NOTE ON COUSIN’S PROBLEMS

By

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1. The Mittag-Leffler’s problem, to find a meromorphic function in a domain $D$ of the complex plane $C$ with given poles at given points in $D$ which do not accumulate in $D$, is extended to several complex variables as the Cousin’s I-problem.

K. Oka proved in [8] that the Cousin’s I-problem is always solvable in a domain of holomorphy of the space $C^n$ of $n$ complex variables.

H. Cartan and J. P. Serre considered this theorem in a holomorphically complete complex manifold (i.e. Stein’s manifold) in [1] and H. Grauert in a holomorphically complete complex space in [3].

It is the purpose of the present paper, to consider Cousin’s problems with respect to meromorphic mappings in the Stoll’s sense of a complex space in a complex Lie group, and to apply H. Grauert’s results [4] and those of the author [7] to these problems.

2. By a complex space we mean the complex space defined by Grauert and Remmert [5], who proved in [6] that their complex space coincides with that of H. Cartan.

A subset $M$ of a complex space $X$ is called thin (dünn), if for each point of $M$ there exists a neighbourhood $U$ and an analytic set in $U$ which is nowhere dense in $U$ and contains $U \cap M$. A holomorphic mapping $m : A \to H$ of an open subset of $X$ in a complex space $H$ is called a mapping with thin singularity, if $M = X - A$ is thin.

A mapping $m : A \to H$ with thin singularity $M$ is called meromorphic, if for each point $P_0 \in M$ and for each one dimenisonal complex submanifold $N$ satisfying the condition $\bar{N} \cap M = N \cap M = \{P_0\}$ there exists a point $Q_0$ in $H$ such that $m(P) \to Q_0$ for $P \to P_0$ with $P \in N \cap A$.

The above definition of a meromorphic mapping is introduced by Stoll [10].

1) In [10] a meromorphic mapping is defined only in a separable complex space.
Let $X$ be a complex space and $L$ be a complex Lie group. If $m_1: U \to L$ and $m_2: U \to L$ are meromorphic mappings of an open subset $U$ of $X$ in $L$ with thin singularities $M_1$ and $M_2$, then the mappings $m_1, m_2: U\setminus (M_1 \cup M_2) \to L$ and $m^{-1}: U\setminus M_1 \to L$ with thin singularities $M_1 \cup M_2$ and $M_1$ defined by the assignment $(m_1m_2)(P)=m_1(P)m_2(P)$ for $P \in U\setminus (M_1 \cup M_2)$ and $m_1^{-1}(P)=(m_1(P))^{-1}$ for $P \in U\setminus M_1$ yield meromorphic mappings of $U$ in $L$ as we shall show it below.

Let $P_0$ be any point in $M_1 \cup M_2$ and $N$ be any one dimensional submanifold $N$ of $U$ such that $N \cap (M_1 \cup M_2) = \tilde{N} \cap (M_1 \cup M_2) = \{P_0\}$. If $P_0$ is a point in $M_1 \cap M_2$, it follows that $\tilde{N} \cap M_1 = N \cap M_1 = \{P_0\}$ and $N \cap M_2 = \tilde{N} \cap M_2 = \{P_0\}$. Therefore, as $m_1$ and $m_2$ are meromorphic mappings with thin singularities $M_1$ and $M_2$, there exist points $Q_1$ and $Q_2$ in $L$ such that $m_1(P) \to Q_1$ for $P \to P_0$ with $P \in N \cap (U-M_1)$ and $m_2(P) \to Q_1$ for $P \to P_0$ with $P \in N \cap (U-M_2)$. Hence it follows that $m_1(P)m_2(P) \to Q_1Q_2$ for $P \to P_0$ with $P \in N \cap (U-M_1 \cup M_2)$.

On the other hand, if $P_0$ is a point in $M_1-M_1 \cap M_2$, it follows that $N \cap M_1 = \tilde{N} \cap M_1 = \{P_0\}$ and $P_0 \in U$. Therefore, as $m_1$ is a meromorphic mapping with thin singularity $M_1$ there exists a point $Q_1$ in $L$ such that $m_1(P) \to Q_1$ for $P \to P_0$ with $P \in N \cap (U-M_1)$. Moreover, as $m_2$ is holomorphic in $P_0$, it follows that $m_2(P) \to m_2(P_0)$ for $P \to P_0$ with $P \in U-M_2 \cap U$. Hence it follows that $m_1(P)m_2(P) \to Q_1m_2(P_0)$ for $P \to P_0$ with $P \in N \cap (U-M_1 \cap M_2)$. Thus we have proved that $m_1m_2$ is a meromorphic mapping of $U$ in $L$. The proof that $m_1^{-1}$ is a meromorphic mapping of $U$ in $L$ is quite similar.

In this way, the set of all meromorphic mappings of an open subset $U$ of $X$ in $L$ yields a group.

3. The collection $\{(U_j, m_j); j \in J\}$ of the pairs of open sets $U_j$ and meromorphic mappings $m_j$ of $U_j$ in $L$ is called a right (or left) Cousin's distribution in $X$ with value in $L$ if the following conditions (a) and (b) are fulfilled:

(a) $\cup U_j$ for $j \in J = X$,
(b) either it holds $U_j \cap U_k = \phi$, or $m_jm_k^{-1}$ (or $m_j^{-1}m_k$) is a holomorphic mapping of $U_j \cap U_k$ in $L$ for each $j$ and $k$ in $J$.

The above consistency condition (b) does not depend on the order of $j$ and $k$ as is easily seen from the relation $(m_km_j^{-1})^{-1} = m_jm_k^{-1}$ (or $(m_j^{-1}m_k)^{-1} = m_jm_k^{-1}$).
In general a right (or left) Cousin’s distribution does not always form a left (or right) Cousin’s distribution.

The right (or left) Cousin’s problem in $X$ with value in $L$ is to find a meromorphic mapping $m$ of $X$ in $L$ such that $mm_j^{-1}$ (or $m^{-1}m_j$) is a holomorphic mapping of $U_j$ in $L$ for each $j$ in $J$ for a right (or left) Cousin’s distribution $\{(U_j, m_j); j \in J\}$.

A right Cousin’s distribution in $X$ with value in $L$ which is a left Cousin’s distribution at the same time is called a both-sided Cousin’s distribution in $X$ with value in $L$.

The both-sided Cousin’s problem in $X$ with value in $L$ is to find a meromorphic mapping $m$ of $X$ in $L$ such that $mm_j^{-1}$ and $m^{-1}m_j$ are holomorphic mapping of $U_j$ in $L$ for each $j$ in $J$ for a both-sided Cousin’s problem $\{(U_j, m_j); j \in J\}$.

If $\{(U_j, m_j); j \in J\}$ is a right (or left) Cousin’s distribution in $X$ with value in $L$, then $\{(U_j, m_j^{-1}); j \in J\}$ is a left (or right) Cousin’s distribution as is easily seen from the relation $(m_j^{-1})^{-1}(m_k^{-1}) = m_j m_k^{-1}$ (or $(m_j^{-1})(m_k^{-1})^{-1} = m_j^{-1}m_k$) in $U_j \cap U_k$ for any $j$ and $k$ in $J$ such that $U_j \cap U_k \neq \phi$. Moreover, if $m$ is a solution of the left (or right) Cousin’s problem $\{(U_j, m_j^{-1}); j \in J\}$, $m^{-1}m_j^{-1}$ (or $m(m_j^{-1})^{-1}$) is a holomorphic mapping of $U_j$ in $L$ for each $j$ in $J$.

Therefore, $m^{-1}$ is a solution of the right (or left) Cousin’s problem $\{(U_j, m_j); j \in J\}$. Thus we have shown that a right (or left) Cousin’s problem $\{(U_j, m_j); j \in J\}$ in $X$ with value in $L$ has a solution if and only if the left (or right) Cousin’s problem $\{(U_j, m_j^{-1}); j \in J\}$ has a solution.

In this way all left Cousin’s problems can be reduced to right Cousin’s problems. Hereafter we consider exclusively only right Cousin’s problems and we call a right Cousin’s problem simply by a Cousin’s problem, unless otherwise is stated.

If $\{(U_j, m_j); j \in J\}$ is a Cousin’s distribution in $X$ with value in $L$, then $g_{jk} = m_j m_k^{-1}$ is a holomorphic mapping of $U_j \cap U_k$ in $L$ for each $j$ and $k$ such that $U_j \cap U_k \neq \phi$, and satisfies the following condition:

(c) $g_{jk}g_{ji} = g_{ki}$ in $U_i \cap U_j \cap U_k$ for each $j$, $j$ and $k$ such that $U_i \cap U_j \cap U_k \neq \phi$.

Therefore as in Steenrod [9], there exists one and only one complex-analytic principal fibre-bundle $F$ with $X$ as base space, with $L$ as structure group and with $g_{jk}$'s as coordinate transformation, and is called a prin-
cipal fibre-bundle associated with the Cousin's distribution. We denote its bundle space, projection and coordinate functions by \( B, p : B \rightarrow X \) and \( \phi_i; U_i \times L \rightarrow p^{-1}(U_i) \) respectively.

If \( F \) has holomorphic cross-section \( h : X \rightarrow B \), the mapping \( h_j \) defined by the relation \( h = \phi_j(x, h_j) \) for \( x \in U_j \) and \( j \in J \) yields a well-defined, holomorphic mapping of \( U_j \) in \( L \). Since it follows \( h = \phi_j(x, h_j) = \phi_k(x, h_k) = \phi_j(x, g_{jk}h_k) \) in \( U_j \cap U_k \), \( j \) and \( k \) in \( J \) such that \( U_j \cap U_k \neq \phi \), we have \( h_j = g_{jk}h_k \) in \( U_j \cap U_k \). Hence we have \( m_j^{-1} = h_k^{-1} m_k \) in \( U_j \cap U_k \). If we put \( m = h_j^{-1} m_j \) in \( U_j \), then \( m \) yields a well-defined, meromorphic mapping of \( X \) in \( L \).

Since \( m m_j^{-1} = h_j^{-1} \) is a holomorphic mapping of \( U_j \) in \( L \), \( m \) is a solution of the Cousin's problem \( \{(U_j, m_j) ; j \in J\} \).

Conversely, suppose that the Cousin's problem has a solution \( m \). Then \( m m_j^{-1} \) is a holomorphic mapping of \( U_j \) in \( L \) for each \( j \) in \( L \). Therefore, if we define \( h_j := (m m_j^{-1})^{-1} = m_j m^{-1} \) in \( U_j \), \( h_j \) is a holomorphic mapping of \( U_j \) in \( L \). For each \( j \) and \( k \) in \( J \) such that \( U_j \cap U_k \neq \phi \) it follows that \( h_j = m_j m^{-1} = m_j m_k^{-1} m_k m^{-1} = g_{jk}h_k \) in \( U_j \cap U_k \). Hence it follows that \( \phi(x, h_j) = \phi_j(x, g_{jk}h_k) = \phi_k(x, h_k) \) in \( U_j \cap U_k \). Thus if we define \( h = \phi_j(x, h_j) \) in \( U_j \), \( h \) is a holomorphic cross-section of \( F \).

We summarize this fact in the following proposition.

**PROPOSITION 1.** A Cousin's problem in a complex space with value in a complex Lie group has a solution if and only if its associated complex-analytic principal fibre-bundle has a holomorphic cross-section.

Combining the Grauert's result [4] and Proposition 1, we have

**PROPOSITION 2.** Let \( X \) be a holomorphically complete complex space and \( L \) be a complex Lie group. A Cousin's problem in \( X \) with value in \( L \) has a solution, if and only if its associated complex-analytic principal fibre-bundle has a continuous cross-section.

Especially from the Grauert's result [4] and Proposition 1 we have

**PROPOSITION 3.** Let \( X \) be a holomorphically complete complex space contractible on itself to a point and \( L \) be a complex Lie group. Any Cousin's problem in \( X \) with value in \( L \) is always solvable.

For example, if \( D \) is a convex domain in \( C^n \), then \( D \) is a domain of holomorphy contractible on itself to a point. Since every domain of holo-

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2) For the definition cf. [3].

3) For the definition cf. [9].
morphism in $C^*$ is a holomorphically complete complex space, every Cousin’s problem in $D$ with value in a complex Lie group is always solvable by Proposition 3.

Since every complex-analytic principal fibre-bundle has a holomorphic cross section, if and only if it is analitically trivial, we have the following proposition directly from Proposition 1 and from the result of the author$^4$ [7].

**Proposition 4.** Let $D$ be a domain in a holomorphically convex complex manifold $X$ such that its pseudo-convex hull$^5$ $D^*$ over $X$ is univalent$^6$ with respect to $X$, and $L$ be complex Lie group$^6$ which is a holomorphic call complet complex space.$^5$ Then, if a Cousin’s problem in $X$ with value in $L$ is solvable in $D$, it is also solvable in $D^*$.

**References**


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$4$) Theorem 2 in [7].

$5$) For the definition, cf. [7].

$6$) In this proposition a complex Lie group $L$ needs not be a complex manifold. cf. [7].