SOME PROPERTIES OF CHARACTERS OF THE SYMMETRIC GROUP

By

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Introduction.

By a fundamental theorem in the theory of modular representations of symmetric groups, which was conjectured by Nakayama and proved by Brauer [1] and Robinson [8], it was clarified that there is a one to one correspondence between p -blocks and p -cores. It is widely known that there is a deep connection between the ordinary irreducible characters of the symmetric group and the number of partitions.

In the present paper we investigate some relations between the degrees of some kind of ordinary irreducible characters of the symmetric group S_n and the number of partitions, under the restriction $n < 2p$ where p is a prime number. We analyse the number of all ordinary irreducible characters of S_n belonging to blocks of the lowest kind, and as an immediate consequence we obtain the number of blocks of the highest kind by a theorem of Brauer [2], a result conforming of course with a special case of the explicit formula given by Oshima [7] and Nagao [6].

Our computation reveals that the number of irreducible characters of S_n whose degree f is prime to p and with the longest hook $> p$ is given by $2 \sum_s \tau(s)p(q-s)$ where $n = p + q$, $0 \leq q < p$ and $\tau(n)$ is the number of divisors of n . This is very interesting since this number is independent of the prime p , and since it contains the arithmetic function $\tau(n)$. The intrusion of the function $\tau(n)$ here suggests some inherent relationships yet undiscovered, between modular representations of S_n and the elliptic modular functions.

1.

Throughout the paper l denotes a prime number (we have replaced p of Introduction by the letter l in order to avoid connotation with $p(n)$, the number of partitions), and we suppose $n=l+q$, $0 \leq q < l$.

We denote the partition $T: n = \sum_{i=1}^t m_i$, $m_1 \geq m_2 \geq \dots \geq m_t > 0$ by $[m_1, \dots, m_t]$ and associate with T another schema $L(T) = (\lambda_1, \dots, \lambda_t)$ with $\lambda_i = m_i + t - i$ ($i=1, \dots, t$). If T corresponds with an irreducible representation Z of S_n , we denote $T = T(Z)$ and $L(T) = L(Z)$. Then the degree $f = f(T) = f(Z)$ of the representation Z is given by

$$f = \frac{n! \prod_{i > j} (\lambda_i - \lambda_j)}{\prod_i \lambda_i!}.$$

If we denote by T' the conjugate partition of T , by Z' the associate representation of Z , then $T(Z') = (T(Z))'$ and $f(T) = f(T')$.

LEMMA 1. *If $n < 2l$ and*

$$L(T) = (l+j, l+k, \dots), \quad j > k \geq 0$$

for some j, k , then

$$L(T) = (l+j, l+k, \dots, j, j-1, \dots, 1).$$

PROOF. Denote by r the number of 1's in $T = [m_1, \dots, m_t]$. Then

$$\begin{aligned} n &= m_1 + \dots + m_t = m_1 + \dots + m_{t-r} + r \\ &\geq \lambda_1 - (t-1) + \lambda_2 - (t-2) + 2(t-r-2) + r \\ &= 2l + j + k - r - 1 \\ &\geq n + k + j - r \geq n + j - r. \end{aligned}$$

This proves $r \geq j$, i.e., there are not less than j of 1's in T .

PROPOSITION 1. *The partitions T such that $l \nmid f(T)$ have one of the three types for $L(T)$:*

- (i) $L(T) = (l+j, \lambda_2, \dots)$, $l > \lambda_2$, ($j \geq 1$, j does not appear in $L(T)$).
- (i') $L(T) = (l+j, \dots, l, \dots, j, j-1, \dots, 1)$ ($j \geq 1$).
- (ii) $L(T) = (l, \dots)$.

Conversely for any partition T with $L(T)$ of the form above we have $f(T) \not\equiv 0 \pmod{l}$.

The three classes of partitions T corresponding to (i), (i'), (ii) are disjoint.

If T is from (i), then its associate partition T' is of the form (i'), and conversely.

PROOF. First of all we have $\lambda_1 \geq l$ since otherwise $l \nmid f$. In case $\lambda_1 > l$ and $l > \lambda_2$ (if any), we have $\lambda_1 = l + j$, $j \geq 1$ and j cannot appear in $L(T)$.

In case $\lambda_1 > \lambda_2 \geq l$, then $\lambda_1 = l + j$, $j \geq 1$ and all $j, j-1, \dots, 1$ appear in $L(T)$ by Lemma 1. Thus $L(T)$ must contain l .

Hence if $f(T) \not\equiv 0 \pmod{l}$, then T must have type (i), (i'), or (ii). The converse is obvious. It is also evident that the conditions (i), (i'), and (ii) are incompatible.

Now if $T = [m_1, \dots, m_t]$ is of type (i), then $m_1 - m_2 \geq j$ and, T' has at least j of 1's, i.e., $L(T') = (l + j, \dots, j, \dots)$, which implies that T' is of type (i').

If moreover $T = [m_1, \dots, m_t]$ is of type (i'), then T has at least j of 1's and $T' = [m_1', m_2', \dots]$ we have $m_1' - m_2' \geq j$, i.e., $\lambda_2 < l$ (if any), i.e., T' is of type (i).

COROLLARY. If $l \nmid f(T)$ for a self-associate partition T , then $L(T) = (l, \dots)$.

2.

We need several identities, called "q-identities" in the theory of modular functions.

Let x be a (real or complex) variable and define for $n=1, 2, \dots$ the functions

$$\begin{aligned} [n] &= 1 - x^n & (n \geq 1), \\ [0]_0 &= [n]_0 = 1, & [n]_r = [n][n-1] \cdots [n-r+1] \quad (n \geq r \geq 1), \\ [n]! &= [n]_n, & \begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]_r}{[r]!} \quad (n \geq r \geq 0). \end{aligned}$$

Moreover we put

$$[\infty]! = [1][2] \cdots, \quad \begin{bmatrix} \infty \\ r \end{bmatrix} = \frac{1}{[r]!} \quad (r \geq 0).$$

The following identities of Euler are well-known [4, p. 280].

$$(2.1) \quad (1+z)(1+xz) \cdots (1+x^{n-1}z) = \sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} x^{r(r-1)/2} z^r,$$

$$(2.2) \quad \frac{1}{(1-z)(1-xz)\cdots(1-x^n z)} = \sum_{r=0}^{\infty} \begin{bmatrix} n+r \\ r \end{bmatrix} z^r.$$

Just as with binomial coefficients we have

$$(2.3) \quad \sum_{r=1}^t \begin{bmatrix} r-1 \\ n-1 \end{bmatrix} x^r = \begin{bmatrix} t+n \\ n \end{bmatrix} x^n.$$

We single out a special case of (2.1) for a future use:

$$(2.4) \quad \sum_{r=0}^n (-1)^r \begin{bmatrix} n \\ r \end{bmatrix} x^{r(r-1)/2} = 0 \quad (n \geq 1).$$

These formulas remain valid for $n = \infty$.

Now denote by $p_r^s(n)$ the number of integer solutions $[m_1, \dots, m_r]$ of

$$m_1 + \cdots + m_r = n, \quad s \geq m_1 \geq \cdots \geq m_r \geq 0.$$

If $r \geq n$, then $p_r^s(n)$ does not depend on r and we denote this by $p^s(n)$. Similarly we define $p_r(n)$. If $r \geq n$, $s \geq n$, then $p(n)$ is the number of partitions of n , and $p(0) = 1$.

The generating function $\sum_{n=0}^{\infty} p_r^s(n) x^n$ for $p_r^s(n)$ is given by $\begin{bmatrix} s+r \\ r \end{bmatrix} = \begin{bmatrix} s+r \\ s \end{bmatrix}$, as seen from (2.2). Hence we have $p_s^r(n) = p_r^s(n)$, $p^r(n) = p_r(n)$.

The generating function $\sum_{n=0}^{\infty} p_r(n) x^n$ is therefore given by $\begin{bmatrix} \infty \\ r \end{bmatrix} = \frac{1}{[r]!}$, and the generating function $\sum_{n=0}^{\infty} p(n) x^n$ is given by $\frac{1}{[\infty]!} = \frac{1}{(1-x)(1-x^2)\cdots}$, as is known well.

Denote by $g_r(n)$ the number of partitions of n into numbers $\geq r$ ($r \geq 1$)

$$n = m_1 + \cdots + m_t, \quad m_1 \geq \cdots \geq m_t \geq r.$$

(This is the number of partitions of n containing at least r positive numbers.) Then we have obviously

$$g_r(n) = \sum_{n \geq rt} p_t(n - rt)$$

and

$$\sum_{n=0}^{\infty} g_r(n) x^n = \frac{1}{[r][r+1]\cdots} = \frac{[r-1]!}{[\infty]}.$$

We now state

PROPOSITION 2.

$$\sum_{t=0}^{\infty} (1 - (1-x^{t+1})(1-x^{t+2})\cdots) = \sum_{n=1}^{\infty} \tau(n) x^n,$$

where $\tau(n)$ is the number of divisors of n .

PROOF. We prove

$$\sum_{t=0}^{\infty} \left(1 - \frac{[\infty]!}{[t]!} \right) = \sum_{m=1}^{\infty} \frac{x^m}{[m]}$$

the latter, being Lambert's series equals to $\sum_{n=1}^{\infty} \tau(n)x_n$.

From Euler's identity (2.1) with $n=\infty$, $z=-x$ we have that

$$\frac{[\infty]!}{[t]!} = \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} \infty \\ k \end{bmatrix} x^{k(k+1)/2+tk}$$

and that

$$\begin{aligned} \sum_{t=0}^{\infty} \left(1 - \frac{[\infty]!}{[t]!} \right) &= - \sum_{k=1}^{\infty} (-1)^k \frac{x^{k(k+1)/2}}{[k]!} \sum_{t=0}^{\infty} x^{tk} \\ &= \sum_{k=1}^{\infty} (-1)^k \frac{x^{k(k-1)/2}}{[k]} \begin{bmatrix} \infty \\ k \end{bmatrix} x^k \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{k(k-1)/2}}{[k]} \sum_{m=1}^{\infty} \begin{bmatrix} m-1 \\ k-1 \end{bmatrix} x^m \quad (\text{by (2.3)}) \\ &= \sum_{m=1}^{\infty} \frac{x^m}{[m]} \sum_{k=1}^{\infty} (-1)^{k-1} \begin{bmatrix} m \\ k \end{bmatrix} x^{k(k-1)/2} \quad (\text{by (2.4)}) \\ &= \sum_{m=1}^{\infty} \frac{x^m}{[m]} \end{aligned}$$

REMARK. The meaning of this Proposition is as follows: Let

$$n = m_1 + m_2 + \cdots + m_r, \quad m_1 > m_2 > \cdots > m_r > 0$$

be any partition of n into distinct positive parts. Then the sum $\sum_r (-1)^{r-1} m_r$ extended over all these partitions is equal to $\tau(n)$. Or taking conjugate partitions: If we consider Diophantine equations

$$\xi_1 + 2\xi_2 + \cdots + \xi_v = n, \quad \xi_1 > 0, \xi_2 > 0, \dots, \xi_v > 0,$$

and denote by $s_v(n)$ the sum of values taken by ξ_v , then

$$\tau(n) = s_1(n) - s_2(n) + \cdots.$$

The latter form is attributed to Cesàro in [3, p. 306], but we added a proof for the sake of completeness.

3.

We now proceed to evaluate the number $N_1(n)$ of the partitions T with $L(T)$ of the type (i') as described in Proposition 1.

Let $T=[m_1, \dots, m_t]$, $L(T)=(\lambda_1, \dots, \lambda_t)$ and suppose that $\lambda_1=l+j$, $\lambda_{k+1}=l$, $\lambda_{t+\nu-j}=j-\nu+1$ ($\nu=1, \dots, j$). Then $k \geq 1$ and $l-t+k \geq 0$ and

$$T=[l+j-t+1, m_2, \dots, m_k, l-t+k+1, m_{k+2}, \dots, m_{t-j}, \overbrace{1, \dots, 1}^j].$$

If we put

$$\begin{aligned} m_i &= l-t+k+1+d_i & (i=2, \dots, k), \\ m_i &= 1+d'_i & (i=k+2, \dots, t-j), \\ (3.1) \quad d &= \sum_{i=2}^{k-1} d_i, \end{aligned}$$

$$(3.2) \quad d' = \sum_{i=k+2}^{t-j} d'_i,$$

we have

$$(3.3) \quad j-k \geq d_2 \geq \dots \geq d_k \geq 0,$$

$$(3.4) \quad l-t+k \geq d'_{k+2} \geq \dots \geq d'_{t-j} \geq 0,$$

and the condition is equivalent to that $L(T)$ is of type (i'). Hence $N_1(n)$ is equal to the number of solutions of (3.1), (3.2), (3.3), (3.4), summed up with respect to j, k, t, d, d' .

First consider j, k, t, d, d' are all fixed. Then the number of solutions of (3.1), (3.3) is $p_{k-1}^{j-k}(d)$, and the number of solutions of (3.2), (3.4) is $p_{t-j-k-1}^{l-t+k}(d')$ and hence

$$(3.5) \quad N_1(n) = \sum_{j,k,t,d,d'} p_{k-1}^{j-k}(d) p_{t-j-k-1}^{l-t+k}(d').$$

The summation is to be extended over all j, k, t, d, d' such that

$$k \geq 1, l-t+k \geq 0, n = \sum_{i=1}^t m_i.$$

Now we have

$$\begin{aligned} n &= \sum_{i=1}^t m_i = l+j-t+1+k(l-t+k+1)+d+t-k-1+d' \\ &= l+k(l-t+k)+j+d+d' \\ &= (k-1)(l-t+k)+2l-1+(k+j-t+1)+d+d'. \end{aligned}$$

The last relation together with $n \leq 2l-1$, $d \geq 0$ shows that $t-j-k-1 \geq d'$, and hence that

$$p_{t-j-k-1}^{l-t+k}(d') = p_{l-t+k}(d').$$

If we put $s=l-t+k$, then as is shown above

$$n=l-t+ks+d+d', \quad k \geq 1, \quad s \geq 0.$$

Since $p_{k-1}^{j-k}(d)$ in (3.5) is independent of d' and s , we have

$$\begin{aligned} \sum_{d'} p_{l-t+k}(d') &= \sum_{s \geq 0} p_s(n-l-j-d-ks) \\ &= \sum_{s \geq 0} p_s(q-j-d-ks) = g_k(q-j-d), \end{aligned}$$

and hence

$$(3.6) \quad N_1(n) = \sum_{j, k, d} g_k(q-j-d) p_{k-1}^{j-k}(d).$$

This is the coefficient of x^q in

$$\begin{aligned} & \sum_{j, k=0}^{\infty} x_j \frac{[k-1]!}{[\infty]!} \frac{[j-1]}{[k-1]} \\ &= \frac{1}{[\infty]!} \sum_{k=1}^{\infty} [k-1]! x^k \begin{bmatrix} \infty \\ k \end{bmatrix} = \frac{1}{[\infty]!} \sum_{k=1}^{\infty} \frac{x^k}{[k]} \\ &= \frac{1}{[\infty]!} \sum_{k=1}^{\infty} \tau(k) x^k \end{aligned}$$

by making use of (2.3).

We have proved

PROPOSITION 3. *The number $N_1(n)$ of the partitions T with $L(T)$ of the type (i') is given by*

$$N_1(n) = \sum_{s=1}^q \tau(s) p(q-s).$$

4.

Similarly we prove

PROPOSITION 4. *The number $N_2(n)$ of the partitions T with $L(T)$ of the type (ii) is given by*

$$N_2(n) = lp(q) - 2 \sum_{s=1}^q \tau(s) p(q-s).$$

PROOF. 1) If $T=[m_1 \dots, m_t]$, $L(T)=(\lambda_1, \dots, \lambda_t)$ with $\lambda_1=l$, then $T=[l-t+1, m_2, \dots, m_t]$. If we put $m_i=1+d_i$ ($i=2, \dots, t$) we have

$$(4.1) \quad \sum_{i=2}^t d_i = n-l = q, \quad l-t \geq d_2 \geq \dots \geq d_t \geq 0,$$

and $N_2(n)$ is equal to the number of solutions of (4.1), summed up with respect to t :

$$(4.2) \quad N_2(n) = \sum_{t=1}^l p_{t-1}^{l-t}(q).$$

2) Now we assert here that

$$(4.3) \quad p_{t-1}^{l-t}(q) = p_{t-1}(q) + p_{t-l}(q) - p(q)$$

under our assumptions (i.e., if $q < l$ and $1 \leq t \leq l$). In order to prove (4.3) consider the totality of $p(q)$ partitions of q . Any partition $T' = [m_1', \dots, m_{l-1}']$ ($m_1' \geq \dots \geq m_{l-1}' \geq 0$, $\sum m_i' = q$) of q satisfies either

$$(4.4) \quad t-1 \geq m_1'$$

or

$$(4.5) \quad m_{l-t+1}' = 0.$$

where t is a preassigned number $1 \leq t \leq l$. Indeed otherwise we would have

$$q = \sum_{i=1}^{l-1} m_i' \geq t + (l-t) = l.$$

The number of T' satisfying (4.4) or (4.5) is $p_{t-1}(q)$, or $p_{t-l}(q)$ respectively, and the number of T' satisfying both (4.4) and (4.5) is $p_{t-1}^{l-t}(q)$. Thus we have by the principle of inclusion and exclusion that

$$p(q) = p_{t-1}(q) + p_{t-l}(q) - p_{t-1}^{l-t}(q),$$

which is equivalent to (4.3).

3) Combining (4.2) and (4.3) we have

$$\begin{aligned} N_2(n) &= 2 \sum_{t=1}^l p_{t-1}(q) - lp(q) \\ &= lp(q) - 2 \sum_{t=0}^{\infty} (p(q) - p_t(q)). \end{aligned}$$

This is the coefficient of x^q in the expansion of

$$\begin{aligned} \frac{l}{[\infty]!} - 2 \sum_{t=0}^{\infty} \left(\frac{1}{[\infty]!} - \frac{1}{[t]!} \right) &= \frac{1}{[\infty]!} \left(l - 2 \sum_{t=0}^{\infty} \left(1 - \frac{[\infty]!}{[t]!} \right) \right) \\ &= \frac{l}{[\infty]!} - 2 \sum_{k=1}^{\infty} \frac{x^k}{[k]} \end{aligned}$$

by Proposition 2.

COROLLARY. *The number of all characters contained in blocks of the lowest kind in S_n , where $n = l + q$, $0 \leq q < l$ is given by $lp(q)$. The number of blocks of the highest kind is given by $p(n) - lp(q)$.*

This is of course a special case of [6] and [7].

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