NOTE ON COUSIN'S PROBLEMS

By

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1. The Mittag-Leffler's problem, to find a meromorphic function in a domain D of the complex plane C with given poles at given points in D which do not accumulate in D , is extended to several complex variables as the Cousin's I-problem.

K. Oka proved in [8] that the Cousin's I-problem is always solvable in a domain of holomorphy of the space C^n of n complex variables.

H. Cartan and J. P. Serre considered this theorem in a holomorphically complete complex manifold (i.e. Stein's manifold) in [1] and H. Grauert in a holomorphically complete complex space in [3].

It is the purpose of the present paper, to consider Cousin's problems with respect to meromorphic mappings in the Stoll's sense of a complex space in a complex Lie group, and to apply H. Grauert's results [4] and those of the author [7] to these problems.

2. By a complex space we mean the complex space defined by Grauert and Remmert [5], who proved in [6] that their complex space coincides with that of H. Cartan.

A subset M of a complex space X is called *thin* (*dünn*), if for each point of M there exists a neighbourhood U and an analytic set in U which is nowhere dense in U and contains $U \cap M$. A holomorphic mapping $m: A \rightarrow H$ of an open subset of X in a complex space H is called a *mapping with thin singularity*, if $M = X - A$ is thin.

A mapping $m: A \rightarrow H$ with thin singularity M is called *meromorphic*, if for each point $P_0 \in M$ and for each one dimensional complex submanifold N satisfying the condition $\bar{N} \cap M = N \cap M = \{P_0\}$ there exists a point Q_0 in H such that $m(P) \rightarrow Q_0$ for $P \rightarrow P_0$ with $P \in N \cap A$.

The above definition of a meromorphic mapping is introduced by Stoll [10]¹⁾ and is very convenient for our purpose.

1) In [10] a meromorphic mapping is defined only in a separable complex space.

Let X be a complex space and L be a complex Lie group. If $m_1: U \rightarrow L$ and $m_2: U \rightarrow L$ are meromorphic mappings of an open subset U of X in L with thin singularities M_1 and M_2 , then the mappings $m_1 m_2: U - (M_1 \cup M_2) \rightarrow L$ and $m^{-1}: U - M_1 \rightarrow L$ with thin singularities $M_1 \cup M_2$ and M_1 defined by the assignement $(m_1 m_2)(P) = m_1(P) m_2(P)$ for $P \in U - (M_1 \cup M_2)$ and $m_1^{-1}(P) = (m_1(P))^{-1}$ for $P \in U - M_1$ yield meromorphic mappings of U in L as we shall show it below

Let P_0 be any point in $M_1 \cup M_2$ and N be any one dimensional submanifold N of U such that $N \cap (M_1 \cup M_2) = \bar{N} \cap (M_1 \cup M_2) = \{P_0\}$.

If P_0 is a point in $M_1 \cap M_2$, it follows that $\bar{N} \cap M_1 = N \cap M_1 = \{P_0\}$ and $N \cap M_2 = \bar{N} \cap M_2 = \{P_0\}$. Therefore, as m_1 and m_2 are meromorphic mappings with thin singularities M_1 and M_2 , there exist points Q_1 and Q_2 in L such that $m_1(P) \rightarrow Q_1$ for $P \rightarrow P_0$ with $P \in N \cap (U - M_1)$ and $m_2(P) \rightarrow Q_2$ for $P \rightarrow P_0$ with $P \in N \cap (U - M_2)$. Hence it follows that $m_1(P) m_2(P) \rightarrow Q_1 Q_2$ for $P \rightarrow P_0$ with $P \in N \cap (U - M_1 \cup M_2)$.

On the other hand, if P_0 is a point in $M_1 - M_1 \cap M_2$, it follows that $N \cap M_1 = \bar{N} \cap M_1 = \{P_0\}$ and $P_0 \in U$. Therefore, as m_1 is a meromorphic mapping with thin singularity M_1 there exists a point Q_1 in L such that $m_1(P) \rightarrow Q_1$ for $P \rightarrow P_0$ with $P \in N \cap (U - M_1)$. Moreover, as m_2 is holomorphic in P_0 , it follows that $m_2(P) \rightarrow m_2(P_0)$ for $P \rightarrow P_0$ with $P \in U - M_2 \cap U$. Hence it follows that $m_1(P) m_2(P) \rightarrow Q_1 m_2(P_0)$ for $P \rightarrow P_0$ with $P \in N \cap (U - M_1 \cap M_2)$. Thus we have proved that $m_1 m_2$ is a meromorphic mapping of U in L . The proof that m_1^{-1} is a meromorphic mapping of U in L is quite similar.

In this way, the set of all meromorphic mappings of an open subset U of X in L yields a group.

3. The collection $\{(U_j, m_j); j \in J\}$ of the pairs of open sets U_j and meromorphic mappings m_j of U_j in L is called a *right* (or *left*) *Cousin's distribution in X with value in L* if the following conditions (a) and (b) are fulfilled:

- (a) $U_j \cup U_k$ for $j \in J = X$,
- (b) either it holds $U_j \cap U_k = \emptyset$, or $m_j m_k^{-1}$ (or $m_j^{-1} m_k$) is a holomorphic mapping of $U_j \cap U_k$ in L for each j and k in J .

The above consistency condition (b) does not depend on the order of j and k as is easily seen from the relation $(m_k m_j^{-1})^{-1} = m_j m_k^{-1}$ (or $(m_k^{-1} m_j)^{-1} = m_j m_k^{-1}$).

In general a right (or left) Cousin's distribution does not always form a left (or right) Cousin's distribution.

The *right (or left) Cousin's problem in X with value in L* is to find a meromorphic mapping m of X in L such that mm_j^{-1} (or $m^{-1}m_j$) is a holomorphic mapping of U_j in L for each j in J for a right (or left) Cousin's distribution $\{(U_j, m_j); j \in J\}$.

A right Cousin's distribution in X with value in L which is a left Cousin's distribution at the same time is called a *both-sided Cousin's distribution in X with value in L* .

The *both-sided Cousin's problem in X with value in L* is to find a meromorphic mapping m of X in L such that mm_j^{-1} and $m^{-1}m_j$ are holomorphic mapping of U_j in L for each j in J for a both-sided Cousin's problem $\{(U_j, m_j); j \in J\}$.

If $\{(U_j, m_j); j \in J\}$ is a right (or left) Cousin's distribution in X with value in L , then $\{(U_j, m_j^{-1}); j \in J\}$ is a left (or right) Cousin's distribution as is easily seen from the relation $(m_j^{-1})^{-1}(m_k^{-1}) = m_j m_k^{-1}$ (or $(m_j^{-1}(m_k^{-1})^{-1} = m_j^{-1}m_k)$ in $U_j \cap U_k$ for any j and k in J such that $U_j \cap U_k \neq \phi$. Moreover, if m is a solution of the left (or right) Cousin's problem $\{(U_j, m_j^{-1}); j \in J\}$, $m^{-1}m_j^{-1}$ (or $m(m_j^{-1})^{-1}$) is a holomorphic mapping of U_j in L for each j in J .

Therefore, m^{-1} is a solution of the right (or left) Cousin's problem $\{(U_j, m_j); j \in J\}$. Thus we have shown that a right (or left) Cousin's problem $\{(U_j, m_j); j \in J\}$ in X with value in L has a solution if and only if the left (or right) Cousin's problem $\{(U_j, m_j^{-1}); j \in J\}$ has a solution.

In this way all left Cousin's problems can be reduced to right Cousin's problems. Hereafter we consider exclusively only right Cousin's problems and we call a right Cousin's problem simply by a Cousin's problem, unless otherwise is stated.

If $\{(U_j, m_j); j \in J\}$ is a Cousin's distribution in X with value in L , then $g_{jk} = m_j m_k^{-1}$ is a holomorphic mapping of $U_j \cap U_k$ in L for each j and k such that $U_j \cap U_k \neq \phi$, and satisfies the following condition:

- (c) $g_{kj} g_{ji} = g_{ki}$ in $U_i \cap U_j \cap U_k$ for each j, j and k such that $U_i \cap U_j \cap U_k \neq \phi$.

Therefore as in Steenrod [9], there exists one and only one complex-analytic principal fibre-bundle F with X as base space, with L as structure group and with g_{jk} 's as coordinate transformation, and is called a *prin-*

principal fibre-bundle associated with the Cousin's distribution. We denote its bundle space, projection and coordinate functions by B , $p: B \rightarrow X$ and ϕ_j ; $U_j \times L \rightarrow p^{-1}(U_j)$ respectively.

If F has holomorphic cross-section $h: X \rightarrow B$, the mapping h_j defined by the relation $h = \phi_j(x, h_j)$ for $x \in U_j$ and $j \in J$ yields a well-defined, holomorphic mapping of U_j in L . Since it follows $h = \phi_j(x, h_j) = \phi_k(x, h_k) = \phi_j(x, g_{jk}h_k)$ in $U_j \cap U_k$ for any j and k in J such that $U_j \cap U_k \neq \emptyset$, we have $h_j = g_{jk}h_k$ in $U_j \cap U_k$. Hence we have $h_j^{-1}m_j = h_k^{-1}m_k$ in $U_j \cap U_k$. If we put $m = h_j^{-1}m_j$ in U_j , then m yields a well-defined, meromorphic mapping of X in L .

Since $mm_j^{-1} = h_j^{-1}$ is a holomorphic mapping of U_j in L , m is a solution of the Cousin's problem $\{(U_j, m_j); j \in J\}$.

Conversely, suppose that the Cousin's problem has a solution m . Then mm_j^{-1} is a holomorphic mapping of U_j in L for each j in J . Therefore, if we define $h_j = (mm_j^{-1})^{-1} = m_jm^{-1}$ in U_j , h_j is a holomorphic mapping of U_j in L . For each j and k in J such that $U_j \cap U_k \neq \emptyset$ it follows that $h_j = m_jm^{-1} = m_jm_k^{-1}m_km^{-1} = g_{jk}h_k$ in $U_j \cap U_k$. Hence it follows that $\phi(x, h_j) = \phi_j(x, g_{jk}h_k) = \phi_k(x, h_k)$ in $U_j \cap U_k$. Thus if we define $h = \phi_j(x, h_j)$ in U_j , h is a holomorphic cross-section of F .

We summarize this fact in the following proposition.

PROPOSITION 1. *A Cousin's problem in a complex space with value in a complex Lie group has a solution if and only if its associated complex-analytic principal fibre-bundle has a holomorphic cross-section.*

Combining the Grauert's result [4] and Proposition 1, we have

PROPOSITION 2. *Let X be a holomorphically complete complex space and L be a complex Lie group. A Cousin's problem in X with value in L has a solution, if and only if its associated complex-analytic principal fibre-bundle has a continuous cross-section.*

Especially from the Grauert's result [4] and Proposition 1 we have

PROPOSITION 3. *Let X be a holomorphically complete complex space²⁾ contractible on itself to a point³⁾ and L be a complex Lie group. Any Cousin's problem in X with value in L is always solvable.*

For example, if D is a convex domain in C^n , then D is a domain of holomorphy contractible on itself to a point. Since every domain of holo-

2) For the definition cf. [3].

3) For the definition cf. [9].

morphism in C^n is a holomorphically complete complex space, every Cousin's problem in D with value in a complex Lie group is always solvable by Proposition 3.

Since every complex-analytic principal fibre-bundle has a holomorphic cross section, if and only if it is analytically trivial, we have the following proposition directly from Proposition 1 and from the result of the author⁴⁾ [7].

PROPOSITION 4. *Let D be a domain in a holomorphically convex complex manifold X such that its pseudo-convex hull⁵⁾ D^* over X is univalent⁶⁾ with respect to X , and L be complex Lie group⁶⁾ which is a holomorphically complete complex space.²⁾ Then, if a Cousin's problem in X with value in L is solvable in D , it is also solvable in D^* .*

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4) Theorem 2 in [7].

5) For the definition, cf. [7].

6) In this proposition a complex Lie group L needs not be a complex manifold. cf. [7].