THEORY OF FINITE AUTOMATA

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Contents

§1.	Introduction.
§2.	Definitions in finite automata.
2. 1.	Serial connection. The $*$ -product automaton.
2. 2.	Parallel connection. The $\#$ -product automaton.
2. 3.	The $\wedge$ -product automaton.
2. 4.	The j -th-face automaton.
2. 5.	Finite automaton and its transition diagram.
2. 6.	Examples of some finite automata and face automata.
2. 7.	Representation of a finite automaton.
2. 8.	Addition of finite automata in $\mathfrak{M}_{n,m,p}[X,Y,Z]$.
2. 9.	Definition of the 'inverse' finite automaton.
2.10.	The direct sum of finite automata.
2.11.	The isomorphic finite automata.
§3.	Strongly connected finite automata.
3. 1.	The $*$ -algorithm.
3. 2.	Strong connectedness of a finite automaton and that of its face automaton.
3. 3.	Strong connectedness of the $\#$ -product automaton.
§4.	Indistinguishability and reducibility of finite automata.
§5.	Reversible finite automata and their direct sum decomposition.
§6.	Reduction theorems on finite automata.
§7.	The existence of Scheffer stroke class in the finite automata.
§8.	Finite automata of convolution type.
§9.	Finite automata and Asser computer's programming.
	Acknowledgement
	References

§1. Introduction.

The object of this paper is to develop the theory of finite automata more mathematically than ever. Surely finite automata, investigated by

several mathematical logicians and applied mathematicians, are mathematical systems worth of studying, but, as far as we know, a proper position has not been given to them in the field of applied mathematics, the reason of which, it seems, lies in the lack of refined mathematical theory of them. Yet the tendency toward constructing firm bases of mathematical theory of finite automata may be seen in some groups of applied mathematicians. The first investigator, who regards finite automata as mathematical objects and give them firm bases of study, may be Dr. S. Ginsburg who used freely the theory of semi-group, reformulated the results of Dr's E. F. Moore, G. H. Mealy and D. A. Huffman in some points and extended the applicabilities of the theory of finite automata more efficiently than before. ([6], [7], [8]). For example he solved a problem of interest to switching circuit designers who had been trying to find finite automata with the smallest number of states. His theory itself was complete and beautiful. It seems, however, that the definitions of the input and the output are unnatural in some sense. Properly speaking, the input in his definition is nothing but the input sequence in E. F. Moore's sense, and the same is true about his output. Such definitions have mathematical beauty, but in the view point of real application they seem unnatural in contrast with E. F. Moore's definition. Therefore a theory which is based on E. F. Moore's definition of sequential machines, i. e., finite automata may be desired.

The second investigator who regards finite automata as mathematical objects may be Dr. C. Y. Lee, who pursued the relation between finite automata and his W-machine's programs and showed a program analog to finite automaton. His theory, which is one of a number of approaches to problems on finite automata, not only resembles strongly computer programming, but also offers possibilities of symbol operation and other combinatorial programs, all based on a very simple and elegant program structure ([17]). The theory may be developed to the one using Asser's computer programming.

In this paper we shall show mathematical theory of complete, discrete, synchronous, fixed and deterministic automata, i. e., finite automata. Some authors omit the word 'complete' in the definition of finite automata, but for simplicity we restrict to 'complete' finite automata, not 'partial' finite automata. In section 2 we give the definition of finite automaton and some combinations of finite automata. The definitions which will be used in the

following sections are also introduced in the same section. The properties of strongly connected finite automata are discussed in section 3, and the distinguishabilities of states and its characteristics are defined and discussed in section 4. In the next section we show the properties of finite automata of various types e. g., that of reversible finite automata. It is interesting to study the properties of reduced automaton of a given finite automaton which is shown in section 6. In section 7 we shall define the Scheffer stroke class in the set of (n, m, p) finite automata and show that any finite automaton in the given set of finite automata can be constructed by combining some finite automata that belong to the class. Finite automata of convolution types are discussed in section 8. In the last section we show the relationship between logical designs of finite automata and Asser's computer programmings. It is shown that the logical structure of a given finite automata can be regarded as a special case of normal Markow algorithm in a sense.

§2. Definitions in finite automata.

Let X , Y and Z be three (finite) sets and X^* , Y^* and Z^* be the set of all finite sequences of elements in X , Y and Z , respectively. Let $\#(X)=n$, $\#(Y)=m$ and $\#(Z)=p$, where for example $\#(X)$ means the number of elements in X . A finite automaton $M_{n,m,p}[X,Y,Z]$ is a system of sets $\{X, Y, Z\}$, X , Y , Z being called the set of states, the set of inputs and the set of outputs, respectively, together with three functions defined in the following. The first function, called state transition, assigns to each pair of elements $x (\in X)$, $y (\in Y)$ a state $T_M(x, y)$, called the next state of x transited by y . The second function, called output function, assigns to each state x an output $U_M(x)$. And the third function, called sequential function, assigns to each state x and each finite sequence of inputs $y^* = \{y_{\alpha_1}, y_{\alpha_2}, \dots, y_{\alpha_l}\} \in Y^*$, $\alpha_i, i = 1, 2, \dots, l$, being integers such as $y_{\alpha_i} \in Y$, a finite sequence of outputs $H_M(x, y^*)$. The three functions are required to have the following two properties:

1. Two functions $T_M: X \times Y \rightarrow X$ and $U_M: X \rightarrow Z$ can be chosen arbitrarily and once chosen they are fixed.
2. The function $H_M: X \times Y^* \rightarrow Z^*$ is defined as follows: $H_M(x, y^*) = \{U_M(x), U_M(T_M(x, y_{\alpha_1})), U_M(T_M(T_M(x, y_{\alpha_1}), y_{\alpha_2})), \dots, U_M(T_M(T_M(\dots(T_M(x, y_{\alpha_1}), y_{\alpha_2}), \dots), y_{\alpha_{l-1}})))$ for each $x \in X$ and $y^* = \{y_{\alpha_1}, y_{\alpha_2}, \dots, y_{\alpha_l}\} \in Y^*$ ($y_{\alpha_i} \in Y, i = 1, 2, \dots, l, l \geq 1$).

Let us fix the sets X , Y and Z . The set of finite automata $\mathfrak{M}_{n,m,p}[X,Y,Z]$ is defined to be the set of all finite automata obtained by choosing T_M and U_M arbitrarily.

$$\begin{array}{ccccccc}
 x & \xrightarrow{y_{\alpha_1}} & T_M(x, y_{\alpha_1}) & \xrightarrow{y_{\alpha_2}} & T_M(T_M(x, y_{\alpha_1}), y_{\alpha_2}) & \xrightarrow{y_{\alpha_3}} & \cdots \\
 \downarrow & & \downarrow & & \downarrow & & \\
 U_M(x) & & U_M(T_M(x, y_{\alpha_1})) & & U_M(T_M(T_M(x, y_{\alpha_1}), y_{\alpha_2})) & & \cdots \\
 & \xrightarrow{y_{\alpha_{l-1}}} & T_M(T_M(\cdots(T_M(T_M(x, y_{\alpha_1}), y_{\alpha_2}), \cdots, y_{\alpha_{l-2}}), y_{\alpha_{l-1}})) & & & & \\
 & & \downarrow & \xrightarrow{y_{\alpha_l}} & T_M(T_M(\cdots(T_M(T_M(x, y_{\alpha_1}), y_{\alpha_2}), \cdots), y_{\alpha_l})) & & \\
 & & U_M(T_M(T_M(\cdots(T_M(T_M(x, y_{\alpha_1}), y_{\alpha_2}), \cdots, y_{\alpha_{l-2}}), y_{\alpha_{l-1}}))) & & & &
 \end{array}$$

y : input,
 x : present state,
 $T_M(x, y)$: next state of x transited by y ,
 $U_M(x)$: output.

Fig. 1.

Let $N_p[q, r]$ be the set of integers from decimal number q to decimal number r represented by p -ary number system. Now we consider three bijections $\lambda_X: X \rightarrow N_{10}[0, n-1]$, $\lambda_Y: Y \rightarrow N_{10}[0, m-1]$ and $\lambda_Z: Z \rightarrow N_{10}[0, p-1]$. Then λ_X^{-1} , λ_Y^{-1} and λ_Z^{-1} exist and are also bijections. Let $x_i = \lambda_X^{-1}(i)$, $y_j = \lambda_Y^{-1}(j)$ and $z_k = \lambda_Z^{-1}(k)$ for $i \in N_{10}[0, n-1]$, $j \in N_{10}[0, m-1]$ and $k \in N_{10}[0, p-1]$. When several finite automata are given we can consider some combinations of them. The composite finite automata do not necessarily belong to the original set of finite automata.

2.1. Serial connection. The *-product automaton.

First under the assumption $Z_1 \subset Y_2$ we shall consider a transformation $*$ from $\mathfrak{M}_{n_1, m_1, p_1}[X_1, Y_1, Z_1] \times \mathfrak{M}_{n_2, m_2, p_2}[X_2, Y_2, Z_2]$ into $\mathfrak{M}_{n_1 n_2, m_1 p_2}[X_1 \times X_2, Y_1, Z_2]$, the image of which is denoted by $M_{n_1, m_1, p_1}^{(1)} * M_{n_2, m_2, p_2}^{(2)}$ for $M_{n_1, m_1, p_1}^{(1)} \in \mathfrak{M}_{n_1, m_1, p_1}$ and $M_{n_2, m_2, p_2}^{(2)} \in \mathfrak{M}_{n_2, m_2, p_2}$, and is defined as a finite automaton having the following state transition and output function:

$$\begin{aligned}
T_{M^{(1)} * M^{(2)}}(x, y) &= (T_{M^{(1)}}(x^1, y), T_{M^{(2)}}(x^2, U_{M^{(1)}}(x^1))) \\
&\quad \text{for } x = (x^1, x^2) \in X_1 \times X_2 \text{ and } y \in Y_1, \\
U_{M^{(1)} * M^{(2)}}(x) &= U_{M^{(2)}}(x^2) \quad \text{for } x = (x^1, x^2) \in X_1 \times X_2,
\end{aligned}$$

the corresponding sequential function $H_{M^{(1)} * M^{(2)}}$ being determined from them. We shall call $M_{n_1, m_1, p_1}^{(1)} * M_{n_2, m_2, p_2}^{(2)}$ the $*$ -product automaton of $M_{n_1, m_1, p_1}^{(1)}$ and $M_{n_2, m_2, p_2}^{(2)}$. As will be easily seen, it might be called a serial connection of two automata analogous to the one in network theory (See Fig. 3).

Fig. 3. $M_{n_1, m_1, p_1}^{(1)} * M_{n_2, m_2, p_2}^{(2)}$.

PROPOSITION 2.1.1. Let $M_{n_i, m_i, p_i}^{(i)}[X_i, Y_i, Z_i] \in \mathfrak{M}_{n_i, m_i, p_i}[X_i, Y_i, Z_i]$ ($i=1, 2, 3$). Assume $Z_i \subset Y_{i+1}$ ($i=1, 2$). Then

$$(M_{n_1, m_1, p_1}^{(1)} * M_{n_2, m_2, p_2}^{(2)}) * M_{n_3, m_3, p_3}^{(3)} = M_{n_1, m_1, p_1}^{(1)} * (M_{n_2, m_2, p_2}^{(2)} * M_{n_3, m_3, p_3}^{(3)}).$$

PROOF. By definition of the function $T_{(M^{(1)} * M^{(2)}) * M^{(3)}}: X_1 \times X_2 \times X_3 \times Y_1 \rightarrow X_1 \times X_2 \times X_3$, we have

$$T_{(M^{(1)} * M^{(2)}) * M^{(3)}}(x, y) = (T_{M^{(1)} * M^{(2)}}(x', y), T_{M^{(3)}}(x^3, U_{M^{(1)} * M^{(2)}}(x'))),$$

where $x' = (x^1, x^2)$, $x = (x^1, x^2, x^3)$, $x^1 \in X_1$, $x^2 \in X_2$, $x^3 \in X_3$ and $y \in Y_1$, which can be rewritten as

$$(T_{M^{(1)}}(x^1, y), T_{M^{(2)}}(x^2, U_{M^{(1)}}(x^1)), T_{M^{(3)}}(x^3, U_{M^{(2)}}(x^2))).$$

Also for the function $U_{(M^{(1)} * M^{(2)}) * M^{(3)}}$, we have

$$U_{(M^{(1)} * M^{(2)}) * M^{(3)}}(x) = U_{M^{(3)}}(x).$$

On the other hand since $M^{(2)} * M^{(3)} \in \mathfrak{M}_{n_2 * n_3, m_2, p_3}[X_2 \times X_3, Y_2, Z_3]$, we have for the function $T_{M^{(1)} * (M^{(2)} * M^{(3)})}: X_1 \times X_2 \times X_3 \times Y_1 \rightarrow X_1 \times X_2 \times X_3$

$$T_{M^{(1)} * (M^{(2)} * M^{(3)})}(x, y) = (T_{M^{(1)}}(x^1, y), T_{M^{(2)} * M^{(3)}}(x'', U_{M^{(1)}}(x^1)))$$

where $x'' = (x^2, x^3)$. This can be rewritten by the relation $T_{M^{(2)} * M^{(3)}}(x'', U_{M^{(1)}}(x^1)) = (T_{M^{(2)}}(x^2, U_{M^{(1)}}(x^1)), T_{M^{(3)}}(x^3, U_{M^{(2)}}(x^2)))$ as follows:

$$T_{M^{(1)} * (M^{(2)} * M^{(3)})}(x, y) = (T_{M^{(1)}}(x^1, y), T_{M^{(2)}}(x^2, U_{M^{(1)}}(x^1)), T_{M^{(3)}}(x^3, U_{M^{(2)}}(x^2))).$$

Also we have

$$U_{M^{(1)} * (M^{(2)} * M^{(3)})}(x) = U_{M^{(2)} * M^{(3)}}(x'') = U_{M^{(3)}}(x^3).$$

Thus it is shown that $T_{(M^{(1)} * M^{(2)}) * M^{(3)}} = T_{M^{(1)} * (M^{(2)} * M^{(3)})}$ and $U_{(M^{(1)} * M^{(2)}) * M^{(3)}} = U_{M^{(1)} * (M^{(2)} * M^{(3)})}$. Hence the conclusion.

From this proposition we may omit the parentheses in writing the $*$ -product of three automata.

Similarly the $*$ -product automaton of a finite number of finite automata $M_{n_1, m_1, p_1}^{(1)} * M_{n_2, m_2, p_2}^{(2)} * \cdots * M_{n_l, m_l, p_l}^{(l)}$, where $M_{n_i, m_i, p_i}^{(i)} \in \mathfrak{M}_{n_i, m_i, p_i}[X_i, Y_i, Z_i]$ ($i=1, 2, \dots, l$) and $Z_i \subset Y_{i+1}$ ($i=1, 2, \dots, l-1$) hold, is defined and belongs to the set $\mathfrak{M}_{\prod_{i=1}^l n_i, m_1, p_l}$ $[X_1 \times X_2 \times \cdots \times X_l, Y_1, Z_l]$.

2.2. Parallel connection. The $\#$ -product automaton.

Secondly we shall consider a transformation $\#$ from $\mathfrak{M}_{n_1, m_1, p_1}^{(1)}[X_1, Y_1, Z_1] \times \mathfrak{M}_{n_2, m_2, p_2}^{(2)}[X_2, Y_2, Z_2]$ into $\mathfrak{M}_{n_1 n_2, m_1 m_2, p_1 p_2}[X_1 \times X_2, Y_1 \times Y_2, Z_1 \times Z_2]$, the image of which is denoted by $M_{n_1, m_1, p_1}^{(1)} \# M_{n_2, m_2, p_2}^{(2)}$ for $M_{n_1, m_1, p_1}^{(1)} \in \mathfrak{M}_{n_1, m_1, p_1}^{(1)}$ and $M_{n_2, m_2, p_2}^{(2)} \in \mathfrak{M}_{n_2, m_2, p_2}^{(2)}$, and is defined as a finite automaton having the following state transition and output function:

$$\begin{aligned} T_{M^{(1)} \# M^{(2)}}(x, y) &= (T_{M^{(1)}}(x^1, y^1), T_{M^{(2)}}(x^2, y^2)), \\ U_{M^{(1)} \# M^{(2)}}(x) &= (U_{M^{(1)}}(x^1), U_{M^{(2)}}(x^2)), \end{aligned}$$

where $x = (x^1, x^2) \in X_1 \times X_2$ and $y = (y^1, y^2) \in Y_1 \times Y_2$.

The functions $T_{M^{(1)} \# M^{(2)}}: X_1 \times X_2 \times Y_1 \times Y_2 \rightarrow X_1 \times X_2$ and $U_{M^{(1)} \# M^{(2)}}: X_1 \times X_2 \rightarrow Z_1 \times Z_2$ are defined unambiguously and so is the function $H_{M^{(1)} \# M^{(2)}}$. The

Fig 4. $M_{n_1, m_1, p_1}^{(1)} \# M_{n_2, m_2, p_2}^{(2)}$.

set $\{X_1 \times X_2, Y_1 \times Y_2, Z_1 \times Z_2\}$ together with the three functions $T_{M^{(1)} \# M^{(2)}}$, $U_{M^{(1)} \# M^{(2)}}$ and $H_{M^{(1)} \# M^{(2)}}$ constitutes the finite automaton $M_{n_1, m_1, p_1}^{(1)} \# M_{n_2, m_2, p_2}^{(2)}$, called the $\#$ -product automaton with $M_{n_1, m_1, p_1}^{(1)}$ and $M_{n_2, m_2, p_2}^{(2)}$. It might be called a parallel connection of two automata analogous to the one in network theory (See Fig. 4).

PROPOSITION 2.2.1. Let $M_{n_i, m_i, p_i}^{(i)} \in \mathfrak{M}_{n_i, m_i, p_i}^{(i)} [X_i, Y_i, Z_i]$ ($i=1, 2, 3$).

Then we have

$$(M_{n_1, m_1, p_1}^{(1)} \# M_{n_2, m_2, p_2}^{(2)}) \# M_{n_3, m_3, p_3}^{(3)} = M_{n_1, m_1, p_1}^{(1)} \# (M_{n_2, m_2, p_2}^{(2)} \# M_{n_3, m_3, p_3}^{(3)}).$$

From this proposition we may omit the parentheses in the above product.

Similarly the $\#$ -product automaton of a finite number of finite automata $M_{n_1, m_1, p_1}^{(1)} \# M_{n_2, m_2, p_2}^{(2)} \# \dots \# M_{n_l, m_l, p_l}^{(l)}$, where $M_{n_i, m_i, p_i}^{(i)} \in \mathfrak{M}_{n_i, m_i, p_i}^{(i)} [X_i, Y_i, Z_i]$, is defined and belongs to the set of finite automata $\mathfrak{M}_{\prod_{i=1}^l n_i, \prod_{i=1}^l m_i, \prod_{i=1}^l p_i} [X_1 \times X_2 \times \dots \times X_l, Y_1 \times Y_2 \times \dots \times Y_l, Z_1 \times Z_2 \times \dots \times Z_l]$.

2.3. The $\wedge$ -product automaton.

Let $M_{n, m, p}^{(1)}$ and $M_{n, m, p}^{(2)}$ belong to the same set $\mathfrak{M}_{n, m, p} [X, Y, Z]$. A finite automaton constructed by the following two functions $T_{M^{(1)} \wedge M^{(2)}}: X \times Y \rightarrow X$ and $U_{M^{(1)} \wedge M^{(2)}}: X \rightarrow Z$ is called the $\wedge$ -product automaton, denoted by $M_{n, m, p}^{(1)} \wedge M_{n, m, p}^{(2)}$:

$$\begin{aligned} T_{M^{(1)} \wedge M^{(2)}}(x, y) &= T_{M^{(1)}}(T_{M^{(2)}}(x, y), y), \\ U_{M^{(1)} \wedge M^{(2)}}(x) &= U_{M^{(1)}}(x) \end{aligned}$$

for any $x \in X$ and any $y \in Y$.

The function $H_{M^{(1)} \wedge M^{(2)}}: X \times Y^* \rightarrow Z^*$ is defined unambiguously. The automaton $M_{n, m, p}^{(1)} \wedge M_{n, m, p}^{(2)}$ belongs to the original set $\mathfrak{M}_{n, m, p} [X, Y, Z]$. Let $M_{n, m, p}^{(i)} \in \mathfrak{M}_{n, m, p} [X, Y, Z]$ ($i=1, 2, 3$). By the above definition of $\wedge$ -product automaton we have

$$\begin{aligned} T_{(M^{(1)} \wedge M^{(2)}) \wedge M^{(3)}}(x, y) &= T_{(M^{(1)} \wedge M^{(2)})}(T_{M^{(3)}}(x, y), y) \\ &= T_{M^{(1)}}(T_{M^{(2)}}(T_{M^{(3)}}(x, y), y), y) \end{aligned}$$

and

$$U_{(M^{(1)} \wedge M^{(2)}) \wedge M^{(3)}}(x) = U_{(M^{(1)} \wedge M^{(2)})}(x) = U_{M^{(1)}}(x)$$

for any $x \in X$ and any $y \in Y$.

And, further, we obtain for any $x \in X$ and $y \in Y$

$$\begin{aligned} T_{M^{(1)} \wedge M^{(2)} \wedge M^{(3)}}(x, y) &= T_{M^{(1)}}(T_{M^{(2)} \wedge M^{(3)}}(x, y), y) \\ &= T_{M^{(1)}}(T_{M^{(2)}}(T_{M^{(3)}}(x, y), y), y) \end{aligned}$$

and

$$U_{M^{(1)} \wedge M^{(2)} \wedge M^{(3)}}(x) = U_{M^{(1)}}(x).$$

Hence we have

PROPOSITION 2.3. *The operation $\wedge$ is associative.*

Note that the operation is not commutative.

2.4. The j -th-face automaton.

Let $X^{(r)} \equiv X - \bigcup_{k=n-r}^{n-1} \{x; \lambda_X(x)=k\}$ which is an r -prefix of the set X . Then we define the j -th-face operator $\partial_j: \mathfrak{M}_{n,m,p}[X, Y, Z] \rightarrow \mathfrak{M}_{n-1,m,p}[X^{(1)}, Y, Z]$ ($j=0, 1, 2, \dots, n-1$) in the following way. For any finite automaton $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$, the image $\partial_j M_{n,m,p}$ shall have the following state transition and output function: for any $x \in X^{(1)}$ and any $y \in Y$,

$$\begin{aligned} T_{\partial_j M}(x, y) &= \begin{cases} T_M(x, y) & (\text{if } \lambda_X(x) \leq j-1 \text{ and } \lambda_X(T_M(x, y)) \leq j-1), \\ x & (\text{if } \lambda_X(x) \leq j-1 \text{ and } \lambda_X(T_M(x, y)) = j), \\ x' & (\text{if } \lambda_X(x) \leq j-1 \text{ and } \lambda_X(T_M(x, y)) > j. \\ & (\text{Here } \lambda_X(x') = \lambda_X(T_M(x, y)) - 1.)), \\ T_M(x'', y) & (\text{if } \lambda_X(x) \geq j \text{ and } \lambda_X(T_M(x'', y)) \leq j-1. \\ & (\text{Here } \lambda_X(x'') = \lambda_X(x) + 1.)), \\ x & (\text{if } \lambda_X(x) \geq j \text{ and } \lambda_X(T_M(x'', y)) = j. \\ & (\text{Here } \lambda_X(x'') = \lambda_X(x) + 1.)), \\ x'' & (\text{if } \lambda_X(x) \geq j \text{ and } \lambda_X(T_M(x'', y)) > j. \\ & (\text{Here } \lambda_X(x'') = \lambda_X(x) + 1 \text{ and} \\ & \lambda_X(x''') = \lambda_X(T_M(x'', y)) - 1.)), \end{cases} \\ U_{\partial_j M}(x) &= \begin{cases} U_M(x) & (\text{if } \lambda_X(x) \leq j-1), \\ U_M(x'') & (\text{if } \lambda_X(x) \geq j. \\ & (\text{Here } \lambda_X(x'') = \lambda_X(x) + 1.)). \end{cases} \end{aligned}$$

The sequential function $H_{\partial_j M}: X^{(1)} \times Y^* \rightarrow Z^*$ is defined unambiguously by the above two functions $T_{\partial_j M}: X^{(1)} \times Y \rightarrow X^{(1)}$ and $U_{\partial_j M}: X^{(1)} \rightarrow Z$. The set $\{X^{(1)}, Y, Z\}$, together with the three functions $T_{\partial_j M}$, $U_{\partial_j M}$ and $H_{\partial_j M}$ is called the j -th-face automaton of a finite automaton $M_{n,m,p} (\in \mathfrak{M}_{n,m,p}[X, Y, Z])$, denoted by $\partial_j M_{n,m,p}$ ($j=0, 1, \dots, n-1$).

2.5. Finite automaton and its transition diagram.

Let $M_{n,m,p}[X,Y,Z]$ be a finite automaton constructed by the three functions $T_M: X \times Y \rightarrow X$, $U_M: X \rightarrow Z$ and $H_M: X \times Y^* \rightarrow Z^*$, which belongs to $\mathfrak{M}_{n,m,p}[X,Y,Z]$. From two given functions T_M and U_M we can construct the following $(m+1) \times n$ -matrix R_M , called the transition matrix of a finite automaton $M_{n,m,p}$:

$$R_M = \begin{pmatrix} x_{0,0} & x_{0,1} & \cdots & x_{0,n-1} \\ x_{1,0} & x_{1,1} & \cdots & x_{1,n-1} \\ \dots & \dots & \dots & \dots \\ x_{m-1,0} & x_{m-1,1} & \cdots & x_{m-1,n-1} \\ z_{\mu_0} & z_{\mu_1} & \cdots & z_{\mu_{n-1}} \end{pmatrix},$$

where $x_{i,j} = T_M(x, y)$ ($\lambda_X(x) = j$, $\lambda_Y(y) = i$; $i = 0, 1, \dots, m-1$; $j = 0, 1, \dots, n-1$) and $z_{\mu_i} = U_M(x)$ ($\lambda_X(x) = i$, $\mu_i \in N_{10}[0, p-1]$; $i = 0, 1, \dots, n-1$).

Conversely, when the sets X , Y and Z are fixed, any element of $\mathfrak{M}_{n,m,p}[X,Y,Z]$ may be determined by giving a suitable transition matrix.

The above finite automaton $M_{n,m,p}$ can also be diagrammed as shown below. A symbol $\overline{x/U(x)}$ is drawn corresponding to each state x ($\in X$). Then n symbols are obtained. And if there exists the relation $T_M(x, y) = x'$, two symbols $\overline{x/U(x)}$ and $\overline{x'/U(x')}$ are connected by an arrow which is directed from $\overline{x/U(x)}$ to $\overline{x'/U(x')}$, as

$$\overline{x/U(x)} \xrightarrow{y} \overline{x'/U(x')},$$

the y ($\in Y$) by the arrow shows the input yielding the transition. Referring to the matrix R_M , we can write all arrows among n symbols. The diagram completed is called the transition diagram of a finite automaton $M_{n,m,p}$. The following diagram is an example of the transition diagram of a finite automaton in $\mathfrak{M}_{4,2,2}[X,Y,Z]$, where $X \equiv \{x_0, x_1, x_2, x_3\}$, $Y \equiv \{y_0, y_1\}$ and $Z \equiv \{z_0, z_1\}$.

$$R \equiv \begin{pmatrix} x_3 & x_0 & x_3 & x_1 \\ x_2 & x_2 & x_3 & x_1 \\ z_0 & z_0 & z_0 & z_1 \end{pmatrix}.$$

REMARK. Since the sets X , Y and Z are fixed, we may simply represent R as

$$\tilde{R} \equiv \begin{pmatrix} 3 & 0 & 3 & 1 \\ 2 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We shall use this notation freely in the following.

2.6. Examples of some finite automata and face automata.

About some transition diagrams we shall show the composite finite automata and their face automata. Let $M_{4,2,2}^{(1)}$ and $M_{3,2,2}^{(2)}$ be two finite automata defined by the following diagrams:

$$M_{4,2,2}^{(1)}: X_1 = \{x_0^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}\}, \quad Y_1 = \{y_0^{(1)}, y_1^{(1)}\}, \quad Z_1 = \{z_0^{(1)}, z_1^{(1)}\}.$$

$$M_{3,2,2}^{(2)}: X_2 = \{x_0^{(2)}, x_1^{(2)}, x_2^{(2)}\}, \quad Y_2 = \{y_0^{(2)}, y_1^{(2)}\}, \quad Z_2 = \{z_0^{(2)}, z_1^{(2)}\}.$$

1°. Under the assumption $Z_1 = Y_2$ (i. e. let $z_0^{(1)} = y_0^{(2)}$, $z_1^{(1)} = y_1^{(2)}$) we can construct the $*$ -product automaton $M_{4,2,2}^{(1)} * M_{3,2,2}^{(2)}$, the transition diagram of which is as follows:

2°. The $\#$ -product automaton $M_{4,2,2}^{(1)} \# M_{3,2,2}^{(2)}$ can be constructed as follows, where in order to avoid the complexity the notations $(0, 1)$ etc. by the arrows are used instead of $(y_0^{(1)}, y_1^{(2)})$ etc. in the set $Y_1 \times Y_2$.

3°. To show an example of a $\wedge$ -product automaton, let us consider two finite automata $M_{3,2,2}^{(1)}$ and $M_{3,2,2}^{(2)} \in \mathfrak{M}_{3,2,2}[X, Y, Z]$, where $X = \{x_0, x_1, x_2\}$, $Y = \{y_0, y_1\}$ and $Z = \{z_0, z_1\}$.

Then the $\wedge$ -product automaton $M_{3,2,2}^{(1)} \wedge M_{3,2,2}^{(2)}$ is constructed as follows:

4°. The three face automata of the above automaton $M_{3,2,2}^{(1)}$ are as follows:

2.7. Representation of a finite automaton.

Let $M_{n,m,p}$ belong to $\mathfrak{M}_{n,m,p}[X, Y, Z]$. Let $T_M: X \times Y \rightarrow X$ and $U_M: X \rightarrow Z$ be two functions of the automaton $M_{n,m,p}$. And let us define two numbers $a_{(M)}$ and $b_{(M)}$, associated with $M_{n,m,p}$, as follows:

$$a_{(M)} \equiv \sum_{x \in X} \sum_{y \in Y} \lambda_X(T_M(x, y)) n^{nm-1 - (\lambda_Y(y)n + \lambda_X(x))},$$

$$b_{(M)} \equiv \sum_{x \in X} \lambda_Z(U_M(x)) p^{n-1 - \lambda_X(x)}.$$

Then

$$a_{(M)} \in N_n[0, n^{nm}-1] \quad \text{and} \quad b_{(M)} \in N_p[0, p^n-1].$$

Let $\mathfrak{R}_{n,m,p} \equiv N_n[0, n^{nm}-1] \times N_p[0, p^n-1]$ and let us define a function $\tau: \mathfrak{M}_{n,m,p}[X, Y, Z] \rightarrow \mathfrak{R}_{n,m,p}$ by $\tau(M_{n,m,p}) = (a_{(M)}, b_{(M)})$ for any $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$. The function τ is obviously bijective, so that the inverse function τ^{-1} exists:

$$\tau^{-1}: \mathfrak{R}_{n,m,p} \rightarrow \mathfrak{M}_{n,m,p}[X, Y, Z].$$

In considering the inverse function we should fix the set $\{X, Y, Z\}$ in advance.

We may define a finite automaton $M_{n,m,p}$ as $M_{n,m,p} = \tau^{-1}(a_{(M)}, b_{(M)})$ when the set $\{X, Y, Z\}$ is fixed. The pair $(a_{(M)}, b_{(M)}) \in \mathfrak{R}_{n,m,p}$ is called the representative point of $M_{n,m,p}$ in $\mathfrak{R}_{n,m,p}$. By operating τ^{-1} on each representative point in $\mathfrak{R}_{n,m,p}$, we can redefine any finite automaton in $\mathfrak{M}_{n,m,p}[X, Y, Z]$ and also combinations of them. Let us show some examples.

1°. Face-automata.

Let (a, b) be any point in $\mathfrak{R}_{n,m,p}$:

$$a = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} a_{i,j} n^{nm-1-(jn+i)} \quad (a_{i,j} \in N_{10}[0, n-1]),$$

$$b = \sum_{j=1}^{m-1} b_j p^{n-1-j} \quad (b_j \in N_{10}[0, p-1]).$$

And let

$$\langle a_{i,k} \rangle_j = \begin{cases} a_{i,k} & (a_{i,k} < j), \\ i & (a_{i,k} = j), \\ a_{i,k} - 1 & (a_{i,k} > j). \end{cases}$$

A function $\delta_j: \mathfrak{R}_{n,m,p} \rightarrow \mathfrak{R}_{n-1,m,p}$ assigns to each pair (a, b) a pair (a^{*j}, b^{*j}) , where

$$a^{*j} = \sum_{i=0}^{j-1} \sum_{k=0}^{m-1} \langle a_{i,k} \rangle_j (n-1)^{\langle n-1 \rangle m-1-(k(n-1)+i)} + \sum_{i=j+1}^{n-1} \sum_{k=0}^{m-1} \langle a_{i,k} \rangle_j (n-1)^{\langle n-1 \rangle m-1-(k(n-1)+i-1)},$$

$$b^{*j} = \sum_{i=0}^{j-1} b_i p^{n-2-i} + \sum_{i=j+1}^{n-1} b_i p^{n-1-i}, \quad (j=0, 1, \dots, n-1).$$

By direct calculations we can easily verify the following relations:

$$\delta_i \delta_j (\tau M_{n,m,p}) = \delta_j \delta_{i+1} (\tau M_{n,m,p}), \quad (i \geq j; i, j=0, 1, \dots, n-2),$$

$$\delta_i \delta_j (\tau M_{n,m,p}) = \delta_{j-1} \delta_i (\tau M_{n,m,p}), \quad (i \leq j; i=0, 1, \dots, n-2; j=0, 1, \dots, n-1).$$

Using the function $\delta_j: \mathfrak{R}_{n,m,p} \rightarrow \mathfrak{R}_{n-1,m,p}$ ($j=0, 1, \dots, n-1$), the j -th-face automaton $\partial_j M_{n,m,p}$ ($M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$) is defined by

$$\tau(\partial_j M_{n,m,p}) = \delta_j (\tau M_{n,m,p}),$$

i. e.,

$$\partial_j M_{n,m,p} = \tau^{-1}(\delta_j (\tau M_{n,m,p})).$$

PROPOSITION 2.7.1. *Let $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$. Then*

$$\partial_i \partial_j M_{n,m,p} = \partial_j \partial_{i+1} M_{n,m,p} \quad (i \geq j; i, j=0, 1, \dots, n-2),$$

$$\partial_i \partial_j M_{n,m,p} = \partial_{j-1} \partial_i M_{n,m,p} \quad (i \leq j; i=0, 1, \dots, n-2; j=0, 1, \dots, n-1).$$

PROOF. We shall prove the first formula. The second formula follows from the first formula. By definition we have

$$\tau(\partial_i \partial_j M_{n,m,p}) = \delta_i (\tau(\partial_j M_{n,m,p})) = \delta_i (\delta_j (\tau M_{n,m,p}))$$

and

$$\tau(\partial_j \partial_{i+1} M_{n,m,p}) = \delta_j (\tau(\partial_{i+1} M_{n,m,p})) = \delta_j (\partial_{i+1} (\tau M_{n,m,p})).$$

Hence by the relations of δ_i above mentioned we get $\tau(\partial_i \partial_j M_{n,m,p}) = \tau(\partial_j \partial_{i+1} M_{n,m,p})$. And since the function $\tau: \mathfrak{M}_{n-2,m,p}[X^{(2)}, Y, Z] \rightarrow \mathfrak{R}_{n-2,m,p}$ is bijective, we have the conclusion.

2°. $\wedge$ -product automata.

Let two points (a^1, b^1) and (a^2, b^2) be the representative points of $M_{n,m,p}^{(1)}$ and $M_{n,m,p}^{(2)}$, respectively, in the set $\mathfrak{M}_{n,m,p}[X, Y, Z]$:

$$\tau(M_{n,m,p}^{(1)}) = (a^1, b^1), \quad \tau(M_{n,m,p}^{(2)}) = (a^2, b^2).$$

Let us define $a^1 \wedge a^2$ by $a^1 \wedge a^2 \equiv \langle a_{a_0}^{1_2}, a_{a_1}^{1_2}, \dots, a_{jn+a_{jn+k}^2}^{1_2}, \dots, a_{(m-1)n+a_{nm-1}^2}^{1_2} \rangle$ for $a^i = \langle a_{a_0}^i, a_{a_1}^i, \dots, a_{jn+k}^i, \dots, a_{nm-1}^i \rangle$ ($\in N_n[0, n^{nm}-1]$)*). Then the $\wedge$ -product automaton of two finite automata $M_{n,m,p}^{(1)}$ and $M_{n,m,p}^{(2)}$ may be defined by $M^{(1)} \wedge M^{(2)} = \tau^{-1}((a^1 \wedge a^2, b^1))$.

2.8. Addition of finite automata in $\mathfrak{M}_{n,m,p}[X, Y, Z]$.

We shall introduce an operation, called addition, in the set $\mathfrak{M}_{n,m,p}[X, Y, Z]$. The operation assigns to each pair of finite automata $M_{n,m,p}^{(1)}$ and $M_{n,m,p}^{(2)}$ ($\in \mathfrak{M}_{n,m,p}[X, Y, Z]$) a finite automata, denoted by $M_{n,m,p}^{(1)} + M_{n,m,p}^{(2)}$, which is constructed by the following two functions $T_{M^{(1)} + M^{(2)}}: X \times Y \rightarrow X$ and $U_{M^{(1)} + M^{(2)}}: X \rightarrow Z$:

$$T_{M^{(1)} + M^{(2)}}(x, y) = x'$$

and

$$U_{M^{(1)} + M^{(2)}}(x) = z',$$

where

$$x' = \lambda_X^{-1}[\lambda_X(T_{M^{(1)}}(x, y)) + \lambda_X(T_{M^{(2)}}(x, y))]_n$$

and

$$z' = \lambda_Z^{-1}[\lambda_Z(U_{M^{(1)}}(x)) + \lambda_Z(U_{M^{(2)}}(x))]_p.$$

Here the symbols $[a]_n$ and $[a]_p$ show the representative number of the integer a mod. n and mod. p , respectively. Then the set $\mathfrak{M}_{n,m,p}[X, Y, Z]$, together with the addition $+$, forms an Abelian group. The addition can

*) For example let $a = \sum_{i=0}^{nm-1} a_i n^{nm-1-i} \in N_n[0, n^{nm}-1]$, where $a_i \in N_{10}[0, n-1]$. Then let us write, for simplicity, $a = \langle a_0, a_1, \dots, a_{nm-1} \rangle$.

also be introduced from the idea of the representative points. Let (a, b) and (c, d) belong to $\mathfrak{R}_{n,m,p}$, and define $(a, b) \oplus (c, d)$ by $(a \oplus_n c, b \oplus_p d)$, where for example the operation $a \oplus_n c$ means the addition of corresponding digits in a and c mod. n . Then, letting $M_{n,m,p}^{(i)} = \tau^{-1}((a^i, b^i))$ ($i=1, 2$), the sum $M_{n,m,p}^{(1)} + M_{n,m,p}^{(2)}$ may also be defined by

$$\begin{aligned} M_{n,m,p}^{(1)} + M_{n,m,p}^{(2)} &= \tau^{-1}((a^1 \oplus_n a^2, b^1 \oplus_p b^2)) \\ &= \tau^{-1}((a^1, b^1) \oplus (a^2, b^2)). \end{aligned}$$

It is readily shown that both definitions are identical.

Since

$$\begin{aligned} \tau(M_{n,m,p}^{(1)} + M_{n,m,p}^{(2)}) &= (a^1, b^1) \oplus (a^2, b^2) \\ &= \tau(M_{n,m,p}^{(1)}) \oplus \tau(M_{n,m,p}^{(2)}) \end{aligned}$$

by definition, the set $\mathfrak{R}_{n,m,p}$, together with the operation $\oplus$, forms an Abelian group. Therefore the unit element (say $\Theta_{n,m,p}$) of the Abelian group $\mathfrak{R}_{n,m,p}$ corresponds to the origin in $\mathfrak{R}_{n,m,p}$ by the operation τ^{-1} :

$$\Theta_{n,m,p} \equiv \tau^{-1}((0, 0)).$$

We may write the inverse element of $M_{n,m,p}$ ($\in \mathfrak{M}_{n,m,p}[X, Y, Z]$) by $-M_{n,m,p}$. The two functions $T_{-M}: X \times Y \rightarrow X$ and $U_{-M}: X \rightarrow Z$ associated with the automaton $(-M_{n,m,p})$, are defined as follows:

$$\begin{aligned} \lambda_X(T_{-M}(x, y)) &= \begin{cases} n - \lambda_X(T_M(x, y)), & (\text{if } \lambda_X(T_M(x, y)) \neq 0), \\ 0 & (\text{if } \lambda_X(T_M(x, y)) = 0), \end{cases} \\ \lambda_Z(U_{-M}(x)) &= \begin{cases} p - \lambda_Z(U_M(x)), & (\text{if } \lambda_Z(U_M(x)) \neq 0), \\ 0 & (\text{if } \lambda_Z(U_M(x)) = 0) \end{cases} \end{aligned}$$

for any $x \in X$ and any $y \in Y$.

REMARK. The inverse automaton $-M_{n,m,p}$ of $M_{n,m,p}$ is quite different from the ‘inverse’ finite automaton $M_{n,m,p}^{-1}$, constructed in the next paragraph.

2.9. Definition of the ‘inverse’ finite automaton.

A finite automaton is called the ‘inverse’ finite automaton of a given finite automaton $M_{n,m,p}$ ($\in \mathfrak{M}_{n,m,p}[X, Y, Z]$), denoted by $M_{n,m,p}^{-1}$, if it satisfies the following conditions:

- (i) there exist a number n' and a set X' such that

$$M_{n,m,p}^{-1} \in \mathfrak{M}_{n',p,m}[X', Z, Y],$$

(ii) there exists a state $x' \in X'$ such that

$$H_{M^{-1}}(x', H_M(x, y^*)) = y^*$$

for each $x \in X$ and any $y^* \in Y^*$.

(iii) there exists a state $x \in X$ such that

$$H_M(x, H_{M^{-1}}(x', z^*)) = z^*$$

for each $x' \in X'$ and any $z^* \in Z^*$.

2.10. The direct sum of finite automata.

Assume that the sets X_i ($i=1, 2, \dots, l$) are mutually disjoint. Let $X = \bigcup_{i=1}^l X_i$.

Let $M_{n_i, m, p}^{(i)} \in \mathfrak{M}_{n_i, m, p}[X^{(i)}, Y, Z]$ ($i=1, 2, \dots, l$) and let the state-transition and the output function of $M_{n_i, m, p}^{(i)}$ be $T_{M^{(i)}}$ and $U_{M^{(i)}}$, respectively.

We construct the state-transition $T_M: X \times Y \rightarrow X$ and the output function $U_M: X \rightarrow Z$ in the following way:

$$\begin{aligned} T_M(x, y) &= T_{M^{(i)}}(x, y), \\ U_M(x) &= U_{M^{(i)}}(x) \quad \text{when } x \in X_i. \end{aligned}$$

Then we get a finite automaton $M_{\sum_{i=1}^l n_i, m, p} \in \mathfrak{M}_{\sum_{i=1}^l n_i, m, p}[X, Y, Z]$ with the state-transition T_M and the output function U_M . The automaton $M_{\sum_{i=1}^l n_i, m, p}$ is called the direct sum of the automata $M_{n_i, m, p}^{(i)}$.

We denote it by $M_{\sum_{i=1}^l n_i, m, p} \equiv M_{n_1, m, p}^{(1)} \oplus M_{n_2, m, p}^{(2)} \oplus \dots \oplus M_{n_l, m, p}^{(l)}$.

2.11. The isomorphic finite automata.

Two finite automata $M_{n_1, m, p}^{(1)}$ and $M_{n_2, m, p}^{(2)}$ ($\in \mathfrak{M}_{n_i, m, p}[X, Y, Z]$) are called isomorphic, denoted by $M_{n_1, m, p}^{(1)} \cong M_{n_2, m, p}^{(2)}$, if, and only if, there exists a mapping $\sigma: X \rightarrow X$ such that

$$T_{M^{(1)}}(x, y) = T_{M^{(2)}}(\sigma(x), y)$$

and

$$U_{M^{(1)}}(x) = U_{M^{(2)}}(\sigma(x)).$$

For example the two finite automata drawn below are isomorphic.

EXAMPLE. $X \equiv \{x_0, x_1, x_2\}$, $Y \equiv \{y_0, y_1\}$ and $Z \equiv \{z_0, z_1\}$.

The number of the isomorphic automata of a given finite automaton $M_{n,m,p}[X,Y,Z]$ is $n!$.

§3. Strongly connected finite automata.

A finite automata $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X,Y,Z]$ is called strongly connected if there exists a finite sequence of inputs $y^* \in Y^*$ such that $\pi_M(x, y^*) = x'$ for any x and $x' \in X$, where $\pi_M: X \times Y^* \rightarrow X$ is defined by $\pi_M(x, y^*) = T_M(T_M(\dots(T_M(x, y_{\alpha_1}), y_{\alpha_2}), \dots), y_{\alpha_l})$ for any $x \in X$ and any $y^* = \{y_{\alpha_1}, y_{\alpha_2}, \dots, y_{\alpha_l}\} \in Y^*$ ($y_{\alpha_i} \in Y$, $i=1, 2, \dots, l$; $l \geq 1$). Strongly connected automaton have interesting properties, which will be described in the following.

3.1. The *-algorithm.

Let $\mathfrak{G}_n$ be the set of $n \times n$ matrices consisting of the elements 0 and 1, and $\mathfrak{G}_n^*$ the set of matrices of the same type consisting of the elements 0, 1 and *. Two mappings $c: \mathfrak{M}_{n,m,p}[X,Y,Z] \rightarrow \mathfrak{G}_n$ and $c^*: \mathfrak{M}_{n,m,p}[X,Y,Z] \rightarrow \mathfrak{G}_n^*$ are defined by

$$c(M_{n,m,p}) = E_M = (e_{i,j}) \in \mathfrak{G}_n$$

and

$$c^*(M_{n,m,p}) = E_M^* = (e_{i,j}^*) \in \mathfrak{G}_n^* \quad \text{for any } M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X,Y,Z],$$

where

$$e_{ij} = \begin{cases} 1 & \text{(if there exists at least a direct transition from } x_i \text{ to } x_j), \\ 0 & \text{(otherwise),} \end{cases}$$

$$e_{ij}^* = \begin{cases} * & \text{(if there exists at least a direct transition from } x_i \text{ to } x_j), \\ 0 & \text{(otherwise).} \end{cases}$$

Now let us set the following tables of addition $\oplus$ and multiplication $\otimes$ among 0, 1 and $*$:

$\oplus$	0	1	*
0	0	1	*
1	1	1	*
*	*	*	*

$\otimes$	0	1	*
0	0	0	0
1	0	1	*
*	0	*	*

Using the above tables, the powers of $c(M_{n,m,p})$ are defined for any $M_{n,m,p}[X, Y, Z] \in \mathfrak{M}_{n,m,p}[X, Y, Z]$:

$$\begin{aligned} \{c(M_{n,m,p})\}_{i,j}^1 &\equiv \{c^*(M_{n,m,p}) \cdot c(M_{n,m,p})\}_{i,j} \\ &= \begin{cases} * & \text{(if } e_{ij}^* = *), \\ 0 & \left(\text{if } e_{ij}^* = 0, \sum_{\oplus, k=0}^{n-1} e_{ik}^* \otimes e_{kj} = 0 \right), \\ * & \left(\text{if } e_{ij}^* = 0, \sum_{\oplus, k=0}^{n-1} e_{ik}^* \otimes e_{kj} = * \right), \end{cases} \\ \{c(M_{n,m,p})\}_{i,j}^l &\equiv [\{c(M_{n,m,p})\}_{i,j}^{l-1} \cdot c(M_{n,m,p})]_{i,j} \\ &= \begin{cases} * & (\{c(M_{n,m,p})\}_{i,j}^{l-1} = *), \\ 0 & \left(\{c(M_{n,m,p})\}_{i,j}^{l-1} = 0, \sum_{\oplus, k=0}^{n-1} \{c(M_{n,m,p})\}_{i,k}^{l-1} \otimes e_{kj} = 0 \right), \\ * & \left(\{c(M_{n,m,p})\}_{i,j}^{l-1} = 0, \sum_{\oplus, k=0}^{n-1} \{c(M_{n,m,p})\}_{i,k}^{l-1} \otimes e_{kj} = * \right) \end{cases} \\ &\quad (l=3, 4, 5, \dots), \end{aligned}$$

where $\{c(M_{n,m,p})\}_{i,j}^{l-1}$ means the (i, j) element of the matrix $\{c(M_{n,m,p})\}^{l-1}$.

PROPOSITION 3.1. *A finite automaton $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$ is strongly connected if, and only if, the relation*

$$\{c(M_{n,m,p})\}^n = \begin{pmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ * & * & \cdots & * \end{pmatrix}$$

holds.

PROOF. Obviously it is sufficient. The necessity is proved as follows. Let $M_{n,m,p}$ be strongly connected. Take a pair of indices (i, j) . Then there exists an integer $r (\geq 1)$ such that $\{c(M_{n,m,p})\}_{i,j}^{r-1} = 0$ and $\{c(M_{n,m,p})\}_{i,j}^r = *$. If $1 \leq r \leq n$, then $\{c(M_{n,m,p})\}_{i,j}^s = *$ ($s \geq r$) from the above fact and the definition of the powers of matrix, and the property $\{c(M_{n,m,p})\}_{i,j}^n = *$ follows letting $s = n$. Now assume it may occur that $r > n$ for some pair (i, j) . According to the definition of r it holds that any sequence of transitions less than r steps cannot reach from x_i to x_j , but there exists a sequence of just r steps that reaches to x_j from x_i . From this fact together with the assumption $r > n$ we must conclude that there exists some state x_k which is passed at least twice in the sequence. Omitting the intermediate steps from x_k to x_k there remains a sequence of transitions of a number of steps less than r connecting x_i to x_j , which contradicts the meaning of r . Thus we must have $r \leq n$, hence the conclusion of the proposition.

REMARK. The number n in the proposition can not be decreased, which is shown in the following example.

EXAMPLE. Let $c(M_{3,m,p}) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ ($n=3$).

Then

$$\begin{aligned} \{c(M_{3,m,p})\}^2 &= \begin{pmatrix} 0 & 0 & * \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}, \\ \{c(M_{3,m,p})\}^3 &= \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}. \end{aligned}$$

Hence the above $M_{3,m,p}$ is strongly connected.

3.2. Strong-connectedness of a finite automaton and that of its face automaton.

PROPOSITION 3.2. *If among n face-automata $\partial_j M_{n,m,p}$ ($j=0, 1, \dots, n-1$; $n \geq 3$) of a finite automaton $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$ there exist two that are strongly connected, then $M_{n,m,p}$ itself is strongly connected.*

PROOF. We consider mappings $c: \mathfrak{M}_{n-r,m,p}[X^{(r)}, Y, Z] \rightarrow \mathfrak{G}_{n-r}$ ($r=1, 2, \dots, n-1$). Assume that two face-automata $\partial_i M_{n,m,p}$ and $\partial_k M_{n,m,p}$ of a given

finite automaton $M_{n,m,p}$ are strongly connected. By the proposition 3.1,

$$\{c(\partial_i M_{n,m,p})\}^{n-1} = \{c(\partial_k M_{n,m,p})\}^{n-1} = \begin{pmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{pmatrix}. \text{ Considering from the definition}$$

of face operator and the properties of c , we find that $\{c(\partial_i M_{n,m,p})\}^{n-1}$ is equal to the (i, i) cofactor of $\{c(M_{n,m,p})\}^{n-1}$ and $\{c(\partial_k M_{n,m,p})\}^{n-1}$ to the (k, k) cofactor of the same matrix. As the existence of two sequences of transitions from x_i to x_k and from x_k to x_i follows from the assumption of strong-connectedness of $\partial_i M_{n,m,p}$ and $\partial_k M_{n,m,p}$, we obtain the relation

$$\{c(M_{n,m,p})\}^n = \begin{pmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{pmatrix},$$

which shows the strong connectedness of the automaton $M_{n,m,p}$ itself.

EXAMPLE. Let $X \equiv \{x_0, x_1, x_2, x_3\}$, $Y \equiv \{y_0, y_1\}$ and $Z \equiv \{z_0, z_1\}$.

Let $M_{4,2,2}$ be the first automaton described in 2.5. Then $\partial_0 M_{4,2,2}$ and $\partial_2 M_{4,2,2}$ are strongly connected, so $M_{4,2,2}$ is strongly connected.

3.3. Strong connectedness of the $\#$ -product automaton.

PROPOSITION 3.3. *The $\#$ -product finite automaton $M_{n_1, m_1, p_1}^{(1)} \# M_{n_2, m_2, p_2}^{(2)}$ of two finite automata $M_{n_i, m_i, p_i}^{(i)} \in \mathfrak{M}_{n_i, m_i, p_i}[X_i, Y_i, Z_i]$ ($i=1, 2$) is strongly connected if, and only if, both component automata $M_{n_i, m_i, p_i}^{(i)}$ ($i=1, 2$) are strongly connected.*

PROOF. Let $M_{n_i, m_i, p_i}^{(i)}$ ($i=1, 2$) be strongly connected. If under this assumption the finite automaton $M_{n_1, m_1, p_1}^{(1)} \# M_{n_2, m_2, p_2}^{(2)}$ is not strongly connected, it follows that $\pi_{M^{(1)} \# M^{(2)}}(x, y^*) \neq x'$ for some $x = (x^1, x^2)$, $x' = (x'^1, x'^2) \in X_1 \times X_2$ and any $y^* \in Y^*$ ($= (Y_1 \times Y_2)^*$), where $y^* = \{(y_{\alpha_1}^{(1)}, y_{\beta_1}^{(2)}), (y_{\alpha_2}^{(1)}, y_{\beta_2}^{(2)}), \dots, (y_{\alpha_l}^{(1)}, y_{\beta_l}^{(2)})\}$ ($y_{\alpha_i}^{(1)} \in Y_1$, $y_{\beta_i}^{(2)} \in Y_2$; $i=1, 2, \dots, l$; $l \geq 1$). Then letting $y_1^* = \{y_{\alpha_1}^{(1)}, y_{\alpha_2}^{(1)}, \dots, y_{\alpha_l}^{(1)}\}$ and $y_2^* = \{y_{\beta_1}^{(2)}, y_{\beta_2}^{(2)}, \dots, y_{\beta_l}^{(2)}\}$, we have $\pi_{M^{(1)} \# M^{(2)}}(x, y^*) = (\pi_{M^{(1)}}(x^1, y_1^*), \pi_{M^{(2)}}(x^2, y_2^*)) \neq (x'^1, x'^2)$. Hence at least one of the following relations must hold:

$$\pi_{M^{(1)}}(x^1, y_1^*) \neq x'^1, \quad \pi_{M^{(2)}}(x^2, y_2^*) \neq x'^2.$$

The sequence of inputs y^* being taken arbitrarily, y_1^* and y_2^* are also arbitrary. Therefore, one of two automata $M_{n_1, m_1, p_1}^{(1)}$ and $M_{n_2, m_2, p_2}^{(2)}$ is not strongly connected, which contradicts the assumption. Hence the sufficiency is proved. The necessity is proved immediately.

COROLLARY 3.3.1. *The $\#$ -product automaton of a finite number of finite automata is strongly connected if, and only if, all component automata are strongly connected.*

§4. Indistinguishability and reducibility.

Let $M_{n,m,p}$ be a finite automaton in the set $\mathfrak{M}_{n,m,p}[X, Y, Z]$. Two states x and x' ($\in X$) are called indistinguishable with respect to the automaton $M_{n,m,p}$ if, and only if, $H_M(x, y^*) = H_M(x', y^*)$ for any $y^* \in Y^*$. This relation we denote by $x \stackrel{M}{=} x'(Y^*)$. And two states x and x' ($\in X$) are called distinguishable by y^* with respect to the automaton $M_{n,m,p}$ if, and only if, $H_M(x, y^*) \neq H_M(x', y^*)$ for some $y^* \in Y^*$. This relation we denote by $x \neq x'(y^*)$.

The concept of distinguishability, introduced by E. F. Moore, has an interesting phase in the consideration of finite automata. This concept can be extended to distinguishability between states of two different finite automata. And, further, we can consider distinguishability between two finite automata themselves. It is stated as follows:

Let $M_{n_1,m,p}^{(1)} \in \mathfrak{M}_{n_1,m,p}[X_1, Y, Z]$ and $M_{n_2,m,p}^{(2)} \in \mathfrak{M}_{n_2,m,p}[X_2, Y, Z]$. The above finite automata are called distinguishable each other, either if there exist some $x^{(1)} \in X_1$ and some $y^* \in Y^*$ such that $H_{M^{(1)}}(x^{(1)}, y^*) \neq H_{M^{(2)}}(x^{(2)}, y^*)$ for all $x^{(2)} \in X_2$ or if there exist some $x^{(2)} \in X_2$ and some $y^* \in Y^*$ such that $H_{M^{(1)}}(x^{(1)}, y^*) \neq H_{M^{(2)}}(x^{(2)}, y^*)$ for all $x^{(1)} \in X_1$.

Indistinguishability between two finite automata themselves is defined similarly.

Next if two finite automata $M^{(1)} \in \mathfrak{M}_{n_1,m,p}[X, Y, Z]$ and $M^{(2)} \in \mathfrak{M}_{n_2,m,p}[X^{(n_1-n_2)}, Y, Z]^*$ ($n_1 > n_2$) are indistinguishable each other, the finite automaton $M^{(1)}$ will be said reducible and we shall call the finite automaton $M^{(2)}$ a reduced automaton of $M^{(1)}$. A finite automaton M^* will be called a reduced form of $M^{(1)}$ if M^* is a reduced automaton of $M^{(1)}$ and is no more reducible.

A finite automaton $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$ is called fundamental if, and only if, any two elements in X are distinguishable each other with respect to the automaton. Strongly connected automata have the following properties ([21]).

PROPOSITION 4.1. *Let a finite automaton $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$ be*

*) See p. 104.

strongly connected, and $\mathfrak{G}_M$ the set of finite automata which are indistinguishable from $M_{n,m,p}$. Then there exists a finite automaton $M^* \in \mathfrak{G}_M$ such that $\#(X_{M^*}) \leq \#(X_{\tilde{M}})$ for all $\tilde{M} \in \mathfrak{G}_M$. Such an automaton is determined uniquely up to isomorphism, and called the reduced form of $M_{n,m,p}$. It is strongly connected and fundamental.

PROPOSITION 4.2. *A strongly connected finite automaton is a reduced form if, and only if, it is fundamental.*

From the above definitions, we get immediately the following lemmas.

LEMMA 4.1. *Let $M_{n,m,p}^{(1)}$ and $M_{n',m,p}^{(2)}$ be two finite automata in the set $\mathfrak{M}_{n,m,p}[X_1, Y, Z]$ and $\mathfrak{M}_{n',m,p}[X_2, Y, Z]$, respectively. If a state (say $x \in X_1$) of the automaton $M_{n,m,p}^{(1)}$ and a state (say $x' \in X_2$) of the automaton $M_{n',m,p}^{(2)}$ are indistinguishable each other, then the states $x^1, x^2, \dots, x^{l-1}$ and the states $x'^1, x'^2, \dots, x'^{l-1}$, induced by an element $y^* (\in Y^*)$ of length l , are also indistinguishable each other, respectively.*

LEMMA 4.2. *Let $M_{n,m,p}^{(1)}$ and $M_{n',m,p}^{(2)}$ be two finite automata in the set $\mathfrak{M}_{n,m,p}[X_1 Y, Z]$ and $\mathfrak{M}_{n',m,p}[X_2 Y, Z]$, respectively. If a state (say $x \in X_1$) of the automaton $M_{n,m,p}^{(1)}$ and a state (say $x' \in X_2$) of the automaton $M_{n',m,p}^{(2)}$ are indistinguishable each other, and if $\sum_x \equiv X_1$ and $\sum_{x'} \equiv X_2$, then $M_{n,m,p}^{(1)}$ and $M_{n',m,p}^{(2)}$ are indistinguishable^{*)}.*

As corollary of the above lemma we obtain

LEMMA 4.3. *Let $M_{n,m,p}^{(1)}$ and $M_{n',m,p}^{(2)}$ be two strongly connected finite automata in the set $\mathfrak{M}_{n,m,p}[X_1, Y, Z]$ and $\mathfrak{M}_{n',m,p}[X_2, Y, Z]$, respectively. If a state (say $x \in X_1$) of the automaton $M_{n,m,p}^{(1)}$ and a state (say $x' \in X_2$) of the automaton $M_{n',m,p}^{(2)}$ are indistinguishable each other, then $M_{n,m,p}^{(1)}$ and $M_{n',m,p}^{(2)}$ are also indistinguishable.*

Using lemma 4.1, we obtain the following

THEOREM 4.1. *Let $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$. Then there exists in the set $\mathfrak{M}_{n',m,p}[X/\mathfrak{R}, Y, Z]$ a fundamental finite automaton $M'_{n',m,p}$ which is indistinguishable from $M_{n,m,p}$ and is uniquely determined (up to an isomorphism), where $n' = \#(X/\mathfrak{R})$ and $\mathfrak{R}$ is the equivalence relation as follows: $x\mathfrak{R}x'$ if, and only if, the states x and x' are indistinguishable with respect to $M_{n,m,p}$.*

^{*)} $\sum_x \equiv \{\pi_{M^{(1)}}(x, y^*); y^* \in Y^*\}$ and $\sum_{x'} \equiv \{\pi_{M^{(2)}}(x', y^*); y^* \in Y^*\}$.

PROOF. Let R_x be the equivalence class containing the state $x \in X$. And let us define $T_{M'}: X/\mathfrak{R} \times Y \rightarrow X/\mathfrak{R}$ and $U_{M'}: X/\mathfrak{R} \rightarrow Z$ by setting $T_{M'}(R_x, y) = R_{T_M(x, y)}$ and $U_{M'}(R_x) = U_M(x)$ for $R_x \in X/\mathfrak{R}$ and $y \in Y$, respectively. The two mappings $T_{M'}$ and $U_{M'}$ are defined unambiguously from lemma 4.1 and the fact that the elements of R_x are indistinguishable each other. Then the mapping $H_{M'}: X/\mathfrak{R} \times Y^* \rightarrow Z^*$ is defined unambiguously. Hence the set $\{X/\mathfrak{R}, Y, Z\}$ together with the mappings $\{T_{M'}, U_{M'}, H_{M'}\}$ forms a desired finite automaton $M'_{n', m, p}$ in the set $\mathfrak{M}_{n', m, p}[X/\mathfrak{R}, Y, Z]$. It is easily shown that $M'_{n', m, p}$ is fundamental and uniquely determined (up to an isomorphism).

The 'inverse' automata have the following properties:

THEOREM 4.2. *Let the finite automaton $M_{n, m, p}^{-1} \in \mathfrak{M}_{n', p, m}[X', Z, Y]$ be an 'inverse' automaton of the finite automaton $M_{n, m, p} \in \mathfrak{M}_{n, m, p}[X, Y, Z]$. Then a necessary and sufficient condition that a finite automaton $M_{n'', p, m}^*$ ($\in \mathfrak{M}_{n'', p, m}[X'', Z, Y]$) is another 'inverse' automaton of $M_{n, m, p}$ is that $M_{n'', p, m}^*$ is indistinguishable from $M_{n, m, p}^{-1}$.*

PROOF. Assume that $M_{n'', p, m}^*$ is indistinguishable from $M_{n, m, p}^{-1}$. From the definition of $M_{n, m, p}^{-1}$ there exists a state x' such that $H_M(x', H_M(x, y^*)) = y^*$ for each $x \in X$ and any $y^* \in Y^*$. And from the assumption of M^* there exists a state $x'' \in X''$ such that $H_{M^{-1}}(x', z^*) = H_{M^*}(x'', z^*)$ for any $z^* \in Z^*$. Combining these results, we get the relation

$$H_{M^*}(x'', H_M(x, y^*)) = H_{M^{-1}}(x', H_M(x, y^*)) = y^*$$

for each $x \in X$ and any $y^* \in Y^*$.

Similarly we obtain the formula $H_M(\tilde{x}, H_{M^*}(x'', z^*)) = z^*$ for each $x'' \in X''$ and any $z^* \in Z^*$.

Conversely, assume that $M_{n'', p, m}^*$ is an 'inverse' automaton of $M_{n, m, p}$. Then there exists a state x such that $H_M(x, H_{M^*}(x'', z^*)) = z^*$ for each $x'' \in X''$ and any $z^* \in Z^*$. We get from the definition of $M_{n, m, p}^{-1}$ the relation

$H_{M^{-1}}(x', H_M(x, y^*)) = y^*$ for any $y^* \in Y^*$ and some $x' \in X'$. Then it follows

$$\begin{aligned} H_{M^*}(x'', z^*) &= H_{M^{-1}}(x', H_M(x, H_{M^*}(x'', z^*))) \\ &= H_{M^{-1}}(x', z^*) \quad \text{for any } z^* \in Z^*. \end{aligned}$$

Similarly there exists a state x'' such that

$$H_{M^{-1}}(x', z^*) = H_{M^*}(x'', z^*) \quad \text{for each } x' \in X' \text{ and any } z^* \in Z^*.$$

Finally we can also derive the following theorems on reducibility.

PROPOSITION 4.3. *The reduced finite automaton of the direct sum of a finite number of the same finite automata $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$ is the original automaton $M_{n,m,p}$. Further, if the finite automaton $M_{n,m,p}$ is fundamental, the reduced automaton is in reduced form.*

PROOF. The first part of the proposition is self-evident.

The second part follows from E. F. Moore's theorem on strongly connected automaton. For the detailed proof see [21].

COROLLARY 4.3.1. *If the reduced form of a finite automaton $M_{n,m,p}$ is a finite automaton $\tilde{M}_{\tilde{n},m,p}$, then the reduced form of the direct sum $M_{n,m,p} \oplus \tilde{M}_{\tilde{n},m,p}$ is also the finite automaton $\tilde{M}_{\tilde{n},m,p}$.*

§5. Reversible finite automata and their direct sum decomposition.

Some mathematical models of simple cryptographic devices are considered as special (n, n, n) finite automata. In [10] we have studied the classification and the characteristics of these finite automata. Here we shall consider the properties of a reversible finite automaton. A finite automaton $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$ is called reversible if there exists at least an input sequence $\tilde{y}^*$ such that $\pi_M(\pi_M(x, y^*), \tilde{y}^*) = x$ for any state $x \in X$ and any input sequence $y^* \in Y^*$. The input sequence $\tilde{y}^*$ depends obviously upon x and y^* . By the above definition strongly connected finite automaton is reversible.

THEOREM 5.1. *Let $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$ be reversible. Then it is strongly connected if, and only if, the set $\sum_{\hat{x}}^{\wedge}$ associated with a fixed state $\hat{x} \in X$ is equal to X .*

PROOF. Let us take any two states x and $x' \in X$. From the assumption $\sum_{\hat{x}}^{\wedge} \equiv X$ it follows that for properly chosen input sequences y_1^* and y_2^* $\pi_M(\hat{x}, y_1^*) = x$ and $\pi_M(\hat{x}, y_2^*) = x'$. Since $M_{n,m,p}$ is reversible, it follows that $\pi_M(\pi_M(\hat{x}, y_1^*), y_3^*) = \pi_M(x, y_3^*) = \hat{x}$ and $\pi_M(\pi_M(\hat{x}, y_2^*), y_4^*) = \pi_M(x', y_4^*) = \hat{x}$ for properly chosen input sequences y_3^* and $y_4^* \in Y^*$. Hence we have $\pi_M(x, y_3^*) = \pi_M(x', y_4^*) = \hat{x}$, so $\pi_M(\pi_M(x, y_3^*), y_2^*) = \pi_M(\hat{x}, y_2^*) = x'$ and $\pi_M(\pi_M(x', y_4^*), y_1^*) = \pi_M(\hat{x}, y_1^*) = x$. Therefore the automaton $M_{n,m,p}$ is strongly connected. Conversely, if an automaton is strongly connected, it is also reversible. Let us take any state $\tilde{x} \in X$. Then there exists an input sequence $y^* \in Y^*$ such that $\pi_M(\hat{x}, y^*) = \tilde{x}$ for a fixed state $\hat{x} \in X$. Thus $X \subset \sum_{\hat{x}}^{\wedge}$. On the other hand we have $\sum_{\hat{x}}^{\wedge} \subset X$, hence $\sum_{\hat{x}}^{\wedge} = X$.

REMARK. A necessary and sufficient condition that a finite automaton $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$ is strongly connected is that $\sum_x \equiv X$ for any $x \in X$.

THEOREM 5.2. Any reversible finite automaton in $\mathfrak{M}_{n,m,p}[X, Y, Z]$ may be decomposed into a direct sum of strongly connected automata, and the decomposition is unique (up to an isomorphism).

PROOF. Let us take a state $x \in X$. When $\sum_x \equiv X$, the automaton $M_{n,m,p}$ is strongly connected by theorem 5.1 and its decomposition has only one component, so the conclusion is obvious. So let us consider the case when $\sum_x$ is a proper subset of X . Since $\sum_x$ and $X - \sum_x$ are disjoint, there exists no input sequence $y^* \in Y^*$ such that $\pi_M(\tilde{x}, y^*) = \tilde{x}$ for any $\tilde{x} \in X - \sum_x$ and any $\tilde{x} \in \sum_x$. Because if such an input sequence y^* exists, we should have an input sequence $y^\# \in Y^*$ such that $\pi_M(\pi_M(\tilde{x}, y^*), y^\#) = \tilde{x}$ by the assumption of reversibility of $M_{n,m,p}$ and it would follow that $\pi_M(\tilde{x}, y^\#) = \tilde{x}$, so $\tilde{x} \in \sum_x$. Hence, combined with the fact $\sum_x \subset \sum_x$, we have $\tilde{x} \in \sum_x$, which contradicts the hypothesis. We will show that the automaton having $\sum_x$ as the state-set defines a strongly connected sub-automaton in $M_{n,m,p}$ i. e. a strongly connected finite automaton in $\mathfrak{M}_{n',m,p}[\sum_x, Y, Z]$, where $n' = \#(\sum_x)^{**}$.

From the definition of $\sum_x$, it follows that $\pi_M(x_i, y^*) \in \sum_x$ for any $x_i \in \sum_x$ and any input sequence $y^* \in Y^*$. As there exist two input sequences $y^\#$ and y^b such that $\pi_M(x_i, y^\#) = x$ and $\pi_M(x_j, y^b) = x$ for any x_i and $x_j \in \sum_x$, x being fixed, and y^d such that $\pi_M(\pi_M(x_j, y^b), y^d) = x_j$, we get $\pi_M(\pi_M(x_i, y^\#), y^d) = x_j$. Therefore, the automaton having $\sum_x$ as the state set is a strongly connected sub-automaton in $M_{n,m,p}$.

The above automaton is determined uniquely, To show this, it is sufficient to prove that $\sum_{\hat{x}} \equiv \sum_x$ for any $\hat{x} \in \sum_x$, x being fixed, which is readily shown^{*)}.

Next we consider the state set $\sum_{x^{(1)}}$, $x^{(1)}$ being the state not belonging to $\sum_x$. By the same reason as above, $\sum_{x^{(1)}}$ defines a strongly connected sub-automaton in $M_{n,m,p}$, which is uniquely determined. If the relation

*) A finite automaton $M'_{n',m,p}$ constructed from $\{X', Y, Z; T_{M'}, U_{M'}, H_{M'}\}$ is called a sub-automaton of a finite automaton $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$ if the following three conditions are satisfied:

- 1) $X' \subset X$,
- 2) $T_{M'}$, $U_{M'}$, and $H_{M'}$ are the restrictions of T_M , U_M and H_M to $X' \times Y$, X' and $X' \times Y^*$, respectively.
- 3) $T_M(x, y) \in X'$ for any $x \in X'$ and any $y \in Y$.

**) See [10] p. 155.

$\sum_x \cup \sum_{x^{(1)}} \equiv X$ holds, it represents a required direct sum decomposition of $M_{n,m,p}$. If the above relation does not hold, we repeat the above procedure until the union of sets $\sum_{x^{(i)}}$ becomes equal to X . Then we have the desired conclusion. To prove the next theorem 5.3, let us prepare the following lemma on reducibility of reversible automaton.

LEMMA 5.1. *Let a reversible finite automaton $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$ be the direct sum of two strongly connected finite automata $M_{n_1,m,p}^{(1)} \in \mathfrak{M}_{n_1,m,p}[X_1, Y, Z]$ and $M_{n_2,m,p}^{(2)} \in \mathfrak{M}_{n_2,m,p}[X_2, Y, Z]$ and let us assume $\#(X_1) \geq \#(X_2)$. Then if the state $x_i^{(1)} \in X_1$ is indistinguishable from the state $x_j^{(2)} \in X_2$ (i, j : fixed) with respect to the finite automaton $M_{n,m,p}$, the finite automaton $M_{n_2,m,p}^{(2)}$ is a reduced automaton of $M_{n_1,m,p}^{(1)}$. Therefore the reduced automaton of $M_{n,m,p}$ is the finite automaton $M_{n_2,m,p}^{(2)}$.*

PROOF. Let us take any state $x^{(1)} \in X_1$. From the assumption of $M^{(1)}$ there exists an input sequence y^* such that $\pi_{M^{(1)}}(x_i^{(1)}, y^*) = x^{(1)}$. Putting $\pi_{M^{(2)}}(x_j^{(2)}, y^*) = x^{(2)} \in X_2$, we obtain the relation $\pi_{M^{(1)}}(x_i^{(1)}, y^*) \equiv x^{(2)}$ with respect to the automaton $M_{n,m,p}$. Hence we can always take a state $x^{(2)} (\in X_2)$ indistinguishable from the state $x^{(1)}$. By the same procedure we can take a state $x^{(1)}$ such that, for any $x^{(2)} \in X_2$, $x^{(1)} \equiv x^{(2)}$. Thus $M^{(2)}$ is a reduced automaton of $M^{(1)}$. A reduced automaton of $M_{n,m,p}$ is $M_{n_1,m,p}^{(1)}$. Using corollary 4.3.1, it follows that a reduced automaton of $M_{n,m,p}$ is the finite automaton $M_{n_2,m,p}^{(2)}$.

By the decomposition theorem 5.2 we can easily extend the theorems about strongly connected automata to reversible automata.

THEOREM 5.3. *Let $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X_M, Y, Z]$ be a given reversible finite automaton and $\mathfrak{R}[M_{n,m,p}]$ be a class of all automata indistinguishable from $M_{n,m,p}$ by any input sequence. Then there exists a finite automaton $M^* \in \mathfrak{M}_{n^*,m,p}[X_{M^*}, X, Y]$ such that $\#(X_{M^*}) \leq \#(X_{\tilde{M}})$ for all $\tilde{M} \in \mathfrak{R}[M_{n,m,p}]$. Here $\#(X_M)$ means the number of elements in X_M . This automaton M^* , called the reduced form of $M_{n,m,p}$, is reversible, fundamental and determined uniquely up to an isomorphism.*

PROOF. By the result of theorem 5.2, we have for any $\tilde{M} \in \mathfrak{R}$ the direct sum decomposition:

$$\tilde{M} = \tilde{M}_1 \oplus \tilde{M}_2 \oplus \cdots \oplus \tilde{M}_l,$$

each $\tilde{M}_i$ ($i=1, 2, \dots, l$) being strongly connected. Then we can construct the

direct sum $\hat{M} = \hat{M}_1 \oplus \hat{M}_2 \oplus \cdots \oplus \hat{M}_l$, where each $\hat{M}_i$ is the reduced form of $\tilde{M}_i$. Then $\tilde{M} \equiv \hat{M}$. Let us reduce the direct sum $\hat{M}$ so that it becomes fundamental. Using lemma 5.1, we can reduce the automaton $\hat{M}$ once more. If $\hat{M}_i \equiv \hat{M}_j$ and $\#(X_{\hat{M}_i}) \geq \#(X_{\hat{M}_j})$, we have $r(\hat{M}_i \oplus \hat{M}_j) \equiv \hat{M}_j$, where $r(\hat{M}_i \oplus \hat{M}_j)$ means any one of reduced forms of $\hat{M}_i \oplus \hat{M}_j$. And if $x^{(i)} \equiv x^{(j)}$ (Y^*) for some $x^{(i)} \in X_{\hat{M}_i}$, $x^{(j)} \in X_{\hat{M}_j}$, then $r(\hat{M}_i) \equiv \hat{M}_j$ (if $\#(X_{\hat{M}_i}) \geq \#(X_{\hat{M}_j})$). Hence we have under the above conditions $r(\hat{M}_i \oplus \hat{M}_j) \equiv \hat{M}_j$. Using the above two facts, we can find a reduced form of $\hat{M}$, and we get finally $\hat{\hat{M}} \equiv \sum_{j=1}^{\tilde{l}} \hat{\hat{M}}_j$ ($\tilde{l} \leq l$), where all $\hat{\hat{M}}_j$ are distinguishable each other.

Since $\hat{\hat{M}}_i$ is strongly connected and is in reduced form, this is fundamental. And both $\hat{\hat{M}}_i$ and $\hat{\hat{M}}_k$ ($i \neq k$) being fundamental, both $X_{\hat{\hat{M}}_i}$ and $X_{\hat{\hat{M}}_k}$ have distinguishable states each other. And $X_{\hat{\hat{M}}_i} \cap X_{\hat{\hat{M}}_k} = \phi$. Hence the finite automata $\hat{\hat{M}}_i$ ($i=1, 2, \dots, \tilde{l}$) are all distinguishable, and thus $\hat{\hat{M}}$ is fundamental. Next we shall show that $\hat{\hat{M}}$ is also reversible. As each $\hat{M}_i$ is strongly connected, $\hat{M}$ is reversible. Obviously the relation $\#(X_{\hat{\hat{M}}}) \leq \#(X_{\tilde{M}})$ for any $\tilde{M} \in \mathfrak{R}[M]$ follows.

We notice that a finite automaton $\hat{\hat{M}}$ is determined uniquely by the above reduction starting from any $\tilde{M} \in \mathfrak{R}[M]$, as we shall now show: Let $\mathfrak{R}(\tilde{M})$ be a reduced form of $\tilde{M}$, and let us assume that for $\tilde{M}^{(1)}$ and $\tilde{M}^{(2)} \in \mathfrak{R}[M]$, $r(\tilde{M}^{(1)}) \equiv r(\tilde{M}^{(2)})$. Here the symbol $\equiv$ means that both sides are equivalent up to an isomorphism. Let $r(\tilde{M}^{(1)}) \equiv \sum_{i=1}^{l_1} \hat{M}_i^{(1)}$ and $r(\tilde{M}^{(2)}) \equiv \sum_{i=1}^{l_2} \hat{M}_i^{(2)}$, where $\hat{M}_i^{(1)}$ and $\hat{M}_i^{(2)}$ are strongly connected automata. Furthermore let us assume that $l_1 \geq l_2$. Then by the assumption it holds that $\hat{M}_i^{(1)} \equiv \hat{M}_j^{(2)}$ for an automaton $\hat{M}_i^{(1)}$ and all automata $\hat{M}_j^{(2)}$ ($j=1, 2, \dots, l_2$). Hence $\tilde{M}^{(1)} \equiv \tilde{M}^{(2)}$, which contradicts the hypothesis. Denoting $\hat{\hat{M}}$ by M^* , we get the conclusion.

PROPOSITION 5.1. *A reversible finite automaton $M_{n,m,p}$ is fundamental if, and only if, it is in the reduced form.*

PROOF. The sufficiency easily follows. The necessity follows from the decomposition into direct sum and from E. Moore's theorem 5 ([21]) applied upon each decomposed component automaton.

PROPOSITION 5.2. *Let $M^{(1)} \in \mathfrak{M}_{n_1, m, p}[X_{M^{(1)}}, Y, Z]$ be a reversible finite automaton, and $M^{(2)} \in \mathfrak{M}_{n_2, m, p}[X_{M^{(2)}}, Y, Z]$ be a finite automaton indistinguishable from $M^{(1)}$. Then there exists a state $x_j^{(2)} \in X_{M^{(2)}}$ such that $x_i^{(1)} \equiv x_j^{(2)}$ for any $x_i^{(1)} \in X_{M^{(1)}}$, i. e., $H_{M^{(1)}}(x_i^{(1)}, y^*) = H_{M^{(2)}}(x_j^{(2)}, y^*)$ for any $y^* \in Y^*$ and any $x_i^{(1)} \in X_{M^{(1)}}$.*

PROOF. As any reversible automaton is identified to one strongly connected automaton by the decomposition theorem, when its initial state is given, the proof of the proposition is the same as in E. Moore's theorem 3 ([21]).

It is shown by a slight modification of E. F. Moore's proof, that his other theorems hold for reversible finite automata.

PROPOSITION 5.3. *If $\mathfrak{R}_{n,m,p}$ is a class of all reversible (n, m, p) finite automata in the reduced form, then there exists an input sequence y^* of length $N^{Nm+2} p^N / N!$ such that $H_M(x, y^*) \equiv H_{\tilde{M}}(\tilde{x}, y^*)$ for any $x \in X_M$ and any $\tilde{x} \in X_{\tilde{M}}$ ($M, \tilde{M} \in \mathfrak{R}_{n,m,p}$). Here the number N is the maximum number of states of component finite automata of direct sum decomposition for any finite automaton belonging to $\mathfrak{R}_{n,m,p}$.*

PROPOSITION 5.4. *Let $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$ be a reversible and fundamental finite automaton. Then there exists an input sequence y^* of length $(N-1)$ such that $x_i \equiv x_j(y^*)$ for any $x_i, x_j \in X$. Here N is the same number as in proposition 5.3 for $M_{n,m,p}$.*

§6. Reduction theorems on finite automata.

The object of the paragraph is to study some reduction theorems on finite automata. If a finite automaton $M_{n,m,p}$ is strongly connected, there exists a finite automaton, the number of states of which is minimum in the set of finite automata indistinguishable from $M_{n,m,p}$. Such an automaton, which is the reduced form of M , is also strongly connected and fundamental.

Many researches have been done by various investigators for seeking the reduced form.

Here we shall consider a finite automaton, not necessarily strongly connected, introduce the concept of reducibility and find the relations between the structure of the original finite automaton and that of the reduced one.

In the following let $x_i = \lambda_x^{-1}(i)$, $y_j = \lambda_y^{-1}(j)$ and $z_k = \lambda_z^{-1}(k)$ for $i \in N_{10}[0, n-1]$, $j \in N_{10}[0, m-1]$ and $k \in N_{10}[0, p-1]$. Before stating the definition of reducibility, we shall introduce a finite automaton $\Theta_{n,m,p}^{(s,t;k)}$ in the set $\mathfrak{M}_{n,m,p}[X, Y, Z]$, where s, t and k are fixed integers such that $0 \leq s \leq m-1$, $0 \leq t \leq n-1$ and $0 \leq k \leq p-1$. The structures of $\Theta_{n,m,p}^{(s,t;k)}$ we define by the following mappings:

$$\begin{aligned} T_{\Theta_{n,m,p}^{(s,t;k)}}(x_i, y_s) &= x_{[k-t]_n}, \\ T_{\Theta_{n,m,p}^{(s,t;k)}}(x_i, y_j) &= x_0 \quad (i \neq t, j \neq s), \end{aligned}$$

and

$$U_{\Theta_{n,m,p}^{(s,t;k)}}(x_i) = z_0 \quad (i=0, 1, \dots, n-1).$$

The transition matrix of $\Theta_{n,m,p}^{(s,t;k)}$ is represented by

$$\tilde{R}_{\Theta_{n,m,p}^{(s,t;k)}} = \begin{pmatrix} & \text{---} (t+1)^{\text{th}} \text{ column} & \\ & 0 & \\ & \vdots & \\ & 0 & \\ 0 & \cdots & 0 & [k-t]_n & 0 & \cdots & 0 \\ & 0 & \vdots & 0 & \\ & & & 0 & \end{pmatrix} \begin{matrix} \\ \\ \\ \\ \\ (s+1)^{\text{th}} \text{ row} . \end{matrix}$$

EXAMPLE. We shall show the transition diagrams and transition matrices of $\Theta_{4,2,2}^{(0,1;2)}$, $\Theta_{4,2,2}^{(1,2;1)}$ and $\Theta_{4,2,2}^{(1,3;0)}$ in the set $\mathfrak{M}_{4,2,2}[X, Y, Z]$, where $X = \{x_0, x_1, x_2, x_3\}$, $Y = \{y_0, y_1\}$ and $Z = \{z_0, z_1\}$.

1° . $\Theta_{4,2,2}^{(0,1;2)}$

$$\tilde{R}_{\Theta_{4,2,2}^{(0,1;2)}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2° . $\Theta_{4,2,2}^{(1,2;1)}$

$$\tilde{R}_{\Theta_{4,2,2}^{(1,2;1)}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3° . $\Theta_{4,2,2}^{(1,3;0)}$

$$\tilde{R}_{\Theta_{4,2,2}^{(1,3;0)}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In the examples just mentioned above we obtain the following relations:

$$\partial_0 \Theta_{4,2,2}^{(0,1;2)} = \Theta_{3,2,2}^{(0,1;2)} + \Theta_{3,2,2}^{(1,1;2)} + \Theta_{3,2,2}^{(0,2;1)} + \Theta_{3,2,2}^{(1,2;1)},$$

$$\partial_1 \Theta_{4,2,2}^{(0,1;2)} = \Theta_{3,2,2},$$

$$\partial_2 \Theta_{4,2,2}^{(0,1;2)} = \Theta_{3,2,2}^{(0,1;2)},$$

$$\partial_3 \Theta_{4,2,2}^{(0,1;2)} = \Theta_{3,2,2}^{(0,1;2)},$$

$$\partial_0 \Theta_{4,2,2}^{(1,2;1)} = \Theta_{3,2,2}^{(0,1;2)} + \Theta_{3,2,2}^{(1,1;0)} + \Theta_{3,2,2}^{(0,2;1)} + \Theta_{3,2,2}^{(1,2;1)},$$

$$\partial_1 \Theta_{4,2,2}^{(1,2;1)} = \Theta_{3,2,2}^{(1,1;0)},$$

$$\partial_2 \Theta_{4,2,2}^{(1,2;1)} = \Theta_{3,2,2},$$

$$\partial_3 \Theta_{4,2,2}^{(1,2;1)} = \Theta_{3,2,2}^{(1,2;1)},$$

$$\partial_0 \Theta_{4,2,2}^{(1,3;0)} = \Theta_{3,2,2}^{(0,2;1)} + \Theta_{3,2,2}^{(1,2;2)} + \Theta_{3,2,2}^{(0,1;2)} + \Theta_{3,2,2}^{(1,1;2)},$$

$$\partial_1 \Theta_{4,2,2}^{(1,3;0)} = \Theta_{3,2,2}^{(1,2;0)},$$

$$\partial_2 \Theta_{4,2,2}^{(1,3;0)} = \Theta_{3,2,2}^{(1,2;0)},$$

$$\partial_3 \Theta_{4,2,2}^{(1,3;0)} = \Theta_{3,2,2}^{(1,2;0)},$$

In general, the finite automaton $\Theta_{n,m,p}^{(s,t;k)}$ has the following relations (letting $[k-t]_n \equiv r$);

$$\Theta_{n,m,p}^{(s,t;k)} \equiv \Theta_{n,m,p} \quad \text{when } r=0,$$

$$\partial_i \Theta_{n,m,p}^{(s,t;k)} \equiv \Theta_{n,m,p} \quad \text{when } i=t,$$

$$\partial_0 \Theta_{n,m,p}^{(s,t;k)} = \begin{cases} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \Theta_{n-1,m,p}^{(i,j;[2j]_{n-1})} & (\text{when } r=0 \text{ or } t), \\ \Theta_{n-1,m,p}^{(s,t-1;[k-1]_{n-1})} + \sum_{\substack{i=0 \\ i \neq s}}^{m-1} \Theta_{n-1,m,p}^{(i,t-1;[2(t-1)]_{n-1})} + \sum_{\substack{i=0 \\ j \neq t-1}}^{m-1} \sum_{j=0}^{n-1} \Theta_{n-1,m,p}^{(i,j;[2j]_{n-1})} & (\text{when } r \neq 0, t), \end{cases}$$

$$\partial_i \Theta_{n,m,p}^{(s,t;k)} = \begin{cases} \Theta_{n-1,m,p}^{(s,t;[r+t-1]_{n-1})} & (i > t, r > i), \\ \Theta_{n-1,m,p}^{(s,t;[2t]_{n-1})} & (i > t, r = i), \\ \Theta_{n-1,m,p}^{(s,t;k)} & (i > t, r < i), \\ \Theta_{n-1,m,p}^{(s,t-1;[r+t-2]_{n-1})} & (0 < i < t, r > i), \\ \Theta_{n-1,m,p}^{(s,t-1;[2(t-1)]_{n-1})} & (0 < i < t, r = i), \\ \Theta_{n-1,m,p}^{(s,t-1;[r+t-1]_{n-1})} & (0 < i < t, r < i). \end{cases}$$

In [21], E. F. Moore showed the existence of a reduced form of a strongly connected finite automaton and its uniqueness. Using this fact we presented

a reduced form of reversible automaton and its uniqueness (up to an isomorphism). Now we shall consider reducibility about a more general finite automaton.

THEOREM 6.1. *If in a finite automaton $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$ the state x_j is indistinguishable from the state x_k ($j < k$), then the automaton $M_{n,m,p}$ is indistinguishable from the automaton $\partial_j M_{n,m,p} + \sum_{(s,t)} \Theta_{n-1,m,p}^{(s,t;k-1)}$, where the notation $\sum_{(s,t)}$ means a summation over all the pairs $(\tilde{x}, y)$ of the elements in $\partial_j M_{n,m,p}$ corresponding to the pairs (x, y) such that $T_{M_{n,m,p}}(x, y) = x_j(x \neq x_j)$.*

In order to grasp the meaning of the theorem precisely, we shall explain an example before proving the theorem.

Let

$$X = \{x_0, x_1, x_2\}, \quad Y = \{y_0, y_1\} \quad \text{and} \quad Z = \{z_0, z_1\}.$$

Let us consider the finite automaton $M_{3,2,2}^{(1)}$ in 2.6. The automaton $M_{3,2,2}^{(1)}$ is strongly connected and the states x_0 and x_2 are indistinguishable each other.

In the transition diagram of $M_{3,2,2}^{(1)}$ let us get rid of the symbol $\boxed{x_0/z_0}$ and the associated arrows.

Then we get the following diagram;

Next let the unsettled arrows from the states x_1 and x_2 be directed to the state x_2 , indistinguishable from the excluded state x_0 , and we get the diagram:

Lastly renumbering the states x_1 and x_2 as x_0 and x_1 , we get the following diagram;

The finite automaton thus obtained is indistinguishable from the original automaton. This is nothing but the finite automaton

$$\partial_0 M_{3,2,2}^{(1)} + \Theta_{2,2,2}^{(1,0;1)} + \Theta_{2,2,2}^{(1,1;1)}.$$

Since

$$\Theta_{2,2,2}^{(1,1;1)} \equiv \Theta_{3,2,2}, \quad \partial_0 M_{3,2,2}^{(1)} + \Theta_{2,2,2}^{(1,0;1)} + \Theta_{2,2,2}^{(1,1;1)} = \partial_0 M_{3,2,2}^{(1)} + \Theta_{2,2,2}^{(1,0;1)}.$$

Let us show that the transition diagram of $\partial_0 M_{3,2,2}^{(1)} + \Theta_{2,2,2}^{(1,0;1)}$ is equal to the above diagram. By the definition of 0-face operator ∂_0 , we can get the transition diagram of $\partial_0 M_{3,2,2}^{(1)}$ as follows;

Firstly we get rid of the symbol $\overline{x_0/z_0}$ and the associated arrows, and the unsettled arrows of the remaining symbols are directed to the states from which the arrows started. Lastly the state-numbers are renumbered without change of orders. The following diagrams show the above steps.

Next let us consider the transition diagram of the automaton $\Theta_{2,2,2}^{(1,0;1)}$. By the definition of $\Theta_{2,2,2}^{(1,0;1)}$ we get the following diagram:

The sum $\partial_0 M_{3,2,2}^{(1)} + \Theta_{2,2,2}^{(1,0;1)}$ can be constructed by the definition in §2.8, the transition diagram of which is given as follows;

The diagram is identical with the one constructed before. From these considerations we may describe how to write the transition diagram of $\partial_j M_{n,m,p} + \sum_{(s,t)} \Theta_{n,m,p}^{(s,t;k-1)}$, which is stated as follows:

Firstly we get rid of the symbol $\overline{x_j/U_M(x_j)}$ and the associated arrows in the transition diagram of $M_{n,m,p}$, direct the unsettled arrows of the remaining symbols to the state x_k , and lastly we renumber the state-numbers $\{x_0, x_1, \dots, x_{j-1}, x_{j+1}, x_{n-1}\}$ as $\{x_0, x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_{n-2}\}$. Then we obtain the desired diagram. Notice that only $\partial_j M_{n,m,p}$ is not obviously indistinguishable from $M_{n,m,p}$.

PROOF of THEOREM 6.1. For simplicity let us put $M^{(1)} = M$ and $M^{(2)} = \partial_j M_{n,m,p} + \sum_{(s,t)} \Theta_{n,m,p}^{(s,t;k-1)}$. Let $x_i^{(1)}$ and $x_i^{(2)}$ be elements of the sets $X_{M^{(1)}}$ ($\equiv X$) and $X_{M^{(2)}}$ ($\equiv X^{(1)}$). Firstly we shall introduce the notations $\langle i \rangle_{j,k}$ and $\langle\langle i \rangle\rangle_{k-1, j+1}$ as follows:

$$\langle i \rangle_{j,k} = \begin{cases} i & (i < j), \\ k-1 & (i = j), \\ i-1 & (i > j). \end{cases}$$

$$\langle\langle i \rangle\rangle_{k-1, j+1} = \begin{cases} j & (i < k-1), \\ j & (i = k-1), \\ j+1 & (j > k-1). \end{cases}$$

Then it is sufficient to prove that there exists a state in $X_{M^{(2)}}$ indistinguishable from each state in $X_{M^{(1)}}$ and that at the same time there exists a state in $X_{M^{(1)}}$ indistinguishable from each state in $X_{M^{(2)}}$. We shall show that $x_i^{(1)} \equiv x_{\langle i \rangle_{j,k}}^{(2)}$ for each $x_i^{(1)} \in X_{M^{(1)}}$, and $x_i^{(2)} \equiv x_{\langle\langle i \rangle\rangle_{k-1, j+1}}^{(1)}$ for each state $x_i^{(2)} \in X_{M^{(2)}}$. If $i \neq j$ and $T_{M^{(1)}}(x_i^{(1)}, y_k) = x_{\nu_i^{(1)}(k)}^{(1)}$, then we get $T_{M^{(2)}}(x_{\langle i \rangle_{j,k}}^{(2)}, y_k) = x_{\nu_{\langle i \rangle_{j,k}}^{(2)}(k)}^{(2)} = x_{\nu_i^{(1)}(k)}^{(1)}$ by the construction of the finite automaton $\partial_j M_{n,m,p} + \sum_{(s,t)} \Theta_{n,m,p}^{(s,t;k-1)}$.

And if we take an input sequence y^* by which the state sequence $\{x_{\sigma_s-1}^{(1)}\}$ ($s=1, 2, \dots, l$, $(\sigma_s \neq j (s=0, 1, \dots, l-1); \sigma_0 \equiv i)$) is obtained in the automaton $M^{(1)}$, then it holds for such a sequence that $H_{M^{(1)}}(x_i^{(1)}, y^*) = H_{M^{(2)}}(x_{\langle i \rangle_{j,k}}^{(2)}, y^*)$, since

$U_{M^{(1)}}(x_i^{(1)}) = U_{M^{(2)}}(x_{<i>j,k}^{(2)})$ for all i .

If $T_{M^{(1)}}(x_i^{(1)}, y) = x_j^{(1)}$, then we get $T_{M^{(2)}}(x_{<i>j,k}^{(2)}, y) = x_{k-1}^{(2)}$. Hence, in order to prove that $x_i^{(1)} \equiv x_{<i>j,k}^{(2)}$ for any $x_i^{(1)} \in X_{M^{(1)}}$, it is sufficient to prove that $x_j^{(1)} \equiv x_{k-1}^{(2)}$.

By the construction of the automaton $\partial_i M_{n,m,p} + \sum_{(s,t)} \Theta_{n-1,m,p}^{(s,t;k-1)}$ the arrows directed from the symbol of any states $x_i^{(1)}$ (not x_j) to the symbol of $x_j^{(1)}$ in the transition diagram of $M_{n,m,p}$ are altered into those directed from the symbols $x_{<i>j,k}^{(2)}$ to the symbol of $x_{k-1}^{(2)}$ in the transition diagram of $M^{(2)}$. And the symbol of $x_j^{(1)}$ and the associated arrows are excluded. From the assumption $x_j^{(1)} \equiv x_k^{(1)}$ and the above facts, combined with the result of lemma 4.1, we have

$$\begin{aligned} x_j^{(1)} &\equiv x_k^{(1)} && (\text{in } M^{(1)}), \\ x_k^{(1)} &\equiv x_{<k>j,k}^{(2)} && (\text{between } M^{(1)} \text{ and } M^{(2)}). \end{aligned}$$

Since $\langle k \rangle_{j,k} = k-1$, and $U_{M^{(1)}}(x_j^{(1)}) = U_{M^{(1)}}(x_k^{(1)}) = U_{M^{(2)}}(x_{k-1}^{(2)})$, we have the conclusion.

Similarly we get the relation $x_i^{(2)} \equiv x_{<i>j,k-1,j+1}^{(1)}$ for each state $x_i^{(2)} \in X_{M^{(2)}}$ from the fact $x_j^{(1)} \equiv x_{k-1}^{(2)}$.

As a simple application of the theorem we get

THEOREM 6.2. *If the finite automaton $M_{n,m,p}$ is strongly connected, then, under the assumptions of theorem 6.1, the reduced automaton $M_{n,m,p} + \sum_{(s,t)} \Theta_{n-1,m,p}^{(s,t;k-1)}$ is also strongly connected.*

REMARK. Let a subset $\{x_{\mu_i}\}$ ($i=1, 2, \dots, l$; $\mu_1 < \mu_2 < \dots < \mu_l$) of the state set X of a finite automaton $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$ be an indistinguishable set, i.e., $x_{\mu_1} \equiv x_{\mu_2} \equiv \dots \equiv x_{\mu_l}$. Then the finite automaton $M_{n,m,p}$ is indistinguishable from the finite automaton

$$\begin{aligned} (*) \partial_{\mu_1} \partial_{\mu_2} (\dots (\partial_{\mu_{l-1}} M_{n,m,p} + \sum_{(s^{(1)}, t^{(1)})} \Theta_{n-1,m,p}^{(s^{(1)}, t^{(1)}; \mu_{l-1})}) + \sum_{(s^{(2)}, t^{(2)})} \Theta_{n-2,m,p}^{(s^{(2)}, t^{(2)}; \mu_{l-2})}) \\ + \dots + \dots) + \sum_{(s^{(l-1)}, t^{(l-1)})} \Theta_{n-(l-1),m,p}^{(s^{(l-1)}, t^{(l-1)}; \mu_{l-(l-1)})}, \end{aligned}$$

where the symbol $\sum_{(s^{(i)}, t^{(i)})}$ means a summation over all the pairs $(\tilde{x}, y)$ of the elements in the finite automaton

$$\begin{aligned} \partial_{\mu_{l-i}} (\partial_{\mu_{l-i+1}} (\dots (\partial_{\mu_{l-2}} (\partial_{\mu_{l-1}} M_{n,m,p} + \sum_{(s^{(1)}, t^{(1)})} \Theta_{n-1,m,p}^{(s^{(1)}, t^{(1)}; \mu_{l-1})}) \\ + \sum_{(s^{(2)}, t^{(2)})} \Theta_{n-2,m,p}^{(s^{(2)}, t^{(2)}; \mu_{l-2})}) + \dots + \dots) + \sum_{(s^{(i-1)}, t^{(i-1)})} \Theta_{n-i+1,m,p}^{(s^{(i-1)}, t^{(i-1)}; \mu_{l-i+1})} \end{aligned}$$

corresponding to the pairs (x, y) such that

$$\begin{aligned} \partial_{\mu_{l-i+1}}(\partial_{\mu_{l-i+2}}(\cdots(\partial_{\mu_{l-2}}(\partial_{\mu_{l-1}}M_{n,m,p} + \sum_{(s^{(1)}, t^{(1)})} \Theta_{n-1,m,p}^{(s^{(1)}, t^{(1)}; \mu_{l-1})}) \\ + \cdots) + \sum_{(s^{(i-1)}, t^{(i-1)})} \Theta_{n-1,m,n}^{(s^{(i-1)}, t^{(i-1)}; \mu_{l-i+1})})(x, y) = x_{\mu_i} \end{aligned}$$

for $x(\neq x_i) \in X$ and $y \in Y$, since the automaton $(*)$ is constructed by iterated use of theorem 6.1. As corollaries of the above reduction theorem we obtain the following propositions.

PROPOSITION 6.1. *Let $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$. A necessary and sufficient condition that $\partial_j M_{n,m,p}$ is a reduced finite automaton of $M_{n,m,p}$ under the same conditions of theorem 6.1 is that the state x_j has not the inputs such that $T_M(x, y) = x_j$ for each $x \in X$.*

PROOF. We get from theorem 6.1 and $\partial_j M_{n,m,p} \equiv M_{n,m,p}$ the identity $\sum_{(s,t)} \Theta_{n-1,m,p}^{(s,t;k-1)} = \Theta_{n-1,m,p}$, which shows non-existence of the inputs mentioned in the proposition. This proves the necessity. Conversely, if the state x_j has not the inputs mentioned in the proposition, then evidently the above identity holds. Hence we get the relation $\partial_j M_{n,m,p} + \sum_{(s,t)} \Theta_{n-1,m,p}^{(s,t;k-1)} = \partial_j M_{n,m,p}$, thus $\partial_j M_{n,m,p}$ is indistinguishable from the automaton $M_{n,m,p}$.

PROPOSITION 6.2. *If the finite automaton $M_{n,m,p}$ is strongly connected, then there does not exist a reduced finite automaton of $M_{n,m,p}$ of the form $\partial_j M_{n,m,p}$.*

§7. The existence of Scheffer stroke class in the finite automata.

In previous paragraphs we discussed the structures, the reducibilities, the length of the experiments to distinguish the states and the properties of the composition of finite automata. In this paragraph we will define the Scheffer stroke class and pseudo Scheffer stroke class in $\mathfrak{M}_{n,m,p}$ and show that any finite automata in $\mathfrak{M}_{n,m,p}[X, Y, Z]$ can be represented by combining some finite automata that belong to the class. A method of synthesizing a finite automaton is obtained at the same time. The term ‘‘Scheffer stroke’’ was taken from the one used in many-valued logic. The construction of Scheffer stroke function in many-valued logic was studied by several authors. The method by which the above authors constructed the Scheffer stroke functions was the one such that they constructed the singulary function,

using diagonal method applied on the truth-table. Such a method can not be used in the theory of finite automata, since there exist no finite automata the induced functions of which have such forms as the above singular functions. In order to find the Scheffer stroke class, we, therefore, use another method and show the existence of the class, remarking that there exist some elements in $\mathfrak{M}_{n,m,p}$ to which it seems that they cannot be constructed by combining copies of only one (definite) element in $\mathfrak{M}_{n,m,p}$, if $n > 2$.

Firstly we shall introduce a new notation of finite automaton. If two mappings $T_M: X \times Y \rightarrow X$ and $U_M: X \rightarrow Z$, which constitute a finite automaton $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$, are such that $T_M(x_i, y_j) = x_{\nu_i(j)}$ and $U_M(x_i) = z_{l_i}$ ($i=0, 1, \dots, n-1$; $j=0, 1, \dots, m-1$), we can write the automaton $M_{n,m,p}$ as $M_{n,m,p} = \sum_{i=0}^{n-1} \Phi_{n,m,p}^{(i; \nu_i(0), \nu_i(1), \dots, \nu_i(m-1); l_i)}$ by the definition of sum-automaton, where each term $\Phi_{n,m,p}^{(i; \nu_i(0), \nu_i(1), \dots, \nu_i(m-1); l_i)}$ ($\equiv \Phi^{(i)}$) is a finite automaton whose mappings $T_{\Phi^{(i)}}$ and $U_{\Phi^{(i)}}$ are defined by $T_{\Phi^{(i)}}(x_i, y_j) = x_{\nu_i(j)}$ (i : fixed; $j=0, 1, \dots, m-1$), $T_{\Phi^{(i)}}(x_k, y_j) = x_0$ ($k \neq i$; $j=0, 1, \dots, m-1$), $U_{\Phi^{(i)}}(x_i) = z_{l_i}$ and $U_{\Phi^{(i)}}(x_k) = z_0$ ($k \neq i$).

This notation is convenient for us to treat the general case. We shall show an example of the notation.

EXAMPLE 7.1. Let $X = \{x_0, x_1, x_2\}$, $Y = \{y_0, y_1\}$ and $Z = \{z_0, z_1\}$. Let the transition diagram and the transition matrix of a finite automaton $M_{3,2,2}$ be as follows:

Now we construct the following three transition matrices with respect to the given matrix $R_{M_{3,2,2}}$:

$$R^{(1)} = \begin{pmatrix} x_1 & x_0 & x_0 \\ x_0 & y_0 & x_0 \\ z_0 & z_0 & z_0 \end{pmatrix}, \quad R^{(2)} = \begin{pmatrix} x_0 & x_2 & x_0 \\ x_0 & x_0 & x_0 \\ z_0 & z_1 & z_0 \end{pmatrix}, \quad R^{(3)} = \begin{pmatrix} x_0 & x_0 & x_1 \\ x_0 & x_0 & x_0 \\ z_0 & z_0 & z_0 \end{pmatrix}.$$

The constructions of $R^{(i)}$ from $R_{M_{3,2,2}}$ will be obvious. The associated transition diagram are written as follows:

These are nothing but the transition diagrams of the finite automata $\Phi_{3,2,2}^{(0;1,0;0)}$, $\Phi_{3,2,2}^{(1;2,0;1)}$ and $\Phi_{3,2,2}^{(2;1,0;0)}$, respectively.

Thus we have

$$R_{\Phi_{3,2,2}^{(0;1,0;0)}} \equiv R^{(1)}, \quad R_{\Phi_{3,2,2}^{(1;2,0;1)}} \equiv R^{(2)} \quad \text{and} \quad R_{\Phi_{3,2,2}^{(2;1,0;0)}} \equiv R^{(3)}.$$

Then, by the definition of sum-automata, the relation

$$M_{3,2,2} \equiv \Phi_{3,2,2}^{(0;1,0;0)} + \Phi_{3,2,2}^{(1;2,0;0)} + \Phi_{3,2,2}^{(2;1,0;0)}$$

holds.

Let us give two finite automata defined by

$$M_{n,m,p}^{(1)} \equiv \sum_{i=0}^{n-1} \Phi_{n,m,p}^{(i; \nu_i^{(1)}(0), \nu_i^{(1)}(1), \dots, \nu_i^{(1)}(m-1); i_i^{(1)})},$$

$$M_{n,m,p}^{(2)} \equiv \sum_{i=0}^{n-1} \Phi_{n,m,p}^{(i; \nu_i^{(2)}(0), \nu_i^{(2)}(1), \dots, \nu_i^{(2)}(m-1); i_i^{(2)})},$$

which belong to the same set $\mathfrak{M}_{n,m,p}[X, Y, Z]$. By the definition of the $\wedge$ -product automaton we have the relation

$$(M_{n,m,p}^{(1)} \wedge M_{n,m,p}^{(2)})_{n,m,p} = \sum_{i=0}^{n-1} \Phi_{n,m,p}^{(i; \nu_i^{(1)}(0), \nu_i^{(2)}(0), \nu_i^{(1)}(1), \nu_i^{(2)}(1), \dots, \nu_i^{(1)}(m-1), \nu_i^{(2)}(m-1); i_i^{(1)})}.$$

Clearly the $\wedge$ -product automaton $(M_{n,m,p}^{(1)} \wedge M_{n,m,p}^{(2)})_{n,m,p}$ belongs to the same set $\mathfrak{M}_{n,m,p}[X, Y, Z]$. In considering the Scheffer stroke class the $\wedge$ -product operation plays an important role.

Let $\mathfrak{S}_{n,m,p}$ be a subset of $\mathfrak{M}_{n,m,p}[X, Y, Z]$. The set $\mathfrak{S}_{n,m,p}$ is called the Scheffer-stroke class if the state-transition $T_M: X \times Y \rightarrow X$ of each finite automaton $M_{n,m,p}$ in $\mathfrak{M}_{n,m,p}$ can be represented by the state transition of a finite automaton constructed by $\wedge$ -operation from a finite number of elements in $\mathfrak{S}_{n,m,p}$. It should be noted that in this definition we treat only the state transition from $X \times Y$ into X , the output function not being considered.

Next let us define the selection mapping $\diamond: X \times X \rightarrow X$ defined by $x_i \diamond x_j$

$=x_{\max(i,j)}$ ($i, j=0, 1, \dots, n-1$). We shall call a set $\mathfrak{S}_{n,m,p}$ a pseudo Scheffer stroke class in $\mathfrak{M}_{n,m,p}[X, Y, Z]$ if the set $\mathfrak{S}_{n,m,p}$ becomes the Scheffer stroke class in $\mathfrak{M}_{n,m,p}[X, Y, Z]$ when we admit the operation $\diamond$. The main object of this paragraph is to show the existence of Scheffer stroke class in $\mathfrak{M}_{n,m,p}$. To prove this, we shall at first prove the existence of the pseudo Scheffer stroke class, and then by dint of it obtain the Scheffer stroke class.

Let us define the set $\mathfrak{M}_{n,m,p}^{(i_1, i_2, \dots, i_m; j)}$ to be the set of finite automata in $\mathfrak{M}_{n,m,p}[X, Y, Z]$ such that

$$\#\{T_M(x, y_k): x \in X\} = i_{k+1} \quad (k=0, 1, \dots, m-1)$$

and

$$\#\{U_M(x): x \in X\} = j,$$

where $\#\{ \}$ means the number of elements contained in the set $\{ \}$. Then the set $\mathfrak{M}_{n,m,p}[X, Y, Z]$ can be represented as follows:

$$\mathfrak{M}_{n,m,p}[X, Y, Z] = \bigcup_{\substack{1 \leq i_1, i_2, \dots, i_m \leq n \\ 1 \leq j \leq p}} \mathfrak{M}_{n,m,p}^{(i_1, i_2, \dots, i_m; j)}.$$

We shall firstly seek the Scheffer stroke class $\mathfrak{S}_{n,m,p}^{(i_1, i_2, \dots, i_m; j)}$ of $\mathfrak{M}_{n,m,p}^{(i_1, i_2, \dots, i_m; j)}$, since the union of the Scheffer stroke class of all sets $\mathfrak{M}_{n,m,p}^{(i_1, i_2, \dots, i_m; j)}$ ($1 \leq i_1, i_2, \dots, i_m \leq n, 1 \leq j \leq p$) is also the Scheffer stroke class of $\mathfrak{M}_{n,m,p}[X, Y, Z]$.

The principal result of the paragraph is as follows:

THEOREM 7.1. *Let two finite automata $W_{n,m,p}$ and $T_{n,m,p}^{(i;j,k)}$ ($\in \mathfrak{M}_{n,m,p}[X, Y, Z]$; $j < k$) be defined by the mappings:*

$$T_W(x_i, y_j) = x_{[\max(i,j)-1]_n},$$

$$T_{T^{(i;j,k)}}(x, y) = \begin{cases} x_l & (x=x_l, y=y_i, l=0, 1, 2, \dots, \tilde{j}, \dots, \tilde{k}, \dots, n-1), \\ x_k & (x=x_j, y=y_i), \\ x_j & (x=x_k, y=y_i), \\ x_0 & (y \neq y_i). \end{cases}$$

, i.e., $W_{n,m,p}$ and $T_{n,m,p}^{(i;j,k)}$ be defined by the modified transition matrices:

$$\tilde{K}_W = \begin{pmatrix} n-1 & 0 & 1 & 2 & 3 & \dots & m-2 & \dots & n-3 & n-2 \\ 0 & 0 & 1 & 2 & 3 & \dots & m-2 & \dots & n-3 & n-2 \\ 1 & 1 & 1 & 2 & 3 & \dots & m-2 & \dots & n-3 & n-2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m-2 & m-2 & m-2 & m-2 & m-2 & \dots & m-2 & \dots & n-3 & n-2 \\ * & * & * & * & * & \dots & * & * & * & * \end{pmatrix},$$

$$\tilde{R}_{T^{(i;j,k)}} = \begin{matrix} & \begin{matrix} j+1^{\text{th}} \text{ column} & k+1^{\text{th}} \text{ column} \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 2 & \cdots & j-1 & k & j+1 & \cdots & k-1 & j & k+1 & \cdots & n-1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & \cdots & * & * & * & \cdots & * & * & * & \cdots & * \end{pmatrix} & \begin{matrix} i+1^{\text{th}} \text{ row} \end{matrix} \end{matrix},$$

where $*$ represents any number in $N_{10}[0, p-1]$. Then the set of two finite automata $W_{n,m,p}$ and $T_{n,m,p}^{(0;n-2,n-1)}$ is a Scheffer stroke class in $\mathfrak{M}_{n,m,p}[X, Y, Z]$.

In order to prove this theorem, we shall use the following two lemmas.

LEMMA 7.1. Let a finite automaton $N_{n,m,p}^{(i)}$ ($\in \mathfrak{M}_{n,m,p}[X, Y, Z]$) be defined by the mapping

$$\begin{cases} T_{N^{(i)}}(x_j, y_i) = x_{[j-1]_n} & (j=0, 1, \dots, n-1), \\ T_{N^{(i)}}(x_j, y_k) = x_0 & (k \neq i; j=0, 1, \dots, n-1) \end{cases}$$

i.e., $N_{n,m,p}^{(i)}$ be defined by the modified matrix:

$$\tilde{R}_{N^{(i)}} = \begin{matrix} \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ n-1 & 0 & 1 & 2 & \cdots & n-3 & n-2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ * & * & * & * & \cdots & * & * \end{pmatrix} & \begin{matrix} i+1^{\text{th}} \text{ row} \end{matrix} \end{matrix}.$$

Let $N_{n,m,p} \equiv \sum_{i=0}^{m-1} N_{n,m,p}^{(i)}$ and $T_{n,m,p}^{<j,k>} \equiv \sum_{i=1}^{m-1} T_{n,m,p}^{(i;j,k)}$. Then the set $\{N_{n,m,p}, T_{n,m,p}^{<0,1>}, T_{n,m,p}^{<1,2>}, \dots, T_{n,m,p}^{<n-2,n-1>}\}$ is a pseudo Scheffer stroke class in $\bigcup_{j=1}^p \mathfrak{M}_{n,m,p}^{(n,n,\dots,n;j)}$.

LEMMA 7.2. Let a finite automaton $D_{n,m,p}^{(i)}$ ($\in \mathfrak{M}_{n,m,p}[X, Y, Z]$) be defined by the mapping:

$$\begin{cases} T_{D^{(i)}}(x_0, y_i) = x_1, \\ T_{D^{(i)}}(x_j, y_i) = x_j & (j=1, 2, \dots, n-1), \\ T_{D^{(i)}}(x_j, y_k) = x_0 & (k \neq i; j=0, 1, \dots, n-1), \end{cases}$$

i.e., $D_{n,m,p}^{(i)}$ be defined by the modified transition matrix

$$\tilde{R}_D^{(i)} \equiv \begin{pmatrix} 0 & 0 & 0 \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 \dots & 0 & 0 \\ 1 & 1 & 2 \dots & n-2 & n-1 \\ 0 & 0 & 0 \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 \dots & 0 & 0 \\ * & * & * \dots & * & * \end{pmatrix} i+1^{\text{th}} \text{ row}.$$

$$T_M(x_k, y_i) = x_0 \quad (j \neq i-1, \ i: \text{fixed}; \ k=0, 1, \dots, n-1)$$

and

$$\tilde{R}_M \equiv \begin{pmatrix} 0 & 0 \dots 0 \\ 0 & 0 \dots 0 \\ \vdots & \vdots \\ 0 & 0 \dots 0 \\ \# & \# \dots \# \\ 0 & 0 \dots 0 \\ \vdots & \vdots \\ 0 & 0 \dots 0 \\ * & * \dots * \end{pmatrix},$$

Then the union of the set $\{N_{n,m,p}^{(i)}, T_{n,m,p}^{(i;0,1)}, T_{n,m,p}^{(i;1,2)}, \dots, T_{n,m,p}^{(i;n-2,n-1)}, D_{n,m,p}^{(i)}\}$ and the set $\mathbb{M}_{n,m,p}^{(0)(1,1,\dots,1,\overset{i-1}{i_k+1},1,\dots,1;j)}$ is a pseudo Scheffer stroke class in $\mathbb{M}_{n,m,p}^{(0)(1,1,\dots,1,\overset{i-1}{i_k},1,\dots,1;j)}$ ($i_k < n$).

We shall prove the above two lemmas.

PROOF of LEMMA 7.1. Let us take any finite automaton $M_{n,m,p} \in \mathfrak{M}_{n,m,p}^{(n,n,\dots,n;j)}$, and let us assume $M_{n,m,p} = \sum_{i=0}^{n-1} \Phi_{n,m,p}^{(i; \nu_i(0), \nu_i(1), \dots, \nu_i(m-1); i)}$, i.e.,

$$\tilde{R}_{\mathcal{M}_n, m, p} \equiv \begin{bmatrix} \nu_0(0) & \nu_1(0) & \cdots & \nu_{n-1}(0) \\ \nu_0(1) & \nu_1(1) & \cdots & \nu_{n-1}(1) \\ \vdots & \vdots & \ddots & \vdots \\ \nu_0(m-1) & \nu_1(m-1) & \cdots & \nu_{n-1}(m-1) \\ l_0 & l_1 & \cdots & l_{n-1} \end{bmatrix}$$

$$(\nu_i(j) \in N_{10}[0, n-1], l_k \in N_{10}[0, p-1]; i, k=0, 1, \dots, n-1; l=0, 1, \dots, m-1).$$

To avoid the complexity of the notation we shall use in the proof the symbol $M_{n,m,p}$ itself in place of the mapping $T_{M_{n,m,p}}$ i.e. let any one of finite auto-

mata constructed by the mapping $M_{n,m,p}: X \times Y \rightarrow X$ be $M_{n,m,p}$.

Let us introduce the mapping $E_{n,m,p}^{(i,j;\nu_j^{(i)}) *}) : X \times Y \rightarrow X$ defined by $E_{n,m,p}^{(i,j;\nu_j^{(i)}) *})(x, y) = \begin{cases} x_{\nu_j^{(i)}} & (x = x_j, y = y_i), \\ x_0 & (\text{otherwise}), \end{cases}$ and let us construct the mapping $\phi_{n,m,p}^{(i)} : X \times Y \rightarrow X$ defined by $\phi_{n,m,p}^{(i)}(x, y) = \bigtriangleup_{j=0,1,\dots,n-1} E_{n,m,p}^{(i,j;\nu_j^{(i)})}(x, y) \quad (x \in X, y \in Y)$.

Then, in the above interpretation, we get the transition matrices of the automata $E_{n,m,p}^{(i,j;\nu_j^{(i)})}$ and $\phi_{n,m,p}^{(i)}$;

$$\tilde{R}_{E_{n,m,p}^{(i,j;\nu_j^{(i)})}} \equiv \begin{pmatrix} 0 & 0 \cdots 0 & 0 & 0 \cdots 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 \cdots 0 & 0 & 0 \cdots 0 \\ 0 & 0 \cdots 0 & \nu_j(i) & 0 \cdots 0 \\ 0 & 0 \cdots 0 & 0 & 0 \cdots 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 \cdots 0 & 0 & 0 \cdots 0 \\ * & * \cdots * & * & * \cdots * \end{pmatrix},$$

$$\tilde{R}_{\phi_{n,m,p}^{(i)}} \equiv \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \nu_0(i) & \nu_1(i) & \cdots & \nu_{n-1}(i) \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ * & * & \cdots & * \end{pmatrix}.$$

So defined, we have the relation $M_{n,m,p}(x, y) = \bigtriangleup_{i=0,1,\dots,m-1} \phi_{n,m,p}^{(i)}(x, y)^{**}$.

Let us define the mapping $\Omega_{n,m,p}^{(i,1)} : X \times Y \rightarrow X$ by

$$\Omega_{n,m,p}^{(i,1)}(x, y) = \begin{cases} \phi_{n,m,p}^{(i)}(x, y) & (x = x_0, y = y_i), \\ x_j & (x = x_j, y = y_i; j = 1, 2, \dots, n-1), \\ x_0 & (\text{otherwise}). \end{cases}$$

Then

$$\tilde{R}_{\Omega_{n,m,p}^{(i,1)}} \equiv \begin{pmatrix} 0 & 0 & 0 \cdots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots 0 & 0 \\ \nu_0(i) & 1 & 2 \cdots n-1 & 0 \\ 0 & 0 & 0 \cdots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots 0 & 0 \\ * & * & * \cdots * & * \end{pmatrix}.$$

*) i.e., $T_{E_{n,m,p}^{(i,j;\nu_j^{(i)})}} : X \times Y \rightarrow X$.

**) In considering the mapping T_M , we may omit the last row of $\tilde{R}_M$, since the row consists of $\{U_M(x) \mid x \in X\}$.

We obtain from the definition of $N_{n,m,p}^{(i)}$ the following relation:

$$N_{n,m,p}^{(i)} \wedge N_{n,m,p}^{(i)}(x, y) = \begin{cases} x_{\lfloor j-2 \rfloor n} & (x=x_j, y=y_i; j=0, 1, \dots, n-1), \\ x_0 & (\text{otherwise}), \end{cases}$$

i.e.,

$$\tilde{R}_{N_{n,m,p}^{(i)} \wedge N_{n,m,p}^{(i)}} = \begin{pmatrix} 0 & 0 & 0 & 0 \dots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \dots 0 & 0 \\ n-2 & n-1 & 0 & 1 \dots n-3 & 0 \\ 0 & 0 & 0 & 0 \dots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \dots 0 & 0 \\ * & * & * & * \dots * & * \end{pmatrix} i+1^{\text{th}} \text{ row}.$$

Iterating these processes, we get finally the state transition:

$$\underbrace{N_{n,m,p}^{(i)} \wedge N_{n,m,p}^{(i)} \wedge \dots \wedge N_{n,m,p}^{(i)}}_{n-\nu_0(i)}(x, y) = \begin{cases} x_{\lfloor j-n+\nu_0(i) \rfloor n} & (x=x_j, y=y_i; j=0, 1, \dots, n-1), \\ x_0 & (\text{otherwise}), \end{cases}$$

which we shall write $[N_{n,m,p}^{(i)}]^{n-\nu_0(i)}(x, y)$ for the sake of brevity. Then we have

$$\tilde{R}_{[N_{n,m,p}^{(i)}]^{n-\nu_0(i)}} = \begin{pmatrix} 0 & 0 & 0 & \dots 0 & 0 \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots 0 & 0 \dots & 0 \\ \nu_0(i) & \lfloor \nu_0(i)+1 \rfloor n & \lfloor \nu_0(i)+2 \rfloor n & \dots 0 & 1 \dots \lfloor \nu_0(i)-1 \rfloor n & 0 \\ 0 & 0 & 0 & \dots 0 & 0 \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots 0 & 0 \dots & 0 \\ * & * & * & \dots & * \dots & * \end{pmatrix}.$$

On the other hand, we have $\phi_{n,m,p}^{(i)}(x_0, y_i) = x_{\nu_0(i)}$, which yields the relation $\mathcal{Q}_{n,m,p}^{(i,1)}(x_0, y_i) = [N_{n,m,p}^{(i)}]^{n-\nu_0(i)}(x_0, y_i)$.

Next let us consider the mapping $\mathcal{Q}_{n,m,p}^{(i,k)}: X \times Y \rightarrow X$ defined by

$$\mathcal{Q}_{n,m,p}^{(i,k)}(x, y) = \begin{cases} \phi_{n,m,p}^{(i)}(x, y) & (y=y_i, x=x_0, x_1, \dots, x_{k-1}), \\ x_j & (y=y_i, x=x_j; j=k, k+1, \dots, n-1), \\ x_0 & (\text{otherwise}), \end{cases}$$

i.e.,

$$\tilde{R}_{\mathcal{Q}_{n,m,p}^{(i,k)}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \nu_0(i) & \nu_1(i) & \cdots & \nu_{k-1}(i) & k & k+1 & \cdots & n-1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ * & * & \cdots & * & * & * & \cdots & * \end{pmatrix} \quad (k \geq 2).$$

Let us define the mappings $S_{n,m,p}^{(i)}: X \rightarrow X$ associated with the mapping $M_{n,m,p}: X \times Y \rightarrow X$ as $S_{n,m,p}^{(i)}(x) = M_{n,m,p}(x, y_i)$ ($i=0, 1, \dots, m-1$).

We call $S_{n,m,p}^{(i)}$ the i -th section of the mapping $M_{n,m,p}$.

Since the i -th section of the mapping $M_{n,m,p}$ is bijective by the assumption of the lemma 7.1, the i -th section of the mapping $\mathcal{Q}_{n,m,p}^{(i,n)}$ is bijective, hence the finite automaton $\mathcal{Q}_{n,m,p}^{(i,n)}$ belongs to $\mathfrak{M}_{n,m,p}^{(1,1,\dots,1,n,1,\dots,1;j)}$ for some j . We shall prove that the mapping $\mathcal{Q}_{n,m,p}^{(i,k)}$ can be represented by the mappings $T_{n,m,p}^{(i;l-1,l)}$ and $\mathcal{Q}_{n,m,p}^{(i,k-1)}$ ($l=1, 2, \dots, n-1$). Since $S_{n,m,p}^{(i)}$ is the bijection from X onto X , the states $x_{\nu_j(i)}$ ($j=0, 1, \dots, k-2$) are different from the state $x_{\nu_{k-1}(i)}$. Thus there exists a state x_l such that $\mathcal{Q}_{n,m,p}^{(i,k-1)}(x_l, y_i) = \phi_{n,m,p}^{(i)}(x_{k-1}, y_i)$ ($k-1 \leq l \leq n-1$, $k \geq 2$). If $l=k-1$, we may take $\mathcal{Q}_{n,m,p}^{(i,k)} \equiv \mathcal{Q}_{n,m,p}^{(i,k-1)}$ and if $l \geq k$, we construct the automaton $\Gamma_{n,m,p}^{(i;l,k)}$ such that

$$\Gamma_{n,m,p}^{(i;l,k)} \equiv T_{n,m,p}^{(i;k-1,l)} \wedge T_{n,m,p}^{(i;k,l)} \wedge \cdots \wedge T_{n,m,p}^{(i;l-2,l)} \wedge T_{n,m,p}^{(i;l-1,l)}.$$

Let us write the state transition of the automaton $\Gamma_{n,m,p}^{(i;l,k)}$ as the symbol $\Gamma_{n,m,p}^{(i;l,k)}$ itself. We remark that $T_{n,m,p}^{(i;j,k)}$ ($k-j > 1$) can be represented by $\wedge$ -combinations of the automata $T_{n,m,p}^{(i;0,1)}$, $T_{n,m,p}^{(i;1,2)}$, $\dots$, $T_{n,m,p}^{(i;n-2,n-1)}$. It follows directly from the definition of $T_{n,m,p}^{(i;j,k)}$ that the automaton $\Gamma_{n,m,p}^{(i;l,k)}$ belongs to $\mathfrak{M}_{n,m,p}^{(1,1,\dots,1,n,1,\dots,1;j)}$ for some j and the mapping $\Gamma_{n,m,p}^{(i;l,k)}$ has the property;

$$\Gamma_{n,m,p}^{(i;l,k)}(x, y) = \begin{cases} x_j & (y=y_i, x=x_j; j=0, 1, \dots, k-2, l+1, \dots, n-1), \\ x_l & (y=y_i, x=x_{k-1}), \\ x_{j-1} & (y=y_i, x=x_j; j=k, k+1, \dots, l), \\ x_0 & (y \neq y_i), \end{cases}$$

i.e.,

$$\tilde{R}_{\Gamma_{n,m,p}^{(i;l,k)}} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2 & \cdots & k-2 & l & k-1 & k & \cdots & l-1 & l+1 & \cdots & n-1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & \cdots & * & * & * & \cdots & * & * & * & \cdots & * \end{pmatrix}.$$

Therefore we get the relation

$$\Omega_{n,m,p}^{(i;k-1)} \wedge \Gamma_{n,m,p}^{(i;l,k)}(x, y) = \Omega_{n,m,p}^{(i;k-1)}(\Gamma_{n,m,p}^{(i;l,k)}(x, y)) = \Omega_{n,m,p}^{(i,k)}(x, y).$$

From definitions of $N_{n,m,p}$ and $T_{n,m,p}^{<j,k>}$ the i -th sections of $N_{n,m,p}$ and $T_{n,m,p}^{<j,k>}$ are equal to the i -th sections of $N_{n,m,p}^{(i)}$ and $T_{n,m,p}^{(i;j,k)}$, respectively.

From the above facts, we have

$$M_{n,m,p}(x, y) = \bigcirc_{i=0, 1, \dots, m-1} \Omega_{n,m,p}^{(i,n)}(x, y),$$

the components of which can be represented by $N_{n,m,p}^{(i)}$ and $T_{n,m,p}^{(i;l-1,l)}$, as desired.

PROOF of LEMMA 7.2. Let us take any automaton $\mathcal{A}_{n,m,p}^{(i)} \in \mathfrak{M}_{n,m,p}^{(0)(1,1,\dots,1,i_k,1,\dots,1;j)}$ ($i_k < n$). By the definition of $\mathfrak{M}_{n,m,p}^{(0)(1,1,\dots,1,i_k,1,\dots,1;j)}$, there exist two states x_t and x_s ($\in X$) such that $T_{\mathcal{A}_{n,m,p}^{(i)}}(x_t, y_i) = T_{\mathcal{A}_{n,m,p}^{(i)}}(x_s, y_i)$ ($x_t \neq x_s$), and there exists also the state x_r such that

$$(7.1) \quad T_{\mathcal{A}_{n,m,p}^{(i)}}(x_j, y_i) \neq x_r \quad (j=0, 1, \dots, n-1).$$

Let us introduce the finite automaton $Z_{n,m,p}^{(i)}$ defined by

$$(7.2) \quad T_{Z_{n,m,p}^{(i)}}(x, y) = \begin{cases} x_r & (x = x_t, y = y_i), \\ T_{\mathcal{A}_{n,m,p}^{(i)}}(x, y) & (x \neq x_t), \end{cases}$$

It belongs to $\mathfrak{M}_{n,m,p}^{(0)(1,1,\dots,1,i_k+1,1,\dots,1;j)}$, since we have from (7.1) $T_{\mathcal{A}_{n,m,p}^{(i)}}(x_t, y_i) \neq x_r$.

There exists a finite automaton $\mathcal{E}_{n,m,p}^{(i)}$, belonging to $\mathfrak{M}_{n,m,p}^{(0)(1,1,\dots,1,i-1,1,\dots,1;j)}$, the state transition of which is

$$(7.3) \quad T_{\mathcal{E}_{n,m,p}^{(i)}}(x, y) = \begin{cases} x_0 & (y = y_i, x = x_r), \\ x_1 & (y = y_i, x = T_{\mathcal{A}_{n,m,p}^{(i)}}(x_t, y_i)), \\ x_0 & (y \neq y_i). \end{cases}$$

where $T_{\mathcal{E}_{n,m,p}^{(i)}}(x, y)$ for $x \neq x_r$, $T_{\mathcal{A}_{n,m,p}^{(i)}}(x_t, y_i)$ and $y = y_i$ take $n-2$ elements in $X - \{x_0, x_1\}$.

Then there exists a finite automaton $\Pi_{n,m,p}^{(i)}$ such that

$$T_{\Pi_{n,m,p}^{(i)}}(x, y) = x \quad \text{for } x \in X \text{ and } y \in Y$$

and

$$T_{\Pi_{n,m,p}^{(i)}}(x, y) = \begin{cases} x_r & (y = y_i, x = x_0), \\ T_{\mathcal{A}_{n,m,p}^{(i)}}(x_t, y_i) & (y = y_i, x = x_1), \\ x_0 & (y \neq y_i), \end{cases}$$

since the i -th section of $T_{\Xi_{n,m,p}^{(i)}}$ is bijective. The automaton is determined uniquely and belongs to $\mathfrak{M}_{n,m,p}^{(0)(\overbrace{1,1,\dots,1}^{i-1},n,\dots,1;j)}$. Then we obtain the relation

$$(7.4) \quad T_{\Pi_{n,m,p}^{(i)} \wedge D_{n,m,p}^{(i)} \wedge Z_{n,m,p}^{(i)}}(x, y) = T_{\Pi_{n,m,p}^{(i)}}(T_{D_{n,m,p}^{(i)}}(T_{\Xi_{n,m,p}^{(i)}}(T_{Z_{n,m,p}^{(i)}}(x, y), y), y), y) \\ = T_{\mathcal{A}_{n,m,p}^{(i)}}(x, y).$$

The result is proved as follows.

If $y=y_i$ and $x \neq x_i$, we get from (7.2)

$$(7.5) \quad T_{Z_{n,m,p}^{(i)}}(x, y_i) = T_{\mathcal{A}_{n,m,p}^{(i)}}(x, y_i)$$

and from (7.1)

$$T_{Z_{n,m,p}^{(i)}}(x, y_i) \neq x_r.$$

Since the i -th section of $T_{\Xi_{n,m,p}^{(i)}}$ is bijective, it follows that

$$T_{\Xi_{n,m,p}^{(i)} \wedge Z_{n,m,p}^{(i)}}(x, y_i) \neq T_{\Xi_{n,m,p}^{(i)}}(x, y_i) = x_0.$$

From the definition of $D_{n,m,p}^{(i)}$ we get

$$T_{D_{n,m,p}^{(i)} \wedge \Xi_{n,m,p}^{(i)} \wedge Z_{n,m,p}^{(i)}}(x, y_i) = T_{\Xi_{n,m,p}^{(i)} \wedge Z_{n,m,p}^{(i)}}(x, y_i),$$

hence

$$\begin{aligned} & T_{\Pi_{n,m,p}^{(i)}}(T_{D_{n,m,p}^{(i)}}(T_{\Xi_{n,m,p}^{(i)}}(T_{Z_{n,m,p}^{(i)}}(x, y_i), y_i), y_i), y_i) \\ &= T_{\Pi_{n,m,p}^{(i)}}(T_{\Xi_{n,m,p}^{(i)}}(T_{Z_{n,m,p}^{(i)}}(x, y_i), y_i), y_i) \\ &= T_{Z_{n,m,p}^{(i)}}(x, y_i) \\ &= T_{\mathcal{A}_{n,m,p}^{(i)}}(x, y_i) \quad (\text{by (7.5)}). \end{aligned}$$

If $x=x_i$, we get from (7.2) and (7.3)

$$T_{Z_{n,m,p}^{(i)}}(x_i, y_i) = x_r$$

and

$$T_{\Xi_{n,m,p}^{(i)} \wedge Z_{n,m,p}^{(i)}}(x_i, y_i) = T_{\Xi_{n,m,p}^{(i)}}(x_r, y_i) = x_0,$$

hence we obtain from the definition of $D_{n,m,p}^{(i)}$ and (7.3)

$$\begin{aligned} & T_{\Pi_{n,m,p}^{(i)}}(T_{D_{n,m,p}^{(i)}}(T_{\Xi_{n,m,p}^{(i)}}(T_{Z_{n,m,p}^{(i)}}(x_i, y_i), y_i), y_i), y_i) \\ &= T_{\Pi_{n,m,p}^{(i)}}(T_{D_{n,m,p}^{(i)}}(x_0, y_i), y_i) \\ &= T_{\Pi_{n,m,p}^{(i)}}(x_1, y_i) \\ &= T_{\mathcal{A}_{n,m,p}^{(i)}}(x_i, y_i), \end{aligned}$$

which shows (7.4).

Admitting $\diamond$ -operation, the finite automata $\mathcal{E}_{n,m,p}^{(i)}$ and $\Pi_{n,m,p}^{(i)}$ can be represented by $N_{n,m,p}^{(i)}$ and $T_{n,m,p}^{(i;l-1,l)}$. Thus the conclusion of the lemma 7.2 follows.

From the above two lemmas we have

THEOREM 7.2. *The set $\{D_{n,m,p}^{(0)}, D_{n,m,p}^{(1)}, \dots, D_{n,m,p}^{(m-1)}, N_{n,m,p}^{(0)}, \dots, N_{n,m,p}^{(m-1)}, T_{n,m,p}^{(0;0,1)}, T_{n,m,p}^{(0;1,2)}, \dots, T_{n,m,p}^{(0;n-2,n-1)}, T_{n,m,p}^{(1;0,1)}, T_{n,m,p}^{(1;1,2)}, \dots, T_{n,m,p}^{(1;n-2,n-1)}, \dots, T_{n,m,p}^{(m-1;n-2,n-1)}\}$ is a pseudo Scheffer stroke class in $\mathfrak{M}_{n,m,p}^{(i_1, i_2, \dots, i_m; j)}$ ($i_1, i_2, \dots, i_m < n$).*

PROOF. Each element $M_{n,m,p} \in \mathfrak{M}_{n,m,p}^{(i_1, i_2, \dots, i_m; j)}$ has the state transition such that

$$T_{M_{n,m,p}}(x, y) = \bigtriangleup_{k=0,1,\dots,m-1} T_{M_{n,m,p}}^{(k)}(x, y),$$

where $M_{n,m,p}^{(k)}$ belongs to $\mathfrak{M}_{n,m,p}^{(0)(\overbrace{1,1,\dots,1}^{i-1}, i_k, 1, \dots, 1; j)}$ and $T_{M_{n,m,p}}^{(k)}(x, y) = \begin{cases} x_0 & \text{for } y \neq y_k, \\ T_{M_{n,m,p}}(x, y) & \text{for } y = y_k. \end{cases}$ We obtain the conclusion by using the above two lemmas.

Now we shall prove theorem 7.1.

PROOF OF THEOREM 7.1. The next relations hold:

$$T_{W_{n,m,p}}(x_j, y_0) = x_{[j-1]_n} = S_{N_{n,m,p}}^{(i)}(x_j) \quad (i=0, 1, \dots, m-1),$$

where $S_{N_{n,m,p}}^{(i)}: X \rightarrow X$ is the i -th section of $T_{N_{n,m,p}}^{(i)*}$.

Let us introduce the successor section $s: X \rightarrow X$ defined by $s(x_i) = x_{[i+1]_n}$ ($i=0, 1, \dots, n-1$). Then we have

$$S_{D_{n,m,p}}^{(i)}(x) = s(W_{n,m,p}(x, y_i)) \quad (i=0, 1, \dots, m-1),$$

where $S_{D_{n,m,p}}^{(i)}$ is the i -th section of $T_{D_{n,m,p}}^{(i)}$.

The operation $\diamond$ can be represented by the sections of $W_{n,m,p}$ and s .

$$x_i \diamond x_j = s(S_{W_{n,m,p}}^{(i)}(x_j)) = s(T_{W_{n,m,p}}(x_j, y_i)).$$

Let $G_{n,m,p}$ be the finite automata, every section of which has the same section $s: X \rightarrow X$. From these facts, the set $\{G_{n,m,p}, W_{n,m,p}, T_{n,m,p}^{(0;0,1)}, T_{n,m,p}^{(0;1,2)}, \dots, T_{n,m,p}^{(0;n-2,n-1)}, T_{n,m,p}^{(1;0,1)}, \dots, T_{n,m,p}^{(1;n-2,n-1)}, \dots, T_{n,m,p}^{(m-1;n-2,n-1)}\}$ is a Scheffer stroke class in $\mathfrak{M}_{n,m,p}$.

We can establish the following relations:

$$(7.6) \quad S_{T_{n,m,p}^{(i;l-1,l)}}^{(0)}(x) = s^{l+1}[T_{T_{n,m,p}^{(0;n-2,n-1)}}(T_{[N_{n,m,p}^{(0)}]^{l+1}}(s(T_{W_{n,m,p}}(x, y_0)), y_0), y_0)] \\ (i=0, 1, \dots, m-1; l=1, 2, \dots, n-2).$$

*) We write $S_{N_{n,m,p}}^{(i)}$ in place of $S_{T_{N_{n,m,p}}^{(i)}}$.

It is proved as follows. By definition we have the formula

$$(7.7) \quad \begin{aligned} s(T_{W_{n,m,p}}(x_j, y_0)) &= s(x_{[j-1]_n}) = x_j & (j=0, 1, \dots, n-1), \\ T_{[N_{n,m,p}]^{l+1}}(x_j, y_0) &= x_{[j-l-1]_n} & (j=0, 1, \dots, n-1), \\ T_{T_{n,m,p}^{(0;n-2,n-1)}}(x_{[j-l-1]_n}, y_0) &= \begin{cases} x_{[j-l-1]_n} & ([j-l-1]_n = 0, 1, \dots, n-3), \\ x_{n-1} & ([j-l-1]_n = n-2), \\ x_{n-2} & ([j-l-1]_n = n-1). \end{cases} \end{aligned}$$

We have from definition

$$\begin{aligned} [j-l-1]_n &= \begin{cases} n-l-1+j & (j=0, 1, \dots, l-1, l \ (l \neq 1)), \\ j-l-1 & (j=l+1, \dots, n-1 \ (l \neq 1)), \end{cases} \\ [j-l-1]_n &= j & (j=0, 1, \dots, n-1 \ (l=1)). \end{aligned}$$

If $l \geq 2$, it follows from (7.7) that

$$T_{T_{n,m,p}^{(0;n-2,n-1)}}(x_{[j-l-1]_n}, y_0) = \begin{cases} x_{[j-l-1]_n} & (j=0, 1, 2, \dots, l-2, l+1, \dots, n-1), \\ x_{n-1} & (j=l-1), \\ x_{n-2} & (j=l), \end{cases}$$

and if $l=1$, we have

$$T_{T_{n,m,p}^{(0;n-2,n-1)}}(x_{[j-l-1]_n}, y_0) = \begin{cases} x_{[j-l-1]_n} & (j=2, 3, \dots, n-1), \\ x_{n-1} & (j=0), \\ x_{n-2} & (j=1). \end{cases}$$

And we get the formulae

$$s^{l+1}(T_{T_{n,m,p}^{(0;n-2,n-1)}}(x_{[j-l-1]_n}, y_0)) = \begin{cases} x_j & (j=0, 1, \dots, l-2, l+1, \dots, n-1), \\ x_l & (j=l-1), \\ x_{l-1} & (j=l) \text{ for } l \neq 1, \end{cases}$$

and

$$s^{l+1}(T_{T_{n,m,p}^{(0;n-2,n-1)}}(x_{[j-l-1]_n}, y_0)) = \begin{cases} x_j & (j=2, 3, \dots, n-1), \\ x_1 & (j=0), \\ x_0 & (j=1) \text{ for } l=1. \end{cases}$$

Hence the formula (7.6).

The successor section $s: X \rightarrow X$ can be given by the formula:

$$(7.8) \quad s(x) = T_{[W_{n,m,p}]^{n-1}}(x, y).$$

The formulae (7.6) and (7.8) show the conclusion of the theorem. Thus we

have obtained one of the Scheffer stroke classes in $\mathfrak{M}_{n,m,p}[X,Y,Z]$.

We can show some other Scheffer stroke classes which we have shown in the previous paper ([12]).

Let us introduce a finite automaton $\overset{[a]}{W}_{n,m,p}$ defined by

$$T_{\overset{[a]}{W}_{n,m,p}}(x, y) = x_{[\max(i,j) - a]_n} \quad (x = x_j, y = y_i),$$

a being a fixed positive integer. Then we have the following

THEOREM 7.3. *If two numbers a and n are relatively prime, the set $\{\overset{[a]}{W}_{n,m,p}, T_{n,m,p}^{(0;n-2,n-1)}\}$ is a Scheffer stroke class in $\mathfrak{M}_{n,m,p}[X,Y,Z]$.*

Next let $(a_0, a_1, \dots, a_{n-1})$ be any permutation of $(0, 1, \dots, n-1)$, and let $P_{n,m,p}$ be the finite automaton such that

$$P_{n,m,p}(x, y) = \begin{cases} x_{a_{[i]_n}} & (x = x_j, y = y_i, i \geq j, i = 0, 1, \dots, m-1), \\ x_{a_{[j]_n}} & (x = x_j, y = y_i, i < j), \end{cases}$$

i.e.,

$$\tilde{R}_{P_{n,m,p}} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{m-1} & \cdots & a_{n-1} \\ a_1 & a_1 & a_2 & \cdots & a_{m-1} & \cdots & a_{n-1} \\ a_2 & a_2 & a_2 & \cdots & a_{m-1} & \cdots & a_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m-1} & a_{m-1} & a_{m-1} & \cdots & a_{m-1} & \cdots & a_{n-1} \\ * & * & * & * & * & * & * \end{pmatrix}.$$

Constructing $\wedge$ -product automata by operating the automata of the form $T_{n,m,p}^{<a_k, a_j>}$ on the automata $P_{n,m,p}$, we can prove that the finite automaton $\overset{[a]}{W}_{n,m,p}$ can be constructed by the set $\{P_{n,m,p}, T_{n,m,p}^{<a_k, a_j>}\}_{k,j}$. The automaton $T_{n,m,p}^{<a_k, a_j>}$, however, can be constructed from the set $\{T_{n,m,p}^{<0, 1>}, T_{n,m,p}^{<1, 2>}, \dots, T_{n,m,p}^{<n-2, n-1>}\}$. Then we have

THEOREM 7.4. *The set $\{P_{n,m,p}, T_{n,m,p}^{<l-1, l>}\}_{l=1,2,\dots,n-1}$ is a Scheffer stroke class in $\mathfrak{M}_{n,m,p}[X,Y,Z]$.*

In this theorem, let us put

$$a_0 = 0, \quad a_1 = 1, \quad \dots, \quad a_{n-3} = n-3, \quad a_{n-2} = n-1, \quad a_{n-1} = n-2.$$

The automaton $\hat{P}_{n,m,p}$, thus obtained, has the relation:

$$T_{\hat{P}_{n,m,p}}^{(0;n-2,n-1)}(x, y_0) = T_{\hat{P}_{n,m,p}}(x, y_0).$$

We get immediately from theorem 7.1 the following

THEOREM 7.5. *The set of finite automata $\{W_{n,m,p}, \hat{P}_{n,m,p}\}$ is a Scheffer stroke class in $\mathfrak{M}_{n,m,p}$.*

§8. Finite automata of convolution type.

There are some composite finite automata, the k components of which have $n_1, n_2, \dots$ and n_k states, respectively, and which have $\prod_{i=1}^k n_i$ states as the state-set. Recently E. F. Moore pointed out that the above composite automaton affords interesting subjects of investigation in case all or most of these states are distinguishable, and that there are many problems about the inverse question i. e. decomposition into components. In this paragraph we shall study the properties of finite automata of these types.

Let us give two finite automata $M_{n_1, m_1, p_1} \in \mathfrak{M}_{n_1, m_1, p_1}[X_1, Y_1, Z_1]$ and $M_{n_2, m_2, p_2} \in \mathfrak{M}_{n_2, m_2, p_2}[X_2, Y_2, Z_2]$, and assume their structures as

$$M_{n_1, m_1, p_1} = \sum_{i=0}^{n_1-1} \phi_{n_1, m_1, p_1}^{(i; \nu_i(0), \nu_i(1), \dots, \nu_i(m-1); l_i)} \quad \text{and} \quad M_{n_2, m_2, p_2} = \sum_{i=0}^{n_2-1} \phi_{n_2, m_2, p_2}^{(i; \mu_i(0), \mu_i(1), \dots, \mu_i(m-1); p_i)}$$

respectively. Let us give three sets $X_{\otimes}, Y_{\otimes}$ and $Z_{\otimes}$, such that $\#(X_{\otimes}) = n_1 n_2$, $\#(Y_{\otimes}) = m_1 m_2$, and $\#(Z_{\otimes}) = p_1 p_2$ in advance. The transformation $\otimes$:

$$\mathfrak{M}_{n_1, m_1, p_1}[X_1, Y_1, Z_1] \times \mathfrak{M}_{n_2, m_2, p_2}[X_2, Y_2, Z_2] \rightarrow \mathfrak{M}_{n_1 n_2, m_1 m_2, p_1 p_2}[X_{\otimes}, Y_{\otimes}, Z_{\otimes}]$$

is defined by

$$M_{n_1, m_1, p_1} \otimes M_{n_2, m_2, p_2} = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} \phi_{n_1 n_2, m_1 m_2, p_1 p_2}^{(i n_2 + j; \nu_i(0) n_2 + \mu_j(0), \nu_i(0) n_2 + \mu_j(1), \dots, \nu_i(m-1) n_2 + \mu_j(m-1); \lambda_{i n_2 + j})}$$

for $M_{n_1, m_1, p_1} \in \mathfrak{M}_{n_1, m_1, p_1}[X_1, Y_1, Z_1]$ and $M_{n_2, m_2, p_2} \in \mathfrak{M}_{n_2, m_2, p_2}[X_2, Y_2, Z_2]$, where $U_{M_{n_1, m_1, p_1} \otimes M_{n_2, m_2, p_2}}(x_{i n_2 + j}) = z_{[l_i n_2 + \rho_j] p_1 p_2}$, i.e. $\lambda_{i n_2 + j} = [l_i n_2 + \rho_j]_{p_1 p_2}$. The automaton $M_{n_1, m_1, p_1} \otimes M_{n_2, m_2, p_2}$ is called to be a convolution automaton of M_{n_1, m_1, p_1} and M_{n_2, m_2, p_2} . In the above definition let us rewrite the elements of $X_{\otimes}, Y_{\otimes}$ and $Z_{\otimes}$ as x_i ($i=0, 1, \dots, n_1 n_2 - 1$), y_j ($j=0, 1, \dots, m_1 m_2 - 1$) and z_k ($k=0, 1, \dots, p_1 p_2 - 1$), respectively. Similarly, we can define the convolution automaton of k (≥ 2) finite automata ([13]). The above definition of the convolution automata has the following meaning: Rewriting symbolically the pairs of states $(x_i^{(1)}, x_j^{(2)})$ for $x_i^{(1)} \in X_1$ and $x_j^{(2)} \in X_2$ as $x_i^{(1)} \otimes x_j^{(2)}$ and similarly the pairs of inputs (y_i, y_j) for $y_i^{(1)} \in Y_1$ and $y_j^{(2)} \in Y_2$ as $y_i^{(1)} \otimes y_j^{(2)}$, we can consider the symbols $\{x_i^{(1)} \otimes x_j^{(2)}\}_{i,j}$ as the elements (states) in $X_{\otimes}$, which correspond to $\{x_{i n_2 + j}\}_{i,j}$ in our notation of the finite automaton, and also $\{y_i^{(1)} \otimes y_j^{(2)}\}$ as the

elements (inputs) in $Y_{\otimes}$, which correspond to $\{y_{im_2+j}\}_{i,j}$. Then we obtain the relation

$$\begin{aligned} T_{M_{n_1,m_1,p_1} \otimes M_{n_2,m_2,p_2}}(x_{im_2+j}, y_{i'm_2+j'}) &= T_{M_{n_1,m_1,p_1} \otimes M_{n_2,m_2,p_2}}(x_i^{(1)} \otimes x_j^{(2)}, y_{i'}^{(1)} \otimes y_{j'}^{(2)}) \\ &= T_{M_{n_1,m_1,p_1}}(x_i^{(1)}, y_{i'}^{(1)}) \otimes T_{M_{n_2,m_2,p_2}}(x_j^{(2)}, y_{j'}^{(2)}) \\ &= x_{\nu_i(i')}^{(1)} \otimes x_{\mu_j(j')}^{(2)} \\ &= x_{\nu_i(i')n_2+\mu_j(j')} \in X_{\otimes}. \end{aligned}$$

We obtain some properties of convolution automata, whose complete proofs are in [13].

THEOREM 8.1. *The convolution automaton $M_{n_1,m_1,p_1}^{(1)} \otimes M_{n_2,m_2,p_2}^{(2)}$ is strongly connected if, and only if, the component finite automata $M_{n_1,m_1,p_1}^{(1)}$ and $M_{n_2,m_2,p_2}^{(2)}$ are both strongly connected.*

THEOREM 8.2. *Let us assume that $n_2 < p_2$ and $(n_2, p_1 p_2) = 1$. If the convolution automaton $M_{n_1,m_1,p_1}^{(1)} \otimes M_{n_2,m_2,p_2}^{(2)}$ of two finite automata $M_{n_1,m_1,p_1} \in \mathfrak{M}_{n_1,m_1,p_1}[X_1, Y_1, Z_1]$ and $M_{n_2,m_2,p_2} \in \mathfrak{M}_{n_2,m_2,p_2}[X_2, Y_2, Z_2]$ has a pair of indistinguishable states, then there exist at least n_2 pairs of indistinguishable states. And, under the same assumptions, a necessary and sufficient condition that a convolution automaton is fundamental is that all component finite automata are fundamental.*

Distinguishability about states of a convolution automaton can be investigated in more detail than that of E. F. Moore about general finite automata. It will be stated as follows:

THEOREM 8.3. *If the convolution automaton of two finite automata $M_{n_1,m_1,p_1} \in \mathfrak{M}_{n_1,m_1,p_1}[X_1, Y_1, Z_1]$ and $M_{n_2,m_2,p_2} \in \mathfrak{M}_{n_2,m_2,p_2}[X_2, Y_2, Z_2]$ is fundamental, and if $n_1 n_2 - p_1 p_2 < 1$, then any two states in $X_{\otimes}$ are distinguishable by a suitable chosen input sequence of length $(\max(n_1, n_2) - 1)$.*

§9. Finite automata and Asser computer's programming.

In this paragraph we shall show the relationship between logical designs of finite automata and Asser computer's programming. We shall show that in a given finite automaton there exists such a program that will produce a result corresponding to the output sequence from the information corresponding to the given input sequence. And, further, we shall show the existence of a more interesting program that imitates the behaviours of any given

(n, m, p) finite automata by giving the data of the structure of the automaton and the data corresponding to the input sequence as the information of Asser computer.

Let $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$ be an (n, m, p) finite automaton the structures of which are $T_{M_{n,m,p}}(x_i, y_j) = x_{\nu_i(j)}$ ($i=0, 1, \dots, n-1$; $j=0, 1, \dots, m-1$; $\nu_i(j) \in N_{10}[0, n-1]$) and $U_{M_{n,m,p}}(x_i) = z_{l_i}$ ($i=0, 1, \dots, n-1$; $l_i \in N_{10}[0, p-1]$). Corresponding to the automaton let us consider the rules of normal Markow algorithm $\mathfrak{A}: \mathcal{Q}(A) \rightarrow \mathcal{Q}(A)$ as follows:

$$(9.1) \quad \begin{cases} s_{i+1}s_{n+j+1} \rightarrow s_{u+m+l_i+1}s_{\nu_i(j)+1} & (i=0, 1, \dots, n-1; j=0, 1, \dots, m-1), \\ s_{n+j+1}s_i \rightarrow \cdot A, \end{cases}$$

where A is the set of working alphabets ' $s_1, s_2, \dots, s_{n+m+p}$ ', $\mathcal{Q}(A)$ the set of all words consisting of elements of A and $\cdot A$ is the null word (i.e., the unit element of the semigroup $\mathcal{Q}(A)$ generated by A). We shall use the notation of [1] without further explanation.

Let $\mathcal{Q}^*(A)$ be the subset of $\mathcal{Q}(A)$ the elements of which are the forms consisting of only one symbol s_i ($i \in N_{10}[0, n-1]$) and finite number of symbols s_{n+j+1} ($j \in N_{10}[0, m-1]$) the symbol s_i not appearing at the right-most end of the word. We restrict the algorithm $\mathfrak{A}: \mathcal{Q}(A) \rightarrow \mathcal{Q}(A)$ to the domain $\mathcal{Q}^*(A)$ and let us write $\mathfrak{B} \equiv \mathfrak{A}|_{\mathcal{Q}^*(A)}$. Then the algorithm $\mathfrak{B}: \mathcal{Q}^*(A) \rightarrow \mathcal{Q}(A)$ associated with the rules (9.1) is called a sequential algorithm associated with the given finite automaton $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$.

Given a (n, m, p) finite automaton, we can always consider the above sequential algorithm $\mathfrak{B}$.

Next, in order to clarify the relationship between finite automata and its sequential algorithm, let us define two mappings $\gamma: X \times Y^* \rightarrow \mathcal{Q}(A)$ and $\gamma': Z^* \rightarrow \mathcal{Q}(A)$ as follows:

$$\gamma(x_i, y^*) = s_{i+1}s_{n+\alpha_1+1}s_{n+\alpha_2+1} \cdots s_{n+\alpha_l+1} \quad \text{for } y^* = \{y_{\alpha_1}, y_{\alpha_2}, \dots, y_{\alpha_l}\} \\ (i=0, 1, \dots, n-1; \alpha_j \in N_{10}[0, m-1], l=1, 2, \dots)$$

and

$$\gamma'(z^*) = s_{n+m+\beta_1+1}s_{n+m+\beta_2+1} \cdots s_{n+m+\beta_l+1} \quad \text{for } z^* = \{z_{\beta_1}, z_{\beta_2}, \dots, z_{\beta_l}\} \\ (\beta_j \in N_{10}[0, p-1]; l=1, 2, \dots).$$

Then we have the following theorem.

THEOREM 7.1. *Let $M_{n,m,p}$ be an (n, m, p) finite automaton in $\mathfrak{M}_{n,m,p}$*

$[X, Y, Z]$. Then there exist an Asser computer's program $\wp_{M_{n,m,p}}$ and a set of 'working alphabets' $A' \equiv \{s_0, s_1, \dots, s_{n+m+p}\}$ such that

$$\wp_{M_{n,m,p}}(J_{\gamma(x,y^*)}) \cong J_{\gamma'(H_{n,m,p}(x,y^*))}$$

for any $x \in X$ and any $y^* \in Y^*$, where $J_{\gamma(x,y)}$ means the expression $\bar{s}_0 s_{i+1} s_{n+\alpha_1+1} \dots s_{n+\alpha_l-1+1} s_{n+\alpha_l+1}$ for $\gamma(x, y^*) = s_{i+1} s_{n+\alpha_1+1} \dots s_{n+\alpha_l+1}$.

PROOF. We can construct the required program $\wp_{M_{n,m,p}}$ as follows:

$\wp_{M_{n,m,p}}$	s_0	s_1	$\dots$	s_i	$\dots$	s_n	s_{n+1}	$\dots$
1	$s_0 R1$	$s_0 R2$	$\dots$	$s_0 Ri+1$	$\dots$	$s_0 Rn+1$	*	$\dots$
2	$s_0 Ln+2$	*	$\dots$	*	$\dots$	*	$s_{n+m+l_0} R\nu_0(0)+1$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$i+1$	$s_0 Ln+2$	*	$\dots$	*	$\dots$	*	$s_{n+m+l_{i-1}} R\nu_{i-1}(0)+1$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n+1$	$s_0 Ln+2$	*	$\dots$	*	$\dots$	*	$s_{n+m+l_{n-1}} R\nu_{n-1}(0)+1$	$\dots$
$n+2$	$s_0 N0$	*	$\dots$	*	$\dots$	*	*	$\dots$
<hr/>								
	s_{n+j+1}	$\dots$		s_{n+m}		s_{n+m+1}	$\dots$	s_{n+m+p}
	*	$\dots$		*		*	$\dots$	*
	$s_{n+m+l_0} R\nu_0(j)+1$	$\dots$		$s_{n+m+l_0} R\nu_0(m-1)+1$		*	$\dots$	*
	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$
	$s_{n+m+l_{i-1}} R\nu_{i-1}(j)+1$	$\dots$		$s_{n+m+l_{i-1}} R\nu_{i-1}(m-1)+1$		*	$\dots$	*
	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$
	$s_{n+m+l_{n-1}} R\nu_{n-1}(j)+1$	$\dots$		$s_{n+m+l_{n-1}} R\nu_{n-1}(m-1)+1$		*	$\dots$	*
	*	$\dots$		*		$s_0 Ln+2$	$\dots$	$s_0 Ln+2$

The program $\wp_{M_{n,m,p}}$, thus constructed, yields a resultant corresponding to the output sequence of a given input sequence, which will be clearly observed.

From this theorem we obtain immediately

THEOREM 9.2. Let $M_{n,m,p}$ be an (n, m, p) finite automaton in $\mathfrak{M}_{n,m,p}$ $[X, Y, Z]$, and $\mathfrak{B}$ a sequential algorithm associated with $M_{n,m,p}$. Then there exist an Asser computer's program $\wp_{\mathfrak{B}}$ and a set of 'working alphabets' $A' = \{s_0, s_1, \dots, s_{n+m+p}\}$ such that

$$\wp_{\mathfrak{B}}(J_W) \cong J_{\mathfrak{B}(W)} \quad \text{for any } W \in \Omega^*(A),$$

where $J_W = \bar{s}_0 s_i s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_l}$ for $W = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_l}$ ($s_i, s_{\alpha_j} \in A$).

This theorem is a special case of Asser's theorem on normal Markow algorithm. Therefore, we can regard a logical structure of an (n, m, p) finite automata as a special case of normal Markow algorithm in a sense.

One may therefore study the behaviour and structure of programs rather than work with states. We may consider $\wp_{M_{n,m,p}}$ as a program analog of finite automata $M_{n,m,p}$.

Next let us define two mappings $\omega: \mathfrak{M}_{n,m,p}[X, Y, Z] \times X \times Y^* \rightarrow \mathcal{Q}(A)$ and $\omega': \mathfrak{M}_{n,m,p}[X, Y, Z] \times Z^* \rightarrow \mathcal{Q}(A)$ as follows:

$$\begin{aligned} \omega(M_{n,m,p}, x_i, y^*) &= s_1 s_{n+1} s_{n+m+l_0+1} s_{\nu_0(0)} + 1 s_1 s_{n+2} s_{n+m+l_0+1} s_{\nu_0(1)} + 1 \cdots s_1 s_{n+m} s_{n+m+l_0+1} s_{\nu_0(m-1)} + 1 \\ &\quad s_2 s_{n+1} s_{n+m+l_1+1} s_{\nu_1(0)} + 1 s_2 s_{n+2} s_{n+m+l_1+1} s_{\nu_1(i)} + 1 \cdots s_2 s_{n+m} s_{n+m+l_1+1} s_{\nu_1(m-1)} + 1 \\ &\quad \cdots \cdots \cdots \\ &\quad s_{i+1} s_{n+1} s_{n+m+l_i+1} s_{\nu_i(0)} + 1 s_{i+1} s_{n+2} s_{n+m+l_i+1} s_{\nu_i(1)} + 1 \cdots s_{i+1} s_{n+m} s_{n+m+l_i+1} s_{\nu_i(m-1)} + 1 \\ &\quad \cdots \cdots \cdots \\ &\quad s_n s_{n+1} s_{n+m+l_{n-1}+1} s_{\nu_{n-1}(0)} + 1 s_n s_{n+1} s_{n+m+l_{n-1}+1} s_{\nu_{n-1}(1)} + 1 \cdots s_n s_{n+m} s_{n+m+l_{n-1}+1} \\ &\quad s_{\nu_{n-1}(m-1)} + 1 s_0 s_{i+1} s_{n+\alpha_1+1} s_{n+\alpha_2+1} \cdots s_{n+\alpha_l+1}, \\ \omega'(M_{n,m,p}, z^*) &= s_1 s_{n+1} s_{n+m+l_0+1} s_{\nu_0(0)} + 1 s_1 s_{n+2} s_{n+m+l_0+1} s_{\nu_0(1)} + 1 \cdots s_1 s_{n+m} s_{n+m+l_0+1} s_{\nu_0(m-1)} + 1 \\ &\quad \cdots \cdots \cdots \\ &\quad \bar{s}_0 s_0 s_{n+m+\beta_1+1} s_{n+m+\beta_2+1} \cdots s_{n+m+\beta_l+1} \end{aligned}$$

for $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X, Y, Z]$, $y^* = \{y_{\alpha_1}, y_{\alpha_2}, \dots, y_{\alpha_l}\}$ and $z^* = \{z_{\beta_1}, z_{\beta_2}, \dots, z_{\beta_l}\}$, where

$$\begin{aligned} T_{M_{n,m,p}}(x_i, y_i) &= x_{\nu_i(j)} \\ (i &= 0, 1, \dots, n-1; j = 0, 1, \dots, m-1; \nu_i(j) \in N_{10}[0, n-1]) \end{aligned}$$

and

$$U_{M_{n,m,p}}(x_i) = z_{l_i} \quad (i = 0, 1, \dots, n-1; l_i \in N_{10}[0, p-1]).$$

We have the following theorem.

THEOREM 9.3. *Let $M_{n,m,p}$ be an arbitrarily chosen finite automaton in $\mathfrak{M}_{n,m,p}[X, Y, Z]$.*

Then there exist an Asser computer's program $\wp_{\mathfrak{M}_{n,m,p}}$, independent of the chosen $M_{n,m,p}$, and a set of 'working alphabets' $A = \{s_0, s_1, \dots, s_{n+m+p}, \#\}$ such that

$$\wp_{\mathfrak{M}_{n,m,p}}[J_{\omega(M_{n,m,p}, x, y^*)}] \cong J_{\omega'(M_{n,m,p}, H_M(x, y^*))}$$

for any $M_{n,m,p} \in \mathfrak{M}_{n,m,p}[X,Y,Z]$, any $x \in X$ and any $y^* \in Y^*$, i.e., there exists a universal program of $\mathfrak{M}_{n,m,p}[X,Y,Z]$ which produces a resultant corresponding to the output sequence of any $M_{n,m,p}$ for a given input sequence by inserting a suitable information of Asser computer.

PROOF. Before constructing the entire program $\wp_{\mathfrak{M}_{n,m,p}}$, let us construct several programs having the following characters;

- $\wp_{\rightarrow}$: Move one square to the right.
- $\wp_{\leftarrow}$: Move one square to the left.
- $\wp_{\downarrow s_i}$: Mark the symbol s_i to the square under scan. ($i=0, 1, \dots, n+m+p$)
- $\wp_{\downarrow \#}$: Mark the symbol $\#$ to the square under scan.
- $\wp_{\rightarrow 4m}$: Move $4m$ square to the right.
- $\wp_{\text{stop}}$: Stop the operation of the Asser computer.
- $\wp_{\#-}$: Move left to the symbol $\#$.
- $\wp_{-\#}$: Move right to the symbol $\#$.
- $\wp_{s_0-}$: Move left to the symbol s_0 .
- $\wp_{-s_0}$: Move right to the symbol s_0 .

These programs can be constructed easily as shown below;

$\wp_{\rightarrow}$	s_0	s_1	$\dots$	s_i	$\dots$	s_{n+m+p}	$\#$
1	$s_0 R0$	$s_1 R0$	$\dots$	$s_i R0$	$\dots$	$s_{n+m+p} R0$	$\# R0$
$\wp_{\leftarrow}$	s_0	s_1	$\dots$	s_i	$\dots$	s_{n+m+p}	$\#$
1	$s_0 L0$	$s_1 L0$	$\dots$	$s_i L0$	$\dots$	$s_{n+m+p} L0$	$\# L0$
(i: fixed)							
$\wp_{\downarrow s_i}$	s_0	s_1	$\dots$	s_i	$\dots$	s_{n+m+p}	$\#$
1	$s_i N0$	$s_i N0$	$\dots$	$s_i N0$	$\dots$	$s_i N0$	$s_i N0$
$\wp_{\downarrow \#}$	s_0	s_1	$\dots$	s_i	$\dots$	s_{n+m+p}	$\#$
1	$\# N0$	$\# N0$	$\dots$	$\# N0$	$\dots$	$\# N0$	$\# N0$
$\wp_{\rightarrow 4m} = \underbrace{\wp_{\rightarrow} \wp_{\rightarrow} \dots \wp_{\rightarrow}}_{4m}$							
$\wp_{\text{stop}}$	s_0	s_1	$\dots$	s_i	$\dots$	s_{n+m+p}	$\#$
1	$s_0 N0$	$s_1 N0$	$\dots$	$s_i N0$	$\dots$	$s_{n+m+p} N0$	$\# N0$
$\wp_{\#-}$	s_0	s_1	$\dots$	s_i	$\dots$	s_{n+m+p}	$\#$
1	$s_0 L1$	$s_1 L1$	$\dots$	$s_i L1$	$\dots$	$s_{n+m+p} L1$	$\# N0$

$\rho_{\rightarrow \#}$	s_0	s_1	$\dots$	s_i	$\dots$	s_{n+m+p}	$\#$
1	$s_0 R1$	$s_1 R1$	$\dots$	$s_i R1$	$\dots$	$s_{n+m+p} R1$	$\#N0$
$\rho_{s_0 \leftarrow}$	s_0	s_1	$\dots$	s_i	$\dots$	s_{n+m+p}	$\#$
1	$s_0 N0$	$s_1 L1$	$\dots$	$s_i L1$	$\dots$	$s_{n+m+p} L1$	$\#L1$
$\rho_{\rightarrow s_0}$	s_0	s_1	$\dots$	s_i	$\dots$	s_{n+m+p}	$\#$
1	$s_0 N0$	$s_1 R1$	$\dots$	$s_i R1$	$\dots$	$s_{n+m+p} R1$	$\#R1$

As we see above, the programs, thus constructed, are quite simple ones. By combining the above programs according to the rules of Asser computer's programming we construct at first the following programs $A_{i+1, k+1}$ ($i=0, 1, \dots, n-1$; $k=0, 1, \dots, p-1$), $B_{i+1, j+1}$ ($i=0, 1, \dots, n-1$; $j=0, 1, \dots, m-1$) and C_{i+1} ($i=0, 1, \dots, n-1$);

$A_{i+1, k+1}$

$B_{i+1,j+1}$

$A_{i+1,k+1}$ is the subroutine by which the symbol $\#$ is marked in place of the most left symbol s_{i+1} of the structure code of a given finite automaton, the symbol $s_{n+m+k+1}$ is marked in place of the symbol $\#$ appearing in the data position and the computer moves right one square, where the structure codes mean the form $s_{i+1}s_{n+j+1}s_{n+m+l_i+1}s_{\nu_i(j)+1}$ etc. (when $T_{M_{n,m,p}}(x_i, y_j) = x_{\nu_i(j)}$ and $U_{M_{n,m,p}}(x_i) = z_{l_i}$) and the data positions mean the positions occupied by $J_{\gamma(x, y^*)}$

C_{i+1} Here $\phi_{\bar{1}} = \phi_{\bar{1}} \quad \phi_{\bar{1}}$

(for example) in the information $J_{\omega(M_{n,m,p},x,y^*)}$.

$B_{i+1,j+1}$ is the subroutine by which the symbol s_{i+1} in the probe square is erased, the symbol s_{i+1} is marked in place of the symbol $\#$ in the information and the computer seeks for the square of the structure code in which the symbols $s_{i+1}s_{n+j+1}$ appear consecutively in order, where probe square means just the left square of the most left square of the information. And C_{i+1} is the subroutine by which the symbol s_{i+1} is marked to the probe square and the most left symbol s_{i+1} of the structure code is sought, the symbol $\#$ is marked to this square and the computer moves back to the right square of the original square.

Let us combine the above programs $A_{i+1,k+1}$, $B_{i+1,j+1}$ and C_{i+1} as follows:

The entire program $\mathcal{P}_{\mathfrak{M}_{n,m,p}}$, thus constructed, has the required property of the theorem, which will be proved easily by following the program faithfully.

REMARK. We introduced a new program-combination Q of programs $Q_0, Q_1, \dots, Q_s$, i.e.,

, the meaning of which will be obvious from the other program-combination of Asser's original paper ([1]).

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