STABILITY PROBLEM OF RANDOM LINEAR SYSTEM OF THE FIRST ORDER

By

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§1. Introduction and summary.

Recently many authors such as Samuels, J. C. and Eringen, A. C. [4], [5], and Rosenbloom, A. [3] developed a number of methods for analysis of the random linear systems, which enabled them to determine the conditions for various systems to have a stable stochastic output in terms of mean square.

In this paper, by making use of the theory of the stochastic integral and the theorem on the stochastic integral equation due to Itô, K. [1], [2], we shall give in Theorem 1 the expression of the output of a certain first order random linear system as a sum of mutually uncorrelated stochastic processes each of which is generated by stochastic integration with respect to a common Wiener process, and furthermore we shall show in Theorem 2 the asymptotic behavior of the covariance function of the output particularly for a given stationary input, when the time parameter t becomes large. It is shown that two constants λ^2 and β defined in (2.2) play a decisive role in giving the stability criteria of the solution of the stochastic integral equation (2.2) in terms of variances and correlations as t becomes large.

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§2. On the output of a first order random linear system.

We consider a first order random linear system which is physically shown as

$$(2.1) \quad \begin{cases} \dot{y}(t) + u(t)y(t) = P(t), \\ y(0) = a, \end{cases}$$

where $u(t)$ is a linear function of the random white noise $\dot{\gamma}(t)$, such that $u(t) = \beta - \lambda \dot{\gamma}(t)$, β and λ being constants independent of t .

In order to have a rigorous mathematical discussion, however, we should start with the set-up satisfying the following assumptions:

(A. I) Let $(\mathcal{Q}, \mathfrak{B}, P)$ be a probability space, that is, let $\mathcal{Q}$ be an abstract space, $\mathfrak{B}$ a σ -field of subsets of $\mathcal{Q}$, P a probability measure defined on $\mathfrak{B}$, and ω an element of $\mathcal{Q}$.

(A. II) Let $\{Z(t, \omega), 0 \leq t \leq T\}$ be a Wiener process, i.e. a real gaussian additive process with (i) $Z(0, \omega) = 0$, (ii) $E\{Z(t_2, \omega) - Z(t_1, \omega)\} = 0$, (iii) $E\{(Z(t_2, \omega) - Z(t_1, \omega))^2\} = |t_2 - t_1|$, for any $t_1, t_2, (0 \leq t_1, t_2 \leq T)$, where T is an assigned finite positive number.

With reference to this set-up, let us consider the following stochastic integral equation in the sense of Itô, K. [1]:

$$(2.2) \quad Y(t, \omega) - Y(0, \omega) + \beta \int_0^t Y(s, \omega) ds = \int_0^t p(s, \omega) ds + \lambda \int_0^t Y(s, \omega) dZ(s, \omega),$$

($0 \leq t \leq T$)

where we assume

(B. I) β and λ are both real constants.

(B. II) $Y(0, \omega)$ is a random variable whose first and second moments are both finite.

(B. III) (i) $p(t, \omega), 0 \leq t \leq T, \omega \in \mathcal{Q}$, is (t, ω) -measurable, that is, $p(t, \omega)$ is measurable with respect to the direct product measure $dt \times P(d\omega)$ defined on measurable space $([0, T] \times \mathcal{Q}, \mathfrak{A} \times \mathfrak{B})$, where $\mathfrak{A}$ is a σ -field of subsets of $[0, T]$, $\mathfrak{A} \times \mathfrak{B}$ is a σ -field of subsets of $[0, T] \times \mathcal{Q}$ generated by the class of all sets of the form $A \times B, A \in \mathfrak{A}, B \in \mathfrak{B}$, and dt is concerned with the Lebesgue measure defined on $\mathfrak{A}$. (ii) For each assigned $t, 0 \leq t \leq T, E\{|p(t, \omega)|\} < \infty$, and $E\{|p(t, \omega)|^2\} < \infty$. (iii) $\{p(t, \omega), 0 \leq t \leq T\}$ is continuous in the mean, that is, for every t in $0 < t < T$,

$$(2.3) \quad \lim_{h \rightarrow 0} E\{|p(t+h, \omega) - p(t, \omega)|^2\} = 0,$$

and

$$(2.4) \quad \lim_{h \rightarrow 0+} E\{|p(h, \omega) - p(0, \omega)|^2\} = 0,$$

$$(2.5) \quad \lim_{h \rightarrow 0-} E\{|p(T+h, \omega) - p(T, \omega)|^2\} = 0.$$

(B. IV) For any fixed $t, s, (0 \leq t, s \leq T)$, a vector random variable

$(Y(0, \omega), p(t, \omega))$ is independent of $Z(s, \omega)$.

Before we consider the stochastic integral equation (2.2), we shall prepare ourselves with the definition and the theorems on stochastic integral due to Itô, K. [1], [2].

Let a stochastic process $\{X(t, \omega), 0 \leq t \leq T\}$ be (t, ω) -measurable, and be independent of the Wiener process $\{Z(t, \omega), 0 \leq t \leq T\}$ defined in (A. II), that is to say, let two random variables $X(t, \omega)$ and $Z(s, \omega)$ be mutually independent for any two assigned t and s in $0 \leq t, s \leq T$. We define now a class $\Phi_{X \cdot T}$ of stochastic process $\{\Phi(t, \omega), 0 \leq t \leq T\}$ which satisfies the following three conditions:

(M. I) $\Phi(t, \omega)$ is (t, ω) -measurable.

(M. II) For any t in $0 \leq t \leq T$, $\Phi(t, \omega)$ depends only on $\{Z(\tau, \omega), 0 \leq \tau \leq t\}$ and $\{X(u, \omega), 0 \leq u \leq t\}$, that is, it is a function defined on $R_1^{[0, t]} \times R_2^{[0, t]}$, where each $R_i^{[0, t]}$, ($i=1, 2$), is an infinite product space with the range of $Z(\tau, \omega)$ and $X(u, \omega)$ as their τ - and u -components respectively, and furthermore it is Borel measurable as a function on $R_1^{[0, t]} \times R_2^{[0, t]}$ in the ordinary sense.

$$(M. III) \quad \int_0^T E\{\Phi(t, \omega)^2\} dt < \infty.$$

We also define another class of stochastic processes, $\Psi_{X \cdot T}$ as the set of all stochastic processes $\{\Phi(t, \omega), 0 \leq t \leq T\}$ having the properties (M. I), (M. II) and (M. III)*, where (M. III)* means as follow:

(M. III)* With respect to a suitable subdivision $\mathcal{A}$ of the interval $[0, T]$ such as

$$(2.6) \quad \mathcal{A}: 0 = a_0 < a_1 < \dots < a_m = T,$$

$\Phi(t, \omega), 0 \leq t \leq T$, is expressed by

$$(2.7) \quad \Phi(t, \omega) = \Phi(a_{i-1}, \omega), \quad a_{i-1} \leq t < a_i, \quad i = 1, 2, \dots, m,$$

such that

$$(2.8) \quad E\{\Phi(a_i, \omega)^2\} < \infty, \quad i = 0, 1, \dots, m-1.$$

For a $\Phi(t, \omega) \in \Psi_{X \cdot T}$ having a representation (2.7), we shall define

$$(2.9) \quad I(t; \Phi, \omega) = \sum_{i=1}^{k-1} \Phi(a_{i-1}, \omega) \{Z(a_i, \omega) - Z(a_{i-1}, \omega)\} \\ + \Phi(a_{k-1}, \omega) \{Z(t, \omega) - Z(a_{k-1}, \omega)\},$$

where $a_{k-1} \leq t < a_k$, $0 \leq t \leq T$.

Then we observe

THEOREM A. (Itô, K. [2]). *For each $\Phi(t, \omega) \in \Phi_{X,T}$, there exists a sequence of the stochastic processes $\{\Psi_n(t, \omega), 0 \leq t \leq T\}$, $\Psi_n(t, \omega) \in \Psi_{X,T}$, $n=1, 2, \dots$ with the following two properties (i) and (ii):*

$$(i) \quad \lim_{n \rightarrow \infty} \int_0^T E\{|\Phi(t, \omega) - \Psi_n(t, \omega)|^2\} dt = 0,$$

(ii) *There exists a suitable subsequence $\{\Psi_{n(p)}(t, \omega), 0 \leq t \leq T\}$, $p=1, 2, \dots$ of the sequence $\{\Psi_n(t, \omega), 0 \leq t \leq T\}$, $n=1, 2, \dots$ such that $\{I(t; \Psi_{n(p)}, \omega), 0 \leq t \leq T\}$, $p=1, 2, \dots$ converges uniformly in t over the interval $[0, T]$ with probability one, as p tends to infinity which we denote by $I(t; \Phi, \omega)$, that is to say, there exists a stochastic process $\{\Psi^*(t, \omega), 0 \leq t \leq T\}$ such that we have*

$$(2.10) \quad \begin{aligned} &Pr. \{ \lim_{p \rightarrow \infty} \sup_{0 \leq t \leq T} |I(t; \Psi_{n(p)}, \omega) - \Psi^*(t, \omega)| = 0 \} \\ &= 1. \end{aligned}$$

This $\Psi^(t, \omega)$ is independent of a choice of a sequence $\{\Psi_n(t, \omega), 0 \leq t \leq T\}$, $n=1, 2, \dots$ and that of its subsequence $\{\Psi_{n(p)}(t, \omega), 0 \leq t \leq T\}$, $p=1, 2, \dots$ and therefore is written by $I(t; \Phi, \omega)$.*

Here we introduce

DEFINITION 1. (Itô, K. [2]). Under the same assumptions to Theorem A, the limit $I(t; \Phi, \omega)$, $0 \leq t \leq T$, is called the stochastic integral of $\Phi(t, \omega)$ with respect to the Wiener process $\{Z(t, \omega), 0 \leq t \leq T\}$, and is denoted by

$$(2.11) \quad I(t; \Phi, \omega) = \int_0^t \Phi(\tau, \omega) dZ(\tau, \omega), \quad 0 \leq t \leq T.$$

THEOREM B. (Itô, K. [2]). *For any $\Phi(t, \omega)$ and $\Psi(t, \omega)$ belonging to the class $\Phi_{X,T}$, we have*

$$(P. I) \quad E\{I(t; \Phi, \omega) I(t; \Psi, \omega)\} = \int_0^t E\{\Phi(\tau, \omega) \Psi(\tau, \omega)\} d\tau, \quad 0 \leq t \leq T.$$

$$(P. II) \quad \text{For any } s, u, t, (0 \leq s, u, t \leq T),$$

$$\int_t^s \Phi(\tau, \omega) dZ(\tau, \omega) = \int_t^u \Phi(\tau, \omega) dZ(\tau, \omega) + \int_u^s \Phi(\tau, \omega) dZ(\tau, \omega),$$

$$(P. III) \quad \text{For any assigned } C > 0,$$

$$C^2 \Pr. \{ \sup_{0 \leq t \leq T} |I(t; \Phi, \omega)| > C \} \leq \int_0^T E \{ (\Phi(\tau, \omega))^2 \} d\tau.$$

THEOREM C. (Itô, K. [1]). Let $a(t, x)$ and $b(t, x)$ be continuous functions in t , $0 \leq t \leq T$, for every x in $-\infty < x < \infty$, and let it be assumed that both of them satisfy the Lipschitz condition: There are constants C_1 and C_2 both independent of x , y and t such that

$$(2.12) \quad \begin{cases} |a(t, x) - a(t, y)| \leq C_1 |x - y|, \\ |b(t, x) - b(t, y)| \leq C_2 |x - y|, \end{cases}$$

for any x, y , $-\infty < x, y < \infty$, and any t , $0 \leq t \leq T$.

Then the stochastic integral equation

$$(2.13) \quad X(t, \omega) = X(0, \omega) + \int_0^t a(\tau, X(\tau, \omega)) d\tau + \int_0^t b(\tau, X(\tau, \omega)) dZ(\tau, \omega),$$

$$0 \leq t \leq T,$$

has a solution, namely a stochastic process $\{X(t, \omega), 0 \leq t \leq T\}$ which belongs to $\Phi_{X^*, T}$, where $X^*(t, \omega) \equiv X(0, \omega)$, for every t , $0 \leq t \leq T$, and satisfies (2.13) with probability one for each t , $0 \leq t \leq T$. The solution is unique in the sense that for each t , $0 \leq t \leq T$, any two solutions $X_1(t, \omega)$ and $X_2(t, \omega)$, both belonging to $\Phi_{X^*, T}$, possibly differ only on a set of probability zero in Ω ; that is,

$$(2.14) \quad \Pr. \{X_1(t, \omega) = X_2(t, \omega)\} = 1, \quad \text{for each } t \text{ in } 0 \leq t \leq T.$$

Now we shall proceed to the derivation of the solution of the equation (2.2) in virtue of these theorems. The main purpose of this section is to give a solution of the equation (2.2), which is a special case of (2.14), in a series expansion which will prove itself to be suited for stability discussion.

LEMMA 1. Let $\{Z(t, \omega), 0 \leq t \leq T\}$ and $\{u_0(t, \omega), 0 \leq t \leq T\}$ be mutually independent processes satisfying the conditions (A. II) and (B. III) respectively. Let us define a sequence of the stochastic processes $\{u_\nu(t, \omega), 0 \leq t \leq T\}$, $\nu = 0, 1, 2, \dots$ by the following recurrent relations

$$(2.15) \quad u_{\nu+1}(t, \omega) = \int_0^t u_\nu(s, \omega) dZ(s, \omega), \quad 0 \leq t \leq T, \quad \nu = 0, 1, 2, \dots$$

and let us put

$$(2.16) \quad \rho_{\nu, \mu}(t, s) = E \{u_\nu(t, \omega) u_\mu(s, \omega)\}, \quad 0 \leq t, s \leq T, \quad \nu, \mu = 0, 1, 2, \dots$$

and also

$$(2.17) \quad \rho_\nu(t) = \rho_{\nu,\nu}(t, t), \quad 0 \leq t \leq T, \quad \nu = 0, 1, 2, \dots$$

Then we have

$$(2.18) \quad \rho_{\nu,\mu}(t, s) = \delta_{\nu,\mu} \rho_\nu(t \wedge s), \quad 0 \leq t, s \leq T, \quad \nu, \mu = 0, 1, 2, \dots$$

and

$$(2.19) \quad \rho_\nu(t) = \int_0^t \frac{(t-s)^{\nu-1}}{(\nu-1)!} \rho_0(s) ds, \quad 0 \leq t \leq T, \quad \nu = 1, 2, \dots$$

where $\delta_{\nu,\mu} = 1$ for $\nu = \mu$ and $\delta_{\nu,\mu} = 0$ for $\nu \neq \mu$, and $t \wedge s = \min(t, s)$.

PROOF. Since $\{u_0(t, \omega), 0 \leq t \leq T\}$ is continuous in the mean in the sense (B. III) (iii), and T is finite, it is clear that each $\{u_\nu(t, \omega), 0 \leq t \leq T\}$, $\nu = 0, 1, 2, \dots$ satisfies (M. III), and hence belongs to $\Phi_{u_0 \cdot T}$ for all ν . Therefore Theorem A implies the existence of the sequence of the stochastic processes $\{u_\nu(t, \omega), 0 \leq t \leq T\}$, $\nu = 0, 1, 2, \dots$. From (P. I), we have

$$(2.20) \quad \begin{aligned} \rho_{n,0}(t, t) &= \int_0^t E\{u_0(t, \omega) u_{n-1}(\tau, \omega) dZ(\tau, \omega)\} \\ &= 0, \quad \text{for all } n > 0, n = 1, 2, \dots \end{aligned}$$

Let $A_{t,s}(u)$ be a function defined by

$$(2.21) \quad \begin{aligned} A_{t,s}(u) &= 1, \quad t \leq u \leq s, \quad 0 \leq t, s, u \leq T, \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Then for any t, s , ($0 \leq t < s \leq T$), any μ , ($\mu = 1, 2, \dots$) and any n , ($n = 0, 1, 2, \dots$), from (P. I) and (P. II), we have

$$(2.22) \quad \begin{aligned} \rho_{\mu+n,\mu}(t, s) &= \int_0^s (A_{0,t}(u))^2 \rho_{\mu+n-1,\mu-1}(u, u) du \\ &\quad + \int_0^s A_{0,t}(u) A_{t,s}(u) \rho_{\mu+n-1,\mu-1}(u, u) du \\ &= \int_0^t \frac{(t-u)^{\mu-1}}{(\mu-1)!} \rho_{n,0}(u, u) du, \quad \text{Q.E.D.} \end{aligned}$$

LEMMA 2. Let $\{u_\nu(t, \omega), 0 \leq t \leq T\}$, $\nu = 0, 1, 2, \dots$ be the sequence of the stochastic processes defined in Lemma 1. For any finite real number λ , let us put

$$(2.23) \quad S_n(t, \omega) = \sum_{\nu=0}^n \lambda^\nu u_\nu(t, \omega), \quad 0 \leq t \leq T.$$

Then the stochastic process $\{S_n(t, \omega), 0 \leq t \leq T\}$ converges with probability one for each t in $0 \leq t \leq T$ as n tends to infinity. We denote this limit by

$$(2.24) \quad S(t, \omega) = \lim_{n \rightarrow \infty} S_n(t, \omega), \quad a.s.$$

Then $S(t, \omega) \in \Phi_{u_0, T}$ and we have

$$(2.25) \quad E\{S^2(t, \omega)\} = \rho_0(t) + \lambda^2 \int_0^t e^{\lambda^2(t-s)} \rho_0(s) ds, \quad 0 \leq t \leq T.$$

PROOF. From (2.16), (2.19) and (P. III), for any $\varepsilon > 0$, we have

$$(2.26) \quad \text{Pr.} \left\{ \max_{n \leq \nu < \nu' \leq m} |S_{\nu'}(t, \omega) - S_\nu(t, \omega)| > \varepsilon \right\} \leq \frac{\lambda^2}{\varepsilon^2} \int_0^T \sum_{k=n}^{\infty} \frac{[\lambda^2(T-s)]^k}{k!} \rho_0(s) ds,$$

which tends to zero uniformly in t as n tends to infinity. Therefore this implies the existence of the limit $S(t, \omega)$ with probability one for each assigned t in $0 \leq t \leq T$, and hence that $S(t, \omega) \in \Phi_{u_0, T}$. Furthermore from (2.16) and (2.19), we have, for each assigned t in $0 \leq t \leq T$,

$$(2.27) \quad \begin{aligned} \lim_{n \rightarrow \infty} E\{(S_n(t, \omega))^2\} &= \rho_0(t) + \lambda^2 \lim_{n \rightarrow \infty} \int_0^t \sum_{\nu=0}^n \frac{[\lambda^2(t-s)]^\nu}{\nu!} \rho_0(s) ds \\ &= \rho_0(t) + \lambda^2 \int_0^t e^{\lambda^2(t-s)} \rho_0(s) ds, \quad 0 \leq t \leq T, \end{aligned}$$

which implies (2.25). Q.E.D.

For the sake of simplicity, we make use of the following notation $L(f, \alpha; t)$ defined on continuous function $f(t)$, $0 \leq t \leq T$, such that

$$(2.28) \quad L(f, \alpha; t) = \int_0^t e^{\alpha s} f(s) ds, \quad 0 \leq t \leq T,$$

where α is a constant.

THEOREM 1. Under the assumptions (A. I), (A. II), (B. I), (B. II), (B. III) and (B. IV), let us define stochastic processes $\{u_\nu(t, \omega), 0 \leq t \leq T\}$, $\nu=0, 1, 2, \dots$ and $\{Y(t, \omega), 0 \leq t \leq T\}$ in the following way:

$$(2.29) \quad u_0(t, \omega) = Y(0, \omega) + \int_0^t e^{\beta u} p(u, \omega) du, \quad 0 \leq t \leq T,$$

$$(2.30) \quad u_{\nu+1}(t, \omega) = \int_0^t u_\nu(s, \omega) dZ(s, \omega), \quad 0 \leq t \leq T, \quad \nu=0, 1, 2, \dots,$$

$$(2.31) \quad Y(t, \omega) = e^{-\beta t} \sum_{\nu=0}^{\infty} \lambda^{\nu} u_{\nu}(t, \omega), \quad a.s. \quad 0 \leq t \leq T.$$

Then $Y(t, \omega)$, $0 \leq t \leq T$, is a solution of the stochastic integral equation (2.2) in the interval $0 \leq t \leq T$ and this solution is unique with probability one, that is, any two solution of the equation (2.2) belonging to $\Phi_{u_0, T}$ are identical except possibly for a set of probability zero in Ω for each t in $0 \leq t \leq T$. Furthermore, we have

$$(2.32) \quad E\{Y(t, \omega)Y(s, \omega)\} = e^{-\beta(t+s)}\rho_0(u) + \lambda^2 e^{-\beta(t+s)} e^{-\lambda^2 u} L(\rho_0, -\lambda^2; u), \\ 0 \leq t, s \leq T,$$

where $\rho_0(t)$, $0 \leq t \leq T$, is as defined in Lemma 1, and $u = t \wedge s$.

PROOF. Lemma 1 and 2 imply that $Y(t, \omega) \in \Phi_{u_0, T}$ and the (2.32) holds true. Let us put

$$(2.33) \quad Y_n(t, \omega) = e^{-\beta t} \sum_{\nu=0}^{n_t} \lambda^{\nu} u_{\nu}(t, \omega),$$

$$(2.34) \quad \Delta Y_n(t, \omega) = Y(t, \omega) - Y_n(t, \omega),$$

$$(2.35) \quad Q(t, \omega) = \int_0^t p(s, \omega) ds,$$

and

$$(2.36) \quad \Delta(t, \omega) = Y(t, \omega) - Y(0, \omega) + \beta \int_0^t Y(s, \omega) ds - Q(t, \omega) - \lambda \int_0^t Y(s, \omega) dZ(s, \omega).$$

Then we have

$$(2.37) \quad \lambda \int_0^t Y_n(s, \omega) dZ(s, \omega)$$

$$= Y_{n+1}(t, \omega) - Y_0(t, \omega) + \beta \int_0^t Y_{n+1}(s, \omega) ds - \beta \int_0^t Y_0(s, \omega) ds,$$

$$(2.38) \quad \beta \int_0^t Y_0(s, \omega) ds = Y(0, \omega) - e^{-\beta t} Y(0, \omega) + \beta \int_0^t e^{-\beta s} ds \int_0^s e^{\beta u} p(u, \omega) du,$$

$$(2.39) \quad \beta \int_0^t e^{-\beta s} ds \int_0^s e^{\beta u} p(u, \omega) du = Q(t, \omega) - e^{-\beta t} \int_0^t e^{\beta s} p(s, \omega) ds,$$

and

$$(2.40) \quad \Delta(t, \omega) = \Delta Y_n(t, \omega) - \lambda \int_0^t \Delta Y_n(s, \omega) dZ(s, \omega) + Y_n(t, \omega) \\ - Y_{n+1}(t, \omega) + \beta \int_0^t \Delta Y_{n+1}(s, \omega) ds.$$

From (2.16), (2.19) and (P. III), we have

$$(2.41) \quad \Pr. \left\{ \text{Max}_{n \leq \nu' < \nu \leq m} |\Delta Y_\nu(t, \omega) - \Delta Y_{\nu'}(t, \omega)| > \varepsilon \right\} \\ \leq \frac{\lambda^2}{\varepsilon^2} \int_0^T \sum_{\mu=n}^{\infty} \frac{[\lambda^2(T-s)]^\mu}{\mu!} \rho_0(s) ds ,$$

$$(2.42) \quad \Pr. \left\{ \text{Max}_{n \leq \nu' < \nu \leq m} \left| \int_0^t \Delta Y_\nu(s, \omega) ds - \int_0^t \Delta Y_{\nu'}(s, \omega) ds \right| > \varepsilon \right\} \\ \leq \frac{\lambda^2 T^2}{\varepsilon^2} \int_0^T \sum_{\mu=n-1}^{\infty} \frac{[\lambda^2(T-s)]^\mu}{\mu!} \rho_0(s) ds ,$$

and

$$(2.43) \quad \Pr. \left\{ \text{Max}_{n \leq \nu' < \nu \leq m} \left| \int_0^t \Delta Y_\nu(s, \omega) dZ(s, \omega) - \int_0^t \Delta Y_{\nu'}(s, \omega) dZ(s, \omega) \right| > \varepsilon \right\} \\ \leq \frac{\lambda^2}{\varepsilon^2} \int_0^T \sum_{\mu=n}^{\infty} \frac{[\lambda^2(T-s)]^\mu}{\mu!} \rho_0(s) ds ,$$

and hence $Y_{n+1}(t, \omega) - Y_n(t, \omega)$, $\Delta Y_n(t, \omega)$, $\int_0^t \Delta Y_n(s, \omega) ds$, and $\int_0^t \Delta Y_n(s, \omega) dZ(s, \omega)$ converge to zero with probability one for each t in $0 \leq t \leq T$, as n tends to infinity.

Therefore we have

$$(2.44) \quad \Delta(t, \omega) = 0 ,$$

with probability one, for each t in $0 \leq t \leq T$.

This implies that $Y(t, \omega)$, $0 \leq t \leq T$, satisfies the stochastic integral equation (2.2).

Now, we shall prove the uniqueness of this solution. We assume that there would exist two solutions; say $Y_1(t, \omega)$ and $Y_2(t, \omega)$, both of which belong to $\Phi_{u_0, T}$ and satisfy the stochastic integral equation (2.2). Let us put $Y_3(t, \omega) = Y_2(t, \omega) - Y_1(t, \omega)$, $0 \leq t \leq T$. Then $Y_3(t, \omega)$ belongs to $\Phi_{u_0, T}$ and satisfies the following stochastic integral equation:

$$(2.45) \quad Y_3(t, \omega) = -\beta \int_0^t Y_3(s, \omega) ds + \lambda \int_0^t Y_3(s, \omega) dZ(s, \omega) , \quad 0 \leq t \leq T .$$

From Theorem C, $Y_3(t, \omega)$ has to be equal to zero except possibly for a set of probability zero in Ω , for each t in $0 \leq t \leq T$. This implies the uniqueness of the solution. Q.E.D.

§3. Asymptotic behavior of the covariance function of the output.

We assume that the following relation holds true

$$(B. III)^* \quad p(t, \omega) = m(t) + \int_0^\infty \cos t\xi dZ_1^*(\xi, \omega) - \int_0^\infty \sin t\xi dZ_2^*(\xi, \omega),$$

$$0 \leq t \leq T,$$

where

(i) $m(t)$ is a continuous function in t , $0 \leq t \leq T$,

(ii) $\{Z_i^*(\xi, \omega), 0 \leq \xi < \infty\}$, $i=1, 2$, are mutually independent and real additive processes such that

$$(3.1) \quad E\{Z_i^*(\xi_1, \omega) - Z_i^*(\xi_2, \omega)\} = 0,$$

$$i=1, 2, \text{ for any } \xi_1, \xi_2, 0 \leq \xi_1, \xi_2 < \infty,$$

$$(3.2) \quad E\{(Z_i^*(\xi_1, \omega) - Z_i^*(\xi_2, \omega))^2\} = \int_{\xi_2}^{\xi_1} dF(\xi),$$

$$i=1, 2, \text{ for any } \xi_1, \xi_2, 0 \leq \xi_1, \xi_2 < \infty,$$

with spectral function $F(\xi)$, $0 \leq \xi < \infty$ which is absolutely continuous and $\int_0^\infty dF(\xi) < \infty$.

We shall now prove the following theorem on the asymptotic behavior of the covariance function and the correlation function of the output $Y(t, \omega)$ of the first order random linear system defined in (2.2).

THEOREM 2. *Under the assumptions (A. I), (A. II), (B. I), (B. II), (B. III)* and (B. IV), let us define the covariance function between $Y(t, \omega)$ and $Y(t+h, \omega)$*

$$(3.3) \quad K(t, t+h) = E\{(Y(t, \omega) - E(Y(t, \omega)))(Y(t+h, \omega) - E(Y(t+h, \omega)))\},$$

$$0 \leq t, t+h \leq T,$$

and hence the correlation function

$$(3.4) \quad R(t, t+h) = \frac{K(t, t+h)}{\{K(t, t)K(t+h, t+h)\}^{1/2}}, \quad 0 \leq t, t+h \leq T.$$

Then we have the following results when t becomes large:

(i) When $0 \leq \lambda^2 < 2\beta$, and $\lambda^2 \neq \beta$,

$$(3.5) \quad K(t, t+h) = \frac{2\beta e^{-\beta h}}{2\beta - \lambda^2} \int_0^\infty \frac{dF(\xi)}{\beta^2 + \xi^2} + O(e^{-(2\beta - \lambda^2)t}),$$

and

$$(3.6) \quad R(t, t+h) = e^{-\beta h} + o(1).$$

(ii) When $\lambda^2 = 2\beta$,

$$(3.7) \quad K(t, t+h) = te^{-\beta h} \int_0^\infty \frac{2\beta dF(\xi)}{\beta^2 + \xi^2} + o(1),$$

$$(3.8) \quad R(t, t+h) = e^{-\beta h} + o(1).$$

(iii) When $\lambda^2 > 2\beta$,

$$(3.9) \quad K(t, t+h) = Ce^{-\beta h} e^{(\lambda^2 - 2\beta)t} + O(1),$$

where C is a constant which is independent of t and h , and is not equal to zero,

$$(3.10) \quad R(t, t+h) = e^{-\beta h} + o(1).$$

PROOF. From the assumption (B. II), $Y(0, \omega)$ can be decomposed as following;

$$(3.11) \quad Y(0, \omega) = m + X_0(\omega) + \int_0^\infty g_1(\xi) dZ_1^*(\xi, \omega) + \int_0^\infty g_2(\xi) dZ_2^*(\xi, \omega),$$

where $g_i(\xi)$, $i=1, 2$, are measurable and squarely integrable functions with respect to the measure induced by $F(\xi)$, $0 \leq \xi < \infty$, m is the mean value of $Y(0, \omega)$, and $X_0(\omega)$ has the mean value zero and is uncorrelated with the last two summands in the righthand side of (3.11).

Let us put

$$(3.12) \quad p^*(t, \omega) = \int_0^\infty \cos t\xi dZ_1^*(\xi, \omega) - \int_0^\infty \sin t\xi dZ_2^*(\xi, \omega),$$

$$(3.13) \quad X^*(\omega) = X_0(\omega) + \int_0^\infty g_1(\xi) dZ_1^*(\xi, \omega) + \int_0^\infty g_2(\xi) dZ_2^*(\xi, \omega),$$

and

$$(3.14) \quad \rho_0^*(t) = E \left\{ \left(X^*(\omega) + \int_0^t e^{\beta u} p^*(u, \omega) du \right)^2 \right\}.$$

Then, after some simple calculations, we have

$$(3.15) \quad \rho_0^*(t) = M_1 + M_2 e^{2\beta t} + e^{\beta t} K_3(t),$$

where

$$(3.16) \quad M_1 = E \{ (X_0(\omega))^2 \} + \int_0^\infty \left\{ g_1(\xi) - \frac{\beta}{\beta^2 + \xi^2} \right\}^2 dF(\xi) \\ + \int_0^\infty \left\{ g_2(\xi) - \frac{\xi}{\beta^2 + \xi^2} \right\}^2 dF(\xi),$$

$$(3.17) \quad M_2 = \int_0^\infty \frac{dF(\xi)}{\beta^2 + \xi^2}$$

and

$$(3.18) \quad K_3(t) = o(1),$$

as t becomes large. This can be readily seen because for any function $f(\xi)$ which is integrable with respect to the measure induced by $F(\xi)$,

$$(3.19) \quad \int_0^\infty f(\xi) \sin t\xi dF(\xi) = o(1),$$

$$(3.20) \quad \int_0^\infty f(\xi) \cos t\xi dF(\xi) = o(1),$$

as t becomes large, because $F(\xi)$ is absolutely continuous and its derivative $F'(\xi)$ belongs to $L^1(0, \infty)$.

In Lemma 2, we have already proved

$$(3.21) \quad K(t, t+h) = e^{-2\beta t} e^{-\beta h} \{ \rho_0^*(t) + \lambda^2 e^{\lambda^2 t} L(\rho_0^*, -\lambda^2; t) \},$$

where $\rho_0^*(t)$ is defined (3.14), and hence the asymptotic behaviors of $K(t, t+h)$ and $R(t, t+h)$ depend on those of $L(\rho_0^*, -\lambda^2; t)$ and $\rho_0^*(t)$. On the other hand, from the linearity of the operation L and (3.15),

$$(3.22) \quad L(\rho_0^*, -\lambda^2; t) = M_1 L(1, -\lambda^2; t) + M_2 L(e^{2\beta s}, -\lambda^2; t) + L(e^{\beta s} K_3(s), -\lambda^2; t),$$

where we have

$$(3.23) \quad L(1, -\lambda^2; t) = \frac{1}{\lambda^2} (1 - e^{-\lambda^2 t}), \quad \text{when } \lambda \neq 0, \\ = t, \quad \text{when } \lambda = 0,$$

$$(3.24) \quad L(e^{2\beta s}, -\lambda^2; t) = \frac{1}{2\beta - \lambda^2} \{ e^{(2\beta - \lambda^2)t} - 1 \}, \quad \text{when } \lambda^2 \neq 2\beta, \\ = t, \quad \text{when } \lambda^2 = 2\beta,$$

and

$$(3.25) \quad L(e^{\beta s} K_3(s), -\lambda^2; t) = M_3 + Q_1(t) e^{(\beta - \lambda^2)t}, \quad \text{when } \lambda^2 \neq \beta,$$

with

$$\begin{aligned}
(3.26) \quad M_3 &= 2 \int_0^\infty (\xi B_1(\xi) - \beta B_2(\xi)) A(\xi, \beta) A(\xi, \beta - \lambda^2) dF(\xi), \\
B_1(\xi) &= \xi g_1(\xi) - g_2(\xi) + (1 - \beta) \xi A(\xi, \beta), \\
B_2(\xi) &= g_1(\xi) + \xi g_2(\xi) - (\beta + \xi^2) A(\xi, \beta), \\
\text{and} \quad A(\xi, a) &= (a^2 + \xi^2)^{-1}.
\end{aligned}$$

Consequently, from (3.23), (3.24) and (3.25), we have the following results as t becomes large:

(i) For $\lambda \neq 0$, $\lambda^2 \neq \beta$, $\lambda^2 \neq 2\beta$,

$$\begin{aligned}
(3.27) \quad L(\rho_0^*, -\lambda^2; t) &= \left(\frac{M_1}{\lambda^2} - \frac{M_2}{2\beta - \lambda^2} \right) - \frac{M_1}{\lambda^2} e^{-\lambda^2 t} + \frac{M_2}{2\beta - \lambda^2} e^{(2\beta - \lambda^2)t} \\
&\quad + Q_2(t) e^{(\beta - \lambda^2)t},
\end{aligned}$$

with $Q_2(t) = o(1)$.

(ii) For $\lambda^2 = 2\beta$,

$$(3.28) \quad L(\rho_0^*, -\lambda^2; t) = \frac{M_1}{\lambda^2} - \frac{M_1}{\lambda^2} e^{-\lambda^2 t} + M_2 t + Q_3(t) e^{-\beta t},$$

with $Q_3(t) = o(1)$.

(iii) For $\lambda = 0$,

$$(3.29) \quad L(\rho_0^*, -\lambda^2; t) = M_1 t - \frac{M_2}{2\beta} + \frac{M_2}{2\beta} e^{2\beta t} + Q_4(t) e^{2\beta t},$$

with $Q_4(t) = o(1)$.

From (3.15) combined with (3.27), (3.28) and (3.29) respectively, we have (3.5), (3.6), (3.7), (3.8), (3.9) and (3.10). Q.E.D.

References.

- [1] ITÔ, K., *On a stochastic integral equation*, Proc. Japan Acad., 22 (1946), 32-35.
- [2] ITÔ, K., *Stochastic integral*, Proc. Imp. Acad. Tokyo, 20 (1944), 519-524.
- [3] ROSENBLOOM, A., *Analysis of randomly time varying linear systems*, Ph. D. Thesis, Univ. of Calif., L. A., 1954.
- [4] SAMUELS, J. C. and ERINGEN, A. C., *On stochastic linear systems*, Technical Report No. 11, ONR, Division of Engineering Sciences, Purdue Univ., 1957.
- [5] SAMUELS, J. C. and ERINGEN, A. C., *On stochastic linear systems*, Journ. of Math. and Phys. 38 (1959), 83-103.