SOME RESULTS ON COUSIN—I DOMAINS OVER C^2 —II

By

Joji KAJIWARA

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In the previous paper [2], we proved that any Cousin—I Riemann domain (R, ϕ) over C^2 without isolated removable boundary points coincides with its envelope of holomorphy.

In the present paper we remark that (R, ϕ) is a domain of holomorphy of a holomorphic function which is unbounded in each boundary point of (R, ϕ) .

1. The set l_2 of all double series $\alpha = \{a_{ij}; i, j=1, 2, 3, \dots\}$ of complex numbers a_{ij} such that $\sum_{i,j=1}^{\infty} |a_{ij}|^2 < \infty$ forms a Hilbert space, when $\alpha + \beta$, $\alpha\alpha$ and $\langle \alpha, \beta \rangle$ is defined by $\alpha + \beta = \{a_{ij} + b_{ij}; i, j=1, 2, 3, \dots\}$, $\alpha\alpha = \{\alpha a_{ij}; i, j=1, 2, 3, \dots\}$ and $\langle \alpha, \beta \rangle = \sum_{i,j=1}^{\infty} a_{ij} \bar{b}_{ij}$.

Making use of this l_2 , we can prove the following Lemma.

LEMMA 1. *Let X be a separable complex space and S be an at most countable subset of X . If, for any different points x and y of S , there exists a holomorphic function f in X such that $f(x) \neq f(y)$, then there exists a holomorphic function g in X such that $g(x) \neq g(y)$ for any different points x and y of S .*

PROOF. Without loss of generality we may assume that S is countable. Therefore we may put $S = \{x_i; i=1, 2, 3, \dots, x_i \in R\}$ where $x_i \neq x_j$ for each i and j such that $i \neq j$. From the assumption of our lemma, there exists, for any different i and j , a holomorphic function f_{ij} with $f_{ij}(x_i) \neq f_{ij}(x_j)$. For convenience' sake we put $f_{ij} = f_{ji}$ and $f_{ii} = 0$.

Since X is separable, there exists an increasing sequence $K_1, K_2, K_3, \dots$ of compact sets in X such that $K_1 \subseteq K_2 \subseteq K_3 \subseteq \dots$, $X = \bigcup_{i=1}^{\infty} K_i$.

If we put $\tilde{f}(x) = \left\{ \frac{f_{ij}(x)}{2^{i+j}(1+|f_{ij}(K_{i+j})|)}; i, j=1, 2, 3, \dots \right\}$, where $|f_{ij}(K_{i+j})| = \sup_{x \in K_{i+j}} |f_{ij}(x)|$, then $\tilde{f}(x) \in l_2$ for any $x \in X$. For any $\alpha \in l_2$, the inner product

$\langle \tilde{f}(x), a \rangle = \sum_{i,j=1}^{\infty} \frac{\bar{a}_{ij} f_{ij}(x)}{2^{i+j}(1+|f_{ij}(K_{i+j})|)}$ is a holomorphic function in X .

We denote by l_{ij} ($i \neq j$) the set of all $a \in l_2$ such that $\langle \tilde{f}(x_i), a \rangle = \langle \tilde{f}(x_j), a \rangle$. Since $\tilde{f}(x_i) - \tilde{f}(x_j) \neq 0$ in l_2 as is easily seen from $f_{ij}(x_i) \neq f_{ij}(x_j)$, l_{ij} is a closed subspace of l_2 such that $l_{ij} \subsetneq l_2$. Therefore l_{ij} is nowhere dense in l_2 . Since l_2 is a set of second category, $l_2 - \bigcup_{i \neq j} l_{ij} \neq \emptyset$. Therefore there exists $a_0 \in l_2$ such that $a_0 \notin l_{ij}$ for any different i and j . If we put $g(x) = \langle \tilde{f}(x), a_0 \rangle$, $g(x)$ is a holomorphic function in X such that $g(x_i) \neq g(x_j)$ for any different i and j . Thus we have proved Lemma 1.

In the previous paper [2], we defined a Riemann domain (R, Φ) over C^2 , its boundary point and its envelope of holomorphy. They are, however, defined quite similarly in C^n .

A holomorphic function f in a Riemann domain (R, Φ) over C^n is called *unbounded* in its boundary point r , if there exists a sequence $x_1, x_2, x_3, \dots$ of points in R such that $x_n \rightarrow r$ in $\bar{R} = R \cup \partial R$ and $|f(x_n)| \rightarrow \infty$ as $n \rightarrow \infty$.

A boundary point r of a Riemann domain (R, Φ) over C^n is said to have the *frontier property*, if there exists a holomorphic function f in R which is unbounded in r .

LEMMA 2. *Let (R, Φ) be a Riemann domain over C^n , S be an at most countable subset of R and B be a subset of ∂R . If for any different points x and y of S , there exists a holomorphic function f in R such that $f(x) \neq f(y)$ and if each boundary point of (R, Φ) which belongs to B has the frontier property, then there exists a holomorphic function g in R such that $g(x) \neq g(y)$ for any different points x and y of S and that g is unbounded in each point of B .*

PROOF. Without loss of generality we may assume that S is countable and we may put $S = \{x_i; i=1, 2, 3, \dots\}$.

From Lemma 1 there exists a holomorphic function g_1 in R such that $g_1(x_i) \neq g_1(x_j)$ for any different points x_i and x_j of S . From Theorem 1 at page 84 in [1], there exists a holomorphic function g_2 in R such that g_2 is unbounded in each point of B .

Without loss of generality we may assume that B is at least countable and we may take countable subset $\{r_i; i=1, 2, 3, \dots\}$ of B which is dense in B with respect to the relative topology in B induced from the topology in $\bar{R} = R \cup \partial R$.

For any $(a, b) \in C^2$, $ag_1(x) + bg_2(x)$ is a holomorphic function in R . We denote by l_{ij} ($i \neq j$) the set of all $(a, b) \in C^2$ such that $ag_1(x_i) + bg_2(x_i) = ag_1(x_j) + bg_2(x_j)$ and by l_i the set of all $(a, b) \in C^2$ such that $ag_1(x) + bg_2(x)$ is not unbounded at the boundary point r_i of (R, Φ) .

Since $g_1(x_i) \neq g_1(x_j)$, l_{ij} is a complex 1-dimensional plane in C^2 and therefore nowhere dense in C^2 .

Obviously l_i is a linear subset of C^2 . If $(a_0, b_0) \in l_i$, then for any $b \neq b_0$, $(a_0, b) \notin l_i$ as is easily seen from the relation $a_0 g_1(x) + b g_2(x) = a_0 g_1(x) + b_0 g_2(x) + (b - b_0) g_2(x)$. If we denote by D_i the set of all a such that for some b , $(a, b) \in l_i$, then D_i is a linear subset of C .

Then only two cases occur.

1° D_i consists of only zero.

In this case l_i consists of only zero. Therefore l_i is obviously nowhere dense in C^2 .

2° D_i contains a non-zero element.

In this case D_i coincides with C , as D_i is a linear subset of C .

For any $a \in C$, there exists one and only one b such that $(a, b) \in l_i$. This correspondence is denoted by $b = Ta$. Since l_i is a linear subset of C^2 , T is represented by $Ta = (T1)a$. Therefore l_i is nowhere dense in C^2 as the graph of linear mapping T .

Thus we have proved that l_{ij} and l_i are nowhere dense in C^2 . Since C^2 is a set of second category $C^2 - \bigcup_{i \neq j} l_{ij} - \bigcup_{i=1}^{\infty} l_i \neq \emptyset$. Therefore there exists $(a, b) \in C^2$ such that $(a, b) \notin l_{ij}$ ($i \neq j$) and $(a, b) \notin l_i$.

If we put $g(x) = ag_1(x) + bg_2(x)$, then $g(x_i) \neq g(x_j)$ ($i \neq j$) and g is unbounded in r_i for any i . Since $\{r_i; i=1, 2, 3, \dots\}$ is dense in B , g is also unbounded in each point of B . Thus Lemma 2 is proved.

2. A holomorphic function f' in a Riemann domain (R', Φ') over C^n is called a *prolongation* of a holomorphic function f in a Riemann domain (R, Φ) over C^n , if there exists a holomorphic mapping $\psi: R \rightarrow R'$ such that $\Phi = \Phi' \circ \psi$ and $f = f' \circ \psi$ in R . (R', Φ') is also called a *domain of prolongation* of f .

A Riemann domain (R, Φ) over C^n is called a *domain of holomorphy* of a holomorphic function f in R when the following condition is satisfied:

If a holomorphic function f' in (R', Φ') over C^n is a prolongation of f and $\psi: R \rightarrow R'$ is a holomorphic mapping such that $\Phi = \Phi' \circ \psi$ and $f = f' \circ \psi$ in R , then ψ is a homeomorphism of R onto R' .

A Riemann domain (R, Φ) over C^n is called a *domain of holomorphy*, if it is a domain of holomorphy of some holomorphic function in R .

LEMMA 3. *Let (R, Φ) be a Riemann domain over C^n which coincides with its envelope of holomorphy and all of whose boundary points have the frontier property.*

Then (R, Φ) is a domain of holomorphy of a holomorphic function in R which is unbounded in each boundary point of R .

PROOF. Since (R, Φ) coincide with its envelope of holomorphy, there exists a countable subset S of R such that S is dense in R and that, for any different points x and y of S , there exists a holomorphic function f in R such that $f(x) \neq f(y)$.

From Lemma 2 there exists a holomorphic function g in R such that $g(x) \neq g(y)$ for any different points x and y of S and that g is unbounded in each boundary point of R .

We now prove that (R, Φ) is a domain of holomorphy of g . Let a holomorphic function g' in (R', Φ') over C^n is a prolongation of g and $\psi: R \rightarrow R'$ is a holomorphic mapping such that $\Phi = \Phi' \circ \psi$ and $g = g' \circ \psi$ in R .

Since g is unbounded in each boundary point of R , ψ is a holomorphic mapping of R onto R' .

Since $g(x) \neq g(y)$ for any $x \neq y$ of S , it holds that $\psi(x) \neq \psi(y)$ for any different points x and y of S . As S is dense in R , ψ is an injective mapping. Thus Lemma 3 is proved.

If (R, Φ) is a Cousin-I Riemann domain over C^2 without isolated removable boundary points, we have proved in the proof of Proposition 1 of the previous paper [2] that each boundary point of (R, Φ) has the frontier property. Therefore the following proposition follows immediately from Proposition 1 of [2].

PROPOSITION. *Any Cousin-I Riemann domain (R, Φ) over C^2 without isolated removable boundary points is a domain of holomorphy of a holomorphic function in R which is unbounded in each boundary point of (R, Φ) .*

References.

- [1] S. BOCHNER and W.T. MARTIN, *Several complex variables*, Princeton Univ. Press (1948).
- [2] J. KAJIWARA, *Some results on Cousin—I domains over C^2* , Mem. of the Fac. of Sci., Kyushu Univ. Ser. A, 14 (1960), 122-133.