SOME STUNTED PROJECTIVE SPACES

By

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Introduction

The object of this paper is to study some relations between the stunted complex projective spaces and the stunted quaternion projective spaces.

Using a quotient map, which will be defined in §1, the result about the stunted quaternion projective spaces, which was obtained by I. M. James (2.10) [6]), may be extended for the case of the stunted complex projective spaces.

In §2, we shall study a fiber map $h: S^{11} \rightarrow QP_2$, and determine the order of the homotopy class $E^l h$ ($l \geq 5$). We shall apply to this problem H. Toda's theory about the nonstable homotopy groups $\pi_{n+r}(S^n)$ [17].

In §3, we shall apply the results of §§1-2 to the study of a secondary cohomology operation.

§1. Stunted projective spaces

Let S^{4n+3} denote the $(4n+3)$ -dimensional sphere, which is represented as follows;

$$S^{4n+3} = \left\{ (q_1, q_2, \dots, q_{n+1}) ; \sum_{i=1}^{n+1} |q_i|^2 = 1 \right\},$$

where $q_1, q_2, \dots, q_{n+1}$ are quaternion numbers. The complex projective space CP_{2n+1} is defined by the following identification; for any complex number c with norm 1,

$$(cq_1, cq_2, \dots, cq_n) \equiv (q_1, q_2, \dots, q_{n+1}).$$

Similarly the quaternion projective space QP_n is defined by the identification;

$$(qq_1, qq_2, \dots, qq_{n+1}) \equiv (q_1, q_2, \dots, q_{n+1}),$$

where q is any quaternion number such that $|q|=1$.

As is well known, Hopf's fiber map $S^3 \rightarrow S^2$ is defined by the identification,

$$qc \equiv q \quad \text{for } |q|=1 \quad \text{and} \quad |c|=1.$$

From these definitions, we have the following commutative fiber maps.

$$(1.1) \quad \begin{array}{ccc} & S^{4n+3} & \\ f \swarrow & & \searrow h \\ CP_{2n+1} & \xrightarrow[g]{S^2} & QP_n \end{array}$$

Fiber maps f and h are respectively representatives of homotopy boundaries of $(4n+4)$ -dimensional cells e_c^{4n+4} and e_q^{4n+4} , which are the highest dimensional cells in CP_{2n+2} and QP_{n+1} respectively. In fact, the characteristic maps $f: (e^{4n+4}, S^{4n+3}) \rightarrow (CP_{2n+2}, CP_{2n+1})$ and $g: (e^{4n+4}, S^{4n+3}) \rightarrow (QP_{n+1}, QP_n)$ may be given by the identification maps, more precisely, for any quaternion number q_i and real number t , such that $0 \leq t \leq 1$, $\sum |q_i|^2 + t^2 = 1$, we can define a map

$$(q_1, \dots, q_{n+1}, t) \rightarrow [q_1, \dots, q_{n+1}, t],$$

where $[]$ is a symbol of the identified class by complex or quaternion number with norm 1.

We denote by $[]_C$ or $[]_Q$ the identified class by complex or quaternion number with norm 1.

From the diagram (1.1), the following commutative diagram is obtained.

$$(1.2) \quad \begin{array}{ccc} & S^{4n+7} & \\ \bar{f} \swarrow & & \searrow \bar{h} \\ CP_{2n+3}/CP_{2n-1} & \xrightarrow{\bar{g}} & QP_{n+1}/QP_{n+1} \end{array}$$

The maps $\bar{f}$, $\bar{h}$ are the compositions of f , h by the pinching maps, and $\bar{g}$ is the quotient map.

From these definitions, we have at once the following Proposition.

Proposition 1.

- (1) $\bar{f}$ is a representative of the homotopy boundary βe_c^{4n+8} in the stunted

complex projective space CP_{2n+4}/CP_{2n-1} , and $\bar{h}$ is a representative of the homotopy boundary βe_Q^{4n+8} in the stunted quaternion projective space QP_{n+2}/QP_{n-1} .
 (2) The order of βe_C^{4n+8} in $\pi_{4n+7}(CP_{2n+3}/CP_{2n-1})$ is a multiple of the order βe_C^{4n+8} in $\pi_{4n+7}(QP_{n+1}/QP_{n-1})$.

Once again, we consider the characteristic maps,

$$\begin{array}{ccc} & S^{4n+3} & \\ \bar{f} \swarrow & & \searrow \bar{h} \\ CP_{2n+1}/CP_{2n-1} & \xrightarrow{\bar{g}} & QP_n/QP_{n-1} = S^{4n}. \end{array}$$

The space CP_{2n+1}/CP_{2n-1} may be represented as follows.

$$CP_{2n+1}/CP_{2n-1} = \{[q_1, \dots, q_n, c, t]_C\},$$

where $q_1, \dots, q_n$ are quaternions, c complex and t real, such that $0 \leq t \leq 1$, $\sum_{i=1}^n |q_i|^2 + |c|^2 + t^2 = 1$, if $c=t=0$, then we will identify $[\]_C$ with reference point, which we denote by $*$.

The quotient map $\bar{g}$ is as follows.

$$\bar{g}[q_1, \dots, q_n, c, t]_C = [q_1, \dots, q_n, c + tj]_Q$$

where the symbol j is one of the quaternion units i, j, k .

The map $\bar{g}$ is the identity on S^{4n} , in fact

$$\begin{aligned} \bar{g}: [q_1, \dots, q_n, s]_C &= (q_1, \dots, q_n, s) \longrightarrow \\ [q_1, \dots, q_n, s]_Q &= (q_1, \dots, q_n, s), \end{aligned}$$

where $0 \leq t \leq 1$ and if $s=0$, then $[\]$ =reference point.

Now we can define a homotopic deformation $CP_{2n+1}/CP_{2n-1} \rightarrow S^{4n} \vee S^{4n+2}$ as follows.

Let $K_\theta = S^{4n} \cup_{f_\theta} e_\theta^{4n+2}$ ($0 \leq \theta \leq 1$) be the following cell complex; $K_\theta = S^{4n} \cup_{f_\theta} e_\theta^{4n+2}$,

$$S^{4n} = \left\{ (q_1, \dots, q_n, s), \sum_{i=1}^n |q_i|^2 + s^2 = 1, 0 \leq s \leq 1 \right\}$$

$$\left\{ s=0 \longrightarrow (\) = * \right\}$$

$$e_\theta^{4n+2} = \begin{cases} (q_1, \dots, q_n, c, t+\theta), & \sum_{i=1}^n |q_i|^2 + |c|^2 + t^2 = 1, \\ -\theta \leq t \leq 1, \\ \text{if } |c|^2 + \theta^2 > 0, \text{ then} \\ (q_1, \dots, q_n, c, 0) \\ \equiv (r_\theta(\bar{c} + \theta j)q_1, \dots, r_\theta(\bar{c} + \theta j)q_n, (|c|^2 + \theta^2)r_\theta) \\ \stackrel{f_\theta}{=} r_\theta = (|c|^2 + \theta^2)^{-(1/2)}. \end{cases}$$

Then we have

$$\begin{aligned} K_0 &= CP_{2n+1}/CP_{2n-1}, \\ K_1 &= S^{4n} \vee S^{4n+2}, \end{aligned}$$

where the common point in K_1 , is $[0, \overbrace{\dots}^{4n}, 1]$.

Next we define a homotopic deformation of $\bar{g}$.

Let $\bar{g}_\theta: K_\theta \rightarrow QP_n/QP_{n-1}$ be as follows.

$$\begin{aligned} \bar{g}_\theta|_{S^{4n}} &= \text{identity}, \\ \bar{g}_\theta(q_1, \dots, q_n, c, t+\theta) \\ &= \left\{ \begin{array}{l} [q_1, \dots, q_n, c+tj]_Q \\ c=t=0 \rightarrow [\] = * \end{array} \right\} \quad \text{on } e_\theta^{4n+2}. \end{aligned}$$

Therefore the quotient map $\bar{g} = \bar{g}_0$ is deformed to $\bar{g}_1: S^{4n} \vee S^{4n+2} \rightarrow QP_n/QP_{n-1}$, which is as follows.

$$\begin{aligned} S^{4n+2} &= \left\{ (q_1, \dots, q_n, c, t+1) \mid -1 \leq t \leq 1, \sum_{i=1}^n |q_i|^2 + |c|^2 + t^2 = 1 \right\}, \\ \bar{g}_1|_{S^{4n+2}} &: (q_1, \dots, q_n, c, t+1) \\ &\longrightarrow \left\{ \begin{array}{l} [q_1, \dots, q_n, c+tj]_Q \\ c=t=0 \rightarrow [\] = * \end{array} \right\}. \end{aligned}$$

We can define a homeomorphism $QP_n/QP_{n-1} \rightarrow S^{4n}$ by

$$\left\{ \begin{array}{l} [q_1, \dots, q_n, q_{n+1}]_Q \\ q_{n+1}=0 \rightarrow [\] = * \end{array} \right\} \rightarrow (2q_1\bar{q}_{n+1}, \dots, 2q_n\bar{q}_{n+1}, 2|q_{n+1}|^2 - 1)$$

[6].

Using this identification, the map $\bar{g}_1|_{S^{4n+2}}$ may be considered as a Hopf construction of the map $S^{4n-1} \times S^2 \rightarrow S^{4n-1}$, where S^2 may be considered as a subspace of the symplectic group $Sp(n)$. Since S^2 is contractible in $S^3 = Sp(1) \subset Sp(n)$, we have that the map $\bar{g}_1|_{S^{4n+2}}$ is homotopic zero. Thus we have

the following Proposition.

Proposition 2.

The quotient map $\bar{g}: CP_{2n+1}/CP_{2n-1} \rightarrow QP_n/QP_{n-1}$ is deformable to the pinching map $S^{4n} \vee S^{4n+2} \rightarrow S^{4n}$.

By I. M. James ([6] (2.10)), the homotopy boundary βe^{4n+4} in QP_{n+1}/QP_{n-1} is $n\nu_{4n}$, where ν_{4n} is the generator of the stable homotopy group $\pi_{4n+3}(S^{4n})$. By this result and Prop. 2 we have

Proposition 3.

The homotopy boundary βe_C^{4n+4} in CP_{2n+2}/CP_{2n-1} is $n\nu_{4n} + \eta_{4n+2}$, where ν_{4n} is the same as above and η_{4n+2} is the generator of the stable homotopy group $\pi_{4n+3}(S^{4n+2})$.

Further we prove the following Proposition as an application of Prop. 2.

Proposition 4.

Suppose n be odd ≥ 3 . The quotient map $\bar{g}: CP_{2n+3}/CP_{2n-1} \rightarrow QP_{n+1}/QP_{n-1}$ is homotopic to a map which pinches a subcomplex

$$S^{4n+2} \cup_{\tau\nu_{4n+2}} e^{4n+6}, \quad (r: \text{odd number}).$$

Proof.

In the cohomology ring $H^*(CP; \mathbb{Z}_2)$, we have the following relations [20].

$$(1.3) \quad \begin{cases} Sq^2 y^{2n} = 0 \\ Sq^4 y^{2n} = y^{2n+2} \\ Sq^2 y^{2n+1} = y^{2n+2} \end{cases}$$

$$(1.4.1) \quad Sq^2 y^{2n+2} = 0$$

$$(1.4.2) \quad Sq^6 y^{2n} = 0$$

$$(1.4.3) \quad Sq^4 y^{2n+1} = y^{2n+3}$$

By Prop. 3, the subcomplex CP_{2n+2}/CP_{2n-1} admits a cellular decomposition

$$CP_{2n+2}/CP_{2n-1} = (S^{4n} \vee S^{4n+2}) \cup_{\eta\nu_{4n+2} + \eta_{4n+2}} e_C^{4n+4}.$$

For pairs $\{(S^{4n} \vee S^{4n+2}) \cup_{n\nu_{4n} + \eta_{4n+2}} e^{4n+4}, S^{4n} \vee S^{4n+2}\}$ and $(S^{4n} \vee S^{4n+2}, S^{4n+2})$, we have the following exact sequences,

$$\begin{array}{ccccc}
 \pi_{4n+5}(S^{4n+2}) & \xrightarrow{i_{1*}} & \pi_{4n+5}(S^{4n} \vee S^{4n+2}) & \xrightarrow{j_{1*}} & \pi_{4n+5}(S^{4n} \vee S^{4n+2}, S^{4n+2}) \\
 & & \downarrow i_{2*} & & \downarrow p_{1*} \\
 & & & & \pi_{4n+5}(S^{4n}) \\
 & \longrightarrow & \pi_{4n+5}((S^{4n} \vee S^{4n+2}) \cup_{n\nu_{4n} + \eta_{4n+2}} e^{4n+4}) & & \\
 & & \downarrow j_{2*} & & \\
 & & \pi_{4n+5}((S^{4n} \vee S^{4n+2}) \cup_{n\nu_{4n} + \eta_{4n+2}} e^{4n+4}, S^{4n} \vee S^{4n+2}) & & \\
 & & \downarrow p_{2*} & & \\
 & & \pi_{4n+5}(S^{4n+4}), & &
 \end{array}$$

where p_{1*} is an isomorphism for $n \geq 1$,

$$\pi_{4n+5}(S^{4n}) = 0 \quad \text{for } n \geq 2,$$

$$p_{2*} \text{ is an isomorphism for } n \geq 1,$$

$$\pi_{4n+5}(S^{4n+4}) = \mathbb{Z}_2 \quad \text{for } n \geq 1.$$

The homotopy boundary βe_C^{4n+6} in CP_{2n+3}/CP_{2n-1} is an element of

$$\pi_{4n+5}((S^{4n} \vee S^{4n+2}) \cup_{n\nu_{4n} + \eta_{4n+2}} e^{4n+4}).$$

$$j_{2*}\beta e_C^{4n+6} = 0,$$

then there is an element α in $\pi_{4n+5}(S^{4n} \vee S^{4n+2})$ such as $i_{2*}(\alpha) = \beta e_C^{4n+6}$.

By (1.4.2)

$$j_{1*}\alpha = 0,$$

then there is an element α_0 in $\pi_{4n+5}(S^{4n+2})$ such as $i_{1*}\alpha_0 = \alpha$. Thus we have

$$\beta e_C^{4n+4} = i_{2*} \circ i_{1*} \alpha_0$$

By (1.4.3)

$$\alpha_0 = r\nu_{4n+2}, \quad \text{where } r \text{ is an odd integer.}$$

Therefore the attaching map of the cell e_C^{4n+4} can be deformed into S^{4n+2} , and a complex $S^{4n+2} \cup_{r\nu_{4n+2}} e^{4n+6}$ may be considered as a subcomplex of CP_{2n+3}/CP_{2n-1} . By Prop. 2, there is a map q such that the following diagram is commutative.

$$\begin{array}{ccc}
CP_{2n+3}/CP_{2n-1} & \xrightarrow{\bar{g}} & QP_{n+1}/QP_{n-1} = S^{4n} \\
\downarrow p & & \downarrow q \\
(S^{4n} \cup_{n\nu_{4n}} e^{4n+4}) \vee S^{4n+6} & &
\end{array}$$

where $\bar{g}$ is the quotient map and p is the map which pinches S^{4n+2} .

From the following exact sequence of the pair $(S^{4n} \cup_{n\nu_{4n}} e^{4n+4}, S^{4n})$,

$$\begin{array}{ccc}
\pi_{4n+7}(S^{4n} \cup_{n\nu_{4n}} e^{4n+4}, S^{4n}) & \xrightarrow{\partial} & \pi_{4n+6}(S^{4n}) = \{\nu_{4n} \circ \nu_{4n+3}\} = Z_2 \\
\uparrow \int (n\nu_{4n})_* & & \uparrow (n\nu_{4n})_* \\
\pi_{4n+7}(e^{4n+4}, S^{4n+3}) & \xrightarrow{=} & \pi_{4n+6}(S^{4n+3}) = \{\nu_{4n+3}\} \\
\downarrow i_* & & \downarrow \int p_* \\
\pi_{4n+6}(S^{4n} \cup_{n\nu_{4n}} e^{4n+4}) & \xrightarrow{\quad} & \pi_{4n+6}(S^{4n} \cup_{n\nu_{4n}} e^{4n+4}, S^{4n}) \xrightarrow{\quad} 0 \\
& & \downarrow \int p_* \\
& & \pi_{4n+6}(S^{4n+4}),
\end{array}$$

we have

$$\pi_{4n+6}(S^{4n} \cup_{n\nu_{4n}} e^{4n+4}) \xrightarrow{(p \circ j)_*} \pi_{4n+6}(S^{4n+4}).$$

Therefore the homotopy class $p'_* \circ (q|S^{4n+6})$ may be considered as a restriction of the following quotient map $\bar{g}'$,

$$\begin{array}{ccc}
CP_{2n+3}/CP_{2n-1} & & \\
\downarrow p & \searrow \bar{g} & \\
(S^{4n} \cup_{n\nu_{4n}} e^{4n+4}) \vee S^{4n+6} & \xrightarrow{q} & QP_{n+1}/QP_{n-1} = S^{4n} \cup_{n\nu_{4n}} e^{4n+4} \\
\downarrow p'_1 & & \downarrow p'_2 \\
CP_{2n+3}/CP_{2n+1} = S^{4n+4} \vee S^{4n+6} & \xrightarrow{\bar{g}'} & QP_{n+1}/QP_n = S^{4n+4},
\end{array}$$

where p'_1, p'_2 are the pinching maps of S^{4n} .

Therefore by Prop. 2 the map q may be considered as the pinching map of S^{4n+6} .

Thus the quotient map $\bar{g}$ is deformable to the pinching map of the sub-complex $S^{4n+2} \cup_{n\nu_{4n+4}} e^{4n+6}$. (Q.E.D.)

§ 2. A fiber space

Let $S^{11}/S^3 = QP_2$ denote the fiber space, by which the quaternion projective space QP_3 is obtained. We will denote by μ_4 the homotopy class of this fiber map. We will prove the following Proposition.

Proposition 5.

For any integer $l \geq 5$, the order of $E^l \mu_4 = \mu_{4+l}$ in $\pi_{11+l}(E^l QP^2)$ is $8 \times 9 \times 5$, where E^l is the l -fold suspension.

Proof.

Referring the result about the triad sequence [5], it is sufficient to prove this Proposition for $l=5$.

Consider the following commutative diagram.

$$\begin{array}{ccccc} & & \pi_{12}(S^5) & \xrightarrow{i_*} & \pi_{12}(S^5 \cup_{\nu_5} e^9) \\ & & \downarrow E & & \downarrow E \\ \pi_{14}(S^6 \cup_{\nu_6} e^{10}, S^6) & \xrightarrow{\vartheta} & \pi_{13}(S^6) & \xrightarrow{i_*} & \pi_{13}(S^6 \cup_{\nu_6} e^{10}) \end{array}$$

By (1.27) of [15],

$$0 = \pi_{13}(S^9) \approx \pi_{14}(e^{10}, S^9) \xrightarrow[\approx]{\bar{\nu}_6^*} \pi_{14}(S^6 \cup_{\nu_6} e^{10}, S^6),$$

where $\bar{\nu}_6$ is the characteristic map, therefore, we have

$$i_* : \pi_{13}(S^6) \longrightarrow \pi_{13}(S^6 \cup_{\nu_6} e^{10})$$

is an isomorphism into.

By the results in [17],

$$E : \pi_{12}(S^5) = Z_{30} \longrightarrow \pi_{13}(S^6) = Z_{60}$$

is an isomorphism into.

Thus the injection i_* in the following commutative diagram is an isomorphism into.

$$\begin{array}{ccccccc} \pi_{12}(S^5) & \xrightarrow{i_*} & \pi_{12}(S^5 \cup_{\nu_5} e^9) & \xrightarrow{j_*} & \pi_{12}(S^5 \cup_{\nu_5} e^9, S^5) & \xrightarrow{\vartheta} & \pi_{11}(S^5) = Z_2 = \{\nu_5 \circ \nu_8\} \\ & & & \uparrow \nu_5 & \uparrow \nu_5^* & & \uparrow \nu_{5*} \\ & & & \int \bar{\nu}_5^* & & & \\ \pi_{12}(e^9, S^8) & \xrightarrow[\approx]{} & \pi_{11}(S^8) & \xrightarrow{\vartheta} & \pi_{11}(S^8) = Z_{24} = \{\nu_8\} \end{array}$$

Then we have the following exact sequence,

$$(2.1) \quad 0 \rightarrow Z_{30} \rightarrow \pi_{12}(S^5 \cup_{\nu_5} e^9) \rightarrow Z_{24} \rightarrow Z_2 \rightarrow 0.$$

The calculation by the killing homotopy method ([3], [7], [12], [19]) shows

$$(2.2) \quad {}^2\pi_{12}(S^5 \cup_{\nu_5} e^9) = Z_2, \quad Z_4 \text{ or } Z_8 \text{ (one generator),}$$

$$(2.3) \quad {}^3\pi_{16}(E^5 QP_2) = Z_9, \quad {}^3\pi_{16}(E^5 QP_3) = 0 \\ {}^5\pi_{16}(E^5 QP_2) = Z_5, \quad {}^5\pi_{16}(E^5 QP_3) = 0.$$

Therefore

$$(2.4) \quad {}^3\pi_{12}(S^5 \cup_{\nu_5} e^9) = Z_9 \\ {}^5\pi_{12}(S^5 \cup_{\nu_5} e^9) = Z_5.$$

By (2.1), (2.2) and (2.4), we have

$$\pi_{12}(S^5 \cup_{\nu_5} e^9) = Z_{8 \times 9 \times 5}.$$

By (2.10) of [6], it follows that the homotopy boundary βe_Q^{12} in QP_3/QP_1 is $2\nu_8$.

Therefore by the composite homomorphism

$$\pi_{11}(QP_2) \xrightarrow{j_*} \pi_{11}(QP_2, S^4) \xrightarrow{p_*} \pi_{11}(S^8),$$

we have

$$p_* \circ j_*(\mu_4) = 2\nu_8.$$

We will consider the following commutative diagram.

$$\begin{array}{ccccc} & & & & 0 \\ & & & & \downarrow \\ & & & & \pi_{12}(S^5) = Z_{30}(\tau''') \\ & & & & \downarrow i_* \\ \pi_{11}(QP_2) & \xrightarrow{E} & & \rightarrow & \pi_{12}(EQP_2) \\ \downarrow p_* \circ j_* & & & & \downarrow j_* \\ \pi_{11}(S^8) = Z_{24} & \xrightarrow{E} & \pi_{12}(S^9) & \xleftarrow{p_*} & \pi_{12}(EQP_2, S^5) = Z_{24} \\ & & & & \downarrow \theta \\ & & & & \pi_{11}(S^5) = Z_2 \\ & & & & \downarrow \\ & & & & 0 \end{array}$$

Let ξ and η denote generators of $\pi_{12}(EQP_2, S^5)$ and $\pi_{12}(EQP_2)$ respectively.

By the exactness of (2.1),

$$\begin{aligned} j_*(\eta) &= 2\xi, \\ i_*(\tau''') &= 12\eta. \end{aligned}$$

By the commutativity of the above diagram,

$$\begin{aligned} p_* \circ j_* \circ E(\mu_4) &= E \circ p_* \circ j_*(\mu_4) \\ &= E(2\nu_8) \\ &= 2\nu_9. \end{aligned}$$

Then we have

$$(2.5) \quad \mu_5 = r\eta, \quad \text{where } r \text{ is an odd integer.}$$

By (2.3),

$$r \text{ is not divisible by 3 or 5.}$$

We may consider the following commutative diagram.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_{12}(S^5) & \xrightarrow{i_{5*}} & \pi_{12}(S^5 \cup e^9) & \longrightarrow & \\ & & \downarrow E & & \downarrow \nu_5 E & & \\ 0 & \longrightarrow & \pi_{13}(S^6) & \xrightarrow{i_{6*}} & \pi_{13}(S^6 \cup e^{10}) & \longrightarrow & \\ & & \downarrow E & & \downarrow \nu_6 E & & \\ 0 & \longrightarrow & \pi_{14}(S^7) & \xrightarrow{i_{7*}} & \pi_{14}(S^7 \cup e^{11}) & \longrightarrow & \\ & & \downarrow E & & \downarrow \nu_7 E & & \\ 0 & \longrightarrow & \pi_{15}(S^8) & \xrightarrow{i_{8*}} & \pi_{15}(S^8 \cup e^{12}) & \longrightarrow & \\ & & \downarrow E & & \downarrow \nu_8 E & & \\ 0 & \longrightarrow & \pi_{16}(S^9) & \xrightarrow{i_{9*}} & \pi_{16}(S^9 \cup e^{13}) & \longrightarrow & \\ & & \downarrow \text{Z}_{240}(\tau_9) & & \downarrow \nu_9 & & \end{array}$$

The injections $i_{5*} \sim i_{9*}$ are isomorphic into, and by the results in [17], the left vertical suspensions are isomorphic into.

By the commutativity,

$$\begin{aligned} (2.6) \quad E^4 \circ i_{5*} \tau''' &= E^4(12\eta) \\ &= i_{9*} E^4 \tau''' \\ &= i_{9*}(8\tau_9). \end{aligned}$$

By (2.5), (2.6), we have that $E^4(12\mu_5) = 12\mu_9$ has the order $2 \times 3 \times 5$.

Thus Prop. 5 is proved.

§ 3. Applications

In this section, we study the secondary cohomology operation connected with the relation

$$Sq^1Sq^8 + (Sq^7 + Sq^4Sq^2Sq^1)Sq^2 + (Sq^8 + Sq^6Sq^2)Sq^1 = 0.$$

This operation, which we will denote by Φ , is useful in the study of 2-component ${}^2\pi_{2n+7}(U(n))$ ($n \equiv 2(8)$).

Proposition 6.

Let y denote the basic class of $H^*(CP; Z_2)$, then we have

$$\Phi(y^{8t+2}) = (t+1)y^{8t+6} \quad \text{for } t \geq 1.$$

Proof

We will prove a following formula.

$$\Phi(E^l y^2) = E^l y^6 \quad \text{for } l \geq 5.$$

As in §1, we consider the diagram,

$$(3.1) \quad \begin{array}{ccc} & S^{11} & \\ \bar{f} \swarrow & & \searrow \bar{h} \\ CP_5/CP_1 & \xrightarrow{\bar{g}} & QP_2 \end{array}$$

and the suspended diagram

$$(3.2) \quad \begin{array}{ccc} & S^{11+l} & \\ E^l \bar{f} \swarrow & & \searrow E^l \bar{h} \\ E^l(CP_5/CP_1) & \xrightarrow{E^l \bar{g}} & E^l QP_2. \end{array}$$

By the killing homotopy method, we have

$$(3.3) \quad {}^2\pi_{11+i}(E^l(CP_5/CP_1)) = Z_2 + Z_4 + Z_{16} \text{ (order } 2^7),$$

$$(3.4) \quad \text{if } \Phi \neq 0, \\ {}^2\pi_{11+i}(E^l(CP_6/CP_1)) = Z_2 + Z_8 \text{ or } Z_4 + Z_4 \text{ (order } 2^4),$$

$$(3.5) \quad \text{if } \Phi = 0 \\ {}^2\pi_{11+i}(E(CP_6/CP_1)) = Z_4 + Z_8, Z_2 + Z_{16} \text{ or } Z_4 + Z_{16} \text{ (order } \geq 2^5).$$

By Prop. 5 in §2, we have that the order of the class $\{E^l \bar{h}\}$ in ${}^2\pi_{11+i}(E^l QP_2)$ is 8.

By §1 Prop. 1. (2), we have that the order of the class $\{E^l \bar{f}\}$ in ${}^2\pi_{11+i}(E^l(CP_5/CP_1)) \geq 8 = 2^3$.

Therefore we have

$$\Phi(E^l y^2) = E^l y^6.$$

By the universal example method, ([1]).

$$\Phi_{0,3}(y^{8t}) = ty^{8t+4}.$$

And by the Cartan-formula for secondary cohomology operations [1], we can complete the proof of Prop. 6.

Next as an application of §1 Prop. 4, we study the group ${}^2\pi_{4n+7}(QP_{n+1}/QP_{n-1})$ ($n=4t+1$).

By the killing homotopy method,

$${}^2\pi_{4n+7}(QP_{n+1}/QP_{n-1}) = Z_4 + Z_{16} \text{ or } Z_2 + Z_{32},$$

and,

$${}^2\pi_{4n+7}(CP_{2n+3}/CP_{2n-1}) = Z_2 + Z_4 + Z_{16}.$$

By §1. Prop. 4, we have

$${}^2\pi_{4n+7}(QP_{n+1}/QP_{n-1}) = Z_4 + Z_{16}.$$

By further calculation, we have

Corollary 7.

$${}^2\pi_{4n+7}(QP_{n+2}/QP_{n-1}) = \begin{cases} Z_8 & \text{if } n=4t+1 \text{ and } t \text{ is even,} \\ Z_{16} & \text{if } n=4t+1 \text{ and } t \text{ is odd.} \end{cases}$$

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