n-2a}, \\ G_2(x, E) &= \Delta^2 x^2 / (1 - \Delta x) + \sum_{a=1}^{\infty} (-)^a \frac{1}{a} \frac{x \partial}{\partial x} \left(\sum_{n=2a}^{\infty} \binom{n-a-1}{a-1} \Delta^{n-2a} x^n \right) \\ &= \Delta^2 x^2 / (1 - \Delta x) - \frac{x \partial}{\partial x} (\log (1 + x^2 / (1 - \Delta x))) \\ &= -2 - \Delta x + \frac{2 - \Delta x}{1 - \Delta x + x^2}, \end{aligned}$$

since $\sum_{n=2a}^{\infty} \binom{n-a-1}{a-1} x^n = (x^2 / (1 - x))^a$.

5.—The operator H .—Let K be a field and x be an indeterminate. In the field of rational functions, $K(x)$, we define the operator H by

$$H(f(x)) = m - \frac{x}{f(x)} \frac{d}{dx} f(x), \quad m = \deg. f(x) \quad (f(x) \neq 0).$$

This is $= m - \frac{xd}{dx} \log f(x)$, if K is the field of complex numbers, or function field over the field of complex numbers. By this operator H the multiplicative group $K(x)^*$ of non-zero elements of $K(x)$ is mapped *homomorphically* into the additive group of the same field:

$$H(f(x)g(x)) = H(f(x)) + H(g(x)) \quad (f(x), g(x) \in K(x)^*).$$

We call functions in the image $H(K(x)^*)$ *H-functions*, and we call $f(x)$ the defining (rational) function of $H(f(x))$. Moreover we define *h-functions* as sum of *H-functions* and a polynomial. It is easily seen that the defining function of an *H-function* is uniquely determined except for a constant factor, and that for any *h-function* there is only one *H-function* such that the difference is a polynomial, which we shall call its *H-form*.

It is to be noted that the operator H was used essentially for the calculation of $G_2(x, E)$, which in its turn establishes a connection between the typical argument of Lemma 1 and the operator H .

6.—The 3-ply staircase.—The following Lemma makes the most conceptual part of our treatment of TOUCHARD's problem.

LEMMA 2. Consider the staircase (Cf. [7])

$$\begin{array}{ccccccc} & 1 & 2 & 3 & \dots & n \\ & 1 & 2 & 3 & \dots & n \\ 1 & 2 & 3 & \dots & n \end{array}$$

and denote by $b_{n,r}$ the number of ways of choosing r columns and $2r$ integers such that each column contains just two integers, from the above staircase. Then

$$b_{n,r} = b_{n-1,r} + 2b_{n-2,r-1} + b_{n-3,r-1} + b_{n-4,r-2}, \quad (n \geq 1),$$

$$b_{0,0} = 1.$$

The corresponding number for the Latin rectangle L^n of the Touchard's problem is given by

$$b_{n,r}^* = b_{n,r} + 2b_{n-2,r-1} + 2b_{n-3,r-1} + 3b_{n-4,r-2}.$$

PROOF. It is convenient to consider another staircase together, which results from the above by shifting the three 1's to the left by a unit length. The number bearing the same meaning for this new staircase as of $b_{n,r}$ will be denoted by $c_{n,r}$. Then we have $b_{n,r} = b_{n-1,r} + 2b_{n-2,r-1} + c_{n-2,r-1}$, corresponding to the three kinds of choices: not containing the integer 1, containing both 1 and 2, and containing 1 but not 2. Similarly $c_{n,r} = b_{n-1,r} + b_{n-2,r-1}$. Elimination of c results in the first formula of the Lemma. In the same manner we find $b_{n,r}^* = b_{n,r} + 2b_{n-2,r-1} + 2c_{n-2,r-1} + b_{n-4,r-2}$. In fact, the number of choices which do not contain the $(n-1)^{\text{st}}$ nor the n^{th} column ("seaming" columns) is $b_{n,r}$, whereas the number of choices containing only one or both

of these two columns is $b_{n-2,r-1} + c_{n-2,r-1}$ or $b_{n-4,r-2}$ respectively.

THEOREM 3. *The function $G_3(x, E)$ is an h -function of x , and has the H -form $H(P(x))$ with*

$$P(x) = (1+x)(1-(1+\Delta)x+(3+\Delta)x^2-x^3), \quad \Delta = E-3.$$

PROOF. The two-way generating function $B(x, y) = \sum b_{n,r} x^n y^r$ is equal to $1/T(x, y)$, with the polynomial $T(x, y) = 1 - x - 2x^2y - x^3y - x^4y^2$, by Lemma 2. Similarly $B^*(x, y) = \sum b_{n,r}^* x^n y^r = \sum J(L^n; 2^r) x^n y^r = (1 + 2x^2y + 2x^3y + 3x^4y^2)/T(x) = H(T(x)) + o_2$, where o_2 denotes a polynomial of degree 2. This is the leading term of the "structure polynomial generating function" $\sum F(L^n; y, z) x^n$ of the sequence of Latin rectangles of TOUCHARD's problem (for $k=3$), which we find to be an h -function. Now consider the remaining terms of the structure polynomial generating function resulting from $J(L^n; 2^r 3^s)$ with $s > 0$. This is calculated by the typical argument of the Lemma 1. We observe that $\sum_r J(L^n; 2^r 3^s) y^r$ is n/s times the coefficient of x^{n-3s} in $(B(x, y))^s$, or n/s times the coefficient of x^n in $(x^3 B(x, y))^s$. This means that the part of the structure polynomial generating function resulting from $J(L^n; 2^r 3^s)$ with $s > 0$ is given by

$$-\frac{xd}{dx} \log(1 - zx^3 B(x, y)) = H(1 - zx^3 B(x, y)) - 4$$

and hence the structure polynomial generating function is

$$H(T(x, y)) + o_2 + H(1 - zx^3 B(x, y)) - 4 = H(T(x, y) - zx^3) + o_2,$$

By the substitution $x \rightarrow \Delta x$, $y \rightarrow -\Delta^{-2}$, $z \rightarrow -2\Delta^{-3}$, we obtain the Theorem.

7.—The Assumption A.—We have seen in §4 and §6 that $G_k(x, E)$ are h -functions for k up to 3. It is natural to replace $G_k(x, E)$ by their H -forms. We do this without changing the notation. Thus we have

$$G_k(x, E) = H(P(x, E)),$$

with

$$\begin{aligned} P(x, E) &= 1 - \Delta x = 1 - (E-1)x && \text{for } k=1, \\ P(x, E) &= 1 - \Delta x + x^2 = 1 - (E-2)x + x^2 && \text{for } k=2, \\ P(x, E) &= 1 - \Delta x + 2x^2 + (\Delta+2)x^3 - x^4 && \text{for } k=3, \\ &= (1+x)(1-(1+\Delta)x+(3+\Delta)x^2-x^3) \\ &= 1 - (E-3)x + 2x^2 + (E-1)x^3 - x^4 \end{aligned}$$

$$= (1+x)(1-(E-2)x+Ex^2-x^3).$$

It is conjectured that $G_k(x, E)$ is an h -function for an arbitrary k . But the author cannot prove this.

Now denote by A the following assumption for the rational functions $G(x, E)$ of two variables x, E .

A1. $G(x, E) = H(P(x, E))$, considered as a function of x , where $P(x, E)$ is a polynomial in x, E .

A2. $P(x, E)$ is linear in E , and has the form

$$P(x, E) = 1 + (-E + a)x + \cdots.$$

A2. is alternately expressed as

$$P(0, E) = -\frac{1}{x} \frac{\partial}{\partial E} P(x, E) \Big|_{x=0} = 1.$$

For the function $G(x, E)$ satisfying the Assumption A , we define $Q_n(E)$ by $G(x, E) = \sum Q_n(E)x^n$. It is readily seen that $Q_n(E)$ is a polynomial in E of degree n .

Moreover we need a derivative of $Q_n(E)$. We introduce $R_n(E)$ by

$${}_n R_{n-1}(E) = \frac{d}{dE} Q_n(E).$$

$R_n(E)$ is a polynomial of degree at most n , and has the generating function $-\frac{1}{xP(x, E)} \frac{\partial P(x, E)}{\partial E}$. Indeed putting this generating function $= K(x, E)$ we have

$$\begin{aligned} \frac{x \partial (xK)}{\partial x} &= \sum {}_n R_{n-1}(E) x^n = \sum \frac{\partial}{\partial E} Q_n(E) x^n = \frac{\partial}{\partial E} (m - x \frac{\partial}{\partial x} \log P) \\ &= -\frac{x \partial}{\partial x} \left(\frac{\partial}{\partial E} \log P \right) = -\frac{x \partial}{\partial x} \frac{1}{P} \frac{\partial P}{\partial E}, \end{aligned}$$

(m is the degree of $P(x, E)$ in x), and hence $xK = -\frac{1}{P} \frac{\partial P}{\partial E}$, since both members are of the form $x + \cdots$ as seen from the Assumption A .

8.—Difference equation for the $Q_n(E)$ and $R_n(E)$.—Now we maintain

THEOREM 4. Under the Assumption A , the $Q_n(E)$ and $R_n(E)$ are combined by a homogeneous linear difference equation of the form

$$\sum a_i Q_{n-i}(E) = \sum b_i R_{n-i}(E), \quad a_0 = b_0 = 1,$$

with integral coefficients a_i, b_i .

PROOF. Write $P(x, E)$ as

$$P(x, E) = U(x) - ExV(x),$$

where $U(x)$ and $V(x)$ are polynomials of x . Note that $U(0) = V(0) = 1$, which is a consequence of the Assumption A . The generating functions of $Q_n(E)$ and $R_n(E)$ are $G(x, E) = M(x, E)/P(x, E)$ and $K(x, E) = V(x)/P(x, E)$ respectively, where

$$\begin{aligned} M(x, E) &= mP(x, E) - \frac{x\partial}{\partial x} P(x, E) \\ &= (mU(x) - xU'(x)) - Ex((m-1)V(x) - xV'(x)), \end{aligned}$$

which means that we have only to find polynomials $A(x)$ and $B(x)$ of x such that

$$A(x)M(x, E) - B(x)V(x) \equiv 0 \pmod{P(x, E)},$$

which is equivalent to find three polynomials $A(x), B(x), C(x)$ such that

$$A(x)M(x, E) - B(x)V(x) = C(x)P(x, E).$$

Such polynomials are given by comparing coefficients of E . For instance

$$\begin{aligned} A(x) &= (V(x))^2, \quad B(x) = U(x)V(x) - x(U'(x)V(x) - U(x)V'(x)), \\ C(x) &= ((m-1)V(x) - xV'(x))V(x). \end{aligned}$$

It follows from $U(0) = V(0) = 1$ that $A(0) = B(0) = 1$.

If $U(x)$ and $V(x)$ has a common divisor, then $A(x)$ and $B(x)$ may be divided by this divisor.

9.—Examples.— $G(x, E) = G_1(x, E)$. We see from §7, that $U = 1 - x$, $V = 1$; $A = B = 1$. The difference equation is

$$(1) \quad Q_n = R_n.$$

Consider the $G(x, E) = G_2(x, E)$. Here $U = 1 - 2x + x^2$, $V = 1$; $A = 1$, $B = 1 - x^2$. The difference equation is

$$(2) \quad Q_n = R_n - R_{n-2}.$$

Next consider the case $G(x, E) = G_3(x, E)$. In this case

$U = 1 + 3x + 2x^2 - x^3 - x^4$, $V = 1 - x^2$; $A = 1 - 2x^2 + x^4$, $B = 1 - 5x^2 - 4x^3 + x^5 - x^6$. But U and V are both divisible by $1 + x$. Hence in the form divided by $1 + x$, we have $A = 1 - x - x^2 + x^3$, $B = 1 - x - 4x^2 + x^4 - x^5$. The difference equation is

$$(3) \quad Q_n - Q_{n-1} - Q_{n-2} + Q_{n-3} = R_n - R_{n-1} - 4R_{n-2} + R_{n-4} - R_{n-5}.$$

In this case $P(x, E)$ is decomposed into $1 + x$ and $\bar{P}(x, E) = 1 - (E - 2)x + Ex^2 - x^3$. For $\bar{G}(x, E) = H(\bar{P}(x, E))$ we have $\bar{U} = 1 + 2x - x^3$, $\bar{V} = 1 - x$; $\bar{A} = 1 - 2x + x^2$, $\bar{B} = 1 - 2x - 2x^2 + 2x^3 - x^4$. The difference equation is

$$(4) \quad \bar{Q}_n - 2\bar{Q}_{n-1} + \bar{Q}_{n-2} = \bar{R}_n - 2\bar{R}_{n-1} - 2\bar{R}_{n-2} + 2\bar{R}_{n-3} - \bar{R}_{n-4}.$$

10.—The meaning of derivative.—There is a curious circumstance concerning the numbers of the form $g(E)0!$, where $g(x)$ is a polynomial.

LEMMA 3. *For any polynomial $g(x)$ we have*

$$g(E)0! - g(0) = g'(E)0!,$$

accent indicating differentiation with respect to E .

PROOF. It is sufficient to prove for $g(x) = x^n$. If $n \geq 1$, then $E^n 0! = n! = n(n-1)! = nE^{n-1}0! = (E^n)'0!$, and if $n = 0$, $E^0 0! = 1$, $(E^0)'0! = 0$.

THEOREM 5. *If the Assumption A is true for $G(x, E)$, then the numbers $N_n = Q_n(E)0!$ satisfy a recursive formula of the form*

$$N_n - (-)^n c_n = nN'_{n-1}, \\ \sum a_i N_{n-i} = \sum b_i N'_{n-i}, \quad a_0 = b_0 = 1,$$

where a_i and b_i are the same as in Theorem 4, and c_n are integers defined by $c_n = Q_n(0)$, or by

$$c_n = \alpha^n + \beta^n + \cdots + \gamma^n,$$

where $\alpha, \beta, \dots, \gamma$ are the m roots of $T(x) = (-x)^m P(-x^{-1}, 0) = 0$.

REMARK. It follows from the Theorem that N'_n are integers.

11.—Recursive formula.—We have essentially proved

THEOREM 6. *Denote by N_n the number of permutations discordant to the $k \times n$ Latin rectangle L^n , corresponding to the Touchard's problem. Denote again by N_n^* the number of permutations which can be formed within the Latin rectangle L^n . Then if*

$k \leq 3$,

$$N_n - (-)^n N_n^*$$

is always divisible by n , and the quotient, N'_{n-1} , satisfies jointly with N_n a homogeneous linear recursive formula with constant (integral) coefficients.

PROOF. We need only remind Theorem 2, which gives a combinatorial meaning to $Q_n(0)$.

More precisely: If $k = 1$, $T(x) = x - 1$, $\alpha = 1$, $N_n^* = 1$. Thus

$$(5) \quad \begin{aligned} N_n - (-)^n 1 &= n N'_{n-1}, \\ N_n &= N'_n. \end{aligned}$$

If $k = 2$, $T(x) = (x - 1)^2$, $N_n^* = 1 + 1 = 2$. Thus

$$(6) \quad \begin{aligned} N_n - (-)^n 2 &= n N'_{n-1}, \\ N_n &= N'_n - N'_{n-2}. \end{aligned}$$

If $k = 3$, $T(x) = (x - 1)^2 (x^2 - x - 1)$, $N_n = 1 + 1 + l_n$, where

$$l_n = \left(\frac{1 + \sqrt{5}}{2} \right)^n + \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

is the so-called series of Lucas (Cf. [3, p. 147]), alternately defined by $l_1 = 1$, $l_2 = 3$, $l_n = l_{n-1} + l_{n-2}$ ($n \geq 3$). Hence

$$(7) \quad \begin{aligned} N_n - (-)^n (2 + l_n) &= n N'_{n-1}, \\ N_n - N_{n-1} - N_{n-2} + N_{n-3} &= N'_n - N'_{n-1} - 4 N'_{n-2} + N'_{n-4} - N'_{n-5}. \end{aligned}$$

A slightly less complicated formula is obtained if we make use of $\bar{Q}_n(E)$.

$$(8) \quad \begin{aligned} \bar{N}_n &= N_n - (-1)^n, \\ \bar{N}_n - (-)^n (1 + l_n) &= n \bar{N}'_{n-1}, \\ \bar{N}_n - 2 \bar{N}_{n-1} + \bar{N}_{n-2} &= \bar{N}'_n - 2 \bar{N}'_{n-1} - 2 \bar{N}'_{n-2} + 2 \bar{N}'_{n-3} - \bar{N}'_{n-4}, \end{aligned}$$

or equivalently

$$(9) \quad \begin{aligned} N_n - (-)^n (2 + l_n) &= n N'_{n-1}, \\ N_n - 2 N_{n-1} + N_{n-2} &= N'_n - 2 N'_{n-1} - 2 N'_{n-2} + 2 N'_{n-3} - N'_{n-4} + (-)^n 4. \end{aligned}$$

The formula (6) is recursive formula of Cayley (Cf. [7]), and (7) and (9) are the natural generalizations.

12.—Asymptotic expansion.—We now proceed to obtain an asymptotic expansion.

THEOREM 7. *Suppose the Assumption A is true for $G_k(x, E)$. Assume moreover that the rational function*

$$\frac{U(x) - V(x)}{xV(x)} - k = Z(x)$$

has a zero-point at $x = 0$. Then we have the asymptotic expansion

$$N_n \sim e^{-k} n! \sum \frac{c_r}{(n-1)_r}$$

with constants c_r defined by

$$\exp.(-Z(x)) = \sum c_r x^r.$$

PROOF. We start from the second formula of Theorem 1:

$$N_n = f(L^n; E) S_0 = Q_n(E+k) S_0.$$

Here we intend to replace S_n by $e^{-k} n!$, i.e., to replace $E^n S_0$ by $e^{-k} E^n 0!$. Now

$$\begin{aligned} \sum Q_n(E+k) x^n &= H(P(x, E+k)) \\ &= \left(\text{const.} + H(U(x) - kxV(x)) \right) - \frac{xd}{dx} \left(\log \left(1 - \frac{ExV(x)}{U(x) - kxV(x)} \right) \right), \end{aligned}$$

where, the first term is a rational function of x , and hence its contribution to $N_n(n!)^{-1}$ is of order $(n)^{-1}$ for an arbitrary r . Neglecting this term we find

$$\begin{aligned} \sum Q_n(E+k) x^n &\sim - \frac{xd}{dx} \left(\log \left(1 - \frac{ExV(x)}{U(x) - kxV(x)} \right) \right) \\ &= \frac{xd}{dx} \left(\sum_{s=1}^{\infty} \frac{1}{s} E^s x^s \left(\frac{U(x) - kxV(x)}{V(x)} \right)^{-s} \right) \\ &= \frac{xd}{dx} \sum_s \frac{E^s x^s}{s} (1 + xZ(x))^{-s} \\ &= \frac{xd}{dx} \sum_s \frac{E^s x^s}{s} \sum_q \binom{-s}{q} x^q (Z(x))^q \\ &= \frac{xd}{dx} \sum_{s,q,r} \frac{E^s x^{s+q+r}}{s} (-)^q \frac{(s+q-1)!}{(s-1)! q!} a_{q,r} \end{aligned}$$

for the $a_{q,r}$ defined as

$$(Z(x))^q = \sum a_{q,r} x^r.$$

Comparing coefficients of x^n , we have

$$\begin{aligned} Q_n(E+k)S_0 &\sim n \sum_{s+q+r=n} (-)^q \frac{E^s}{s} \frac{(s+q-1)!}{(s-1)!q!} a_{q,r} S_0 \\ &\sim e^{-kn} \sum_{q,r} (-)^q \frac{(n-r-1)!}{r!} a_{q,r} \\ &= e^{-kn} n! \sum_r \frac{c_r}{(n-1)_r}, \end{aligned}$$

with

$$c_r = \sum_q (-)^q \frac{a_{q,r}}{r!},$$

i.e., with c_r defined by

$$\sum c_r x^r = \exp. (-Z(x)).$$

13.—Examples.—We note the conditions are satisfied for $k \leq 3$. The function $Z(x)$ is, in fact, $=0$ for $k=1$, $Z(x)=x$ for $k=2$, and $Z(x)=x(3-x)/(1-x)$ for $k=3$, as seen from $U(x)$, $V(x)$ given in §9. The asymptotic expansion is:

$$\begin{aligned} N_n &\sim e^{-1} n! \cdot 1 && \text{for } k=1, \\ N_n &\sim e^{-2} n! \sum (-)^r \frac{1}{r!(n-1)_r} && \text{for } k=2, \\ N_n &\sim e^{-3} n! \sum \frac{C_r}{r!(n-1)_r} && \text{for } k=3, \end{aligned}$$

where

$$C_0 = 1,$$

$$C_n - (2n-5)C_{n-1} + (n-1)(n-4)C_{n-2} + (n-1)(n-2)C_{n-3} = 0.$$

The beginning few coefficients are: $C_0=1$, $C_1=-3$, $C_2=5$, $C_3=-3$, $C_4=9$, $C_5=-3$, $C_6=-51$, $C_7=-675$, $C_8=-5871$, ...

14.—Explicit formula.—Denoting the m roots of the polynomial $Y(x)=x^m P(x^{-1}, E)$ by $\alpha(E)$, $\beta(E)$, ..., $\gamma(E)$ we have

$$H(P(x)) = (1 - \alpha(E)x)^{-1} + (1 - \beta(E)x)^{-1} \dots + (1 - \gamma(E)x)^{-1}.$$

On the other hand we know that $G_k(x, E)$ are of the form for $k \leq 3$. This means that the enumeration polynomial $Q_n(E)$ is given by

$$Q_n(E) = (\alpha(E))^n + (\beta(E))^n + \cdots + (\gamma(E))^n,$$

or that

$$N_n = \left((\alpha(E))^n + (\beta(E))^n + \cdots + (\gamma(E))^n \right) 0!,$$

which may be regarded as an explicit formula.

For $k=1$, we have $Y(x) = x - \Delta$, $\alpha(E) = \Delta = E - 1$, $N_n = (E - 1)^n 0!$.

For $k=2$, we have $Y(x) = x^2 - \Delta x + 1$, $Q_n(E) = (\alpha(E))^n + (\alpha(E))^{-n} = 2 \cos n\psi$, where $\alpha(E) + (\alpha(E))^{-1} = \Delta = E - 2 = \cos \psi$. Hence $Q_n(E) = 2T_n(\Delta/2)$, with a Tchebycheff polynomial. For $\varphi = \psi/2$, we have $Q_n(E) = 2 \cos 2n\varphi$, with $E - 2 = 2 \cos 2\varphi = 4 \cos^2 \varphi - 2$, or with $\varphi = \cos^{-1}(\sqrt{E}/2)$, hence $Q_n(E) = 2T_{2n}(\sqrt{E}/2)$, a result of TOUCHARD [9]. But it is queer that the more natural form $Q_n(E) = 2T_n(E/2 - 1)$ has been overlooked.

For the case $k=3$, we find $Q_n(E) = (-1)^n + (\alpha(E))^n + (\beta(E))^n + (\gamma(E))^n$ with the roots of $x^3 - (E - 2)x^2 + Ex - 1 = 0$.

15.—Conjecture for the general case.—It is highly probable that the structure polynomial generating function, and hence the enumeration polynomial generating function is an h -function in general. The defining polynomial of this generating function, however, does not seem to be linear in E , hence does not seem to satisfy the Assumption A . But it is likely that it can be decomposed into linear factors in E , satisfying the condition A , which means that N_n has an explicit formula of the form given in §13, and that N_n satisfies a recursive formula of the type developed in §11.

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Appendix. Tables for $k=3$.

n	N_n^*	N_n	N'_n
3	6	0	-2
4	9	1	3
5	13	2	0
6	20	20	25
7	31	144	152
8	49	1265	1350
9	78	12072	12644
10	125	1 26565	1 31391
11	201	14 45100	14 89568
12	324	178 75140	183 29481
13	523	2382 82730	2433 65514
14	845	34071 18041	34689 69962
15	1366	5 20345 48064	5 28480 96274
16	2209	84 55695 42593	85 70732 95427
17	3573	1457 02460 18686	1474 42896 90560
18	5780	26539 72144 35860	26820 27906 90465
19	9351	5 09585 30231 09484	5 14386 17025 23924
20	15129	102 87723 40504 93609	103 74642 26990 53582