HOPF SPACE $\text{mod } \mathcal{C}$

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Introduction.

Let $\mathcal{C}$ be a class of abelian groups in the sense of J.-P. Serre. We consider a topological space with a multiplication and a fixed element which plays a role of the unit in the multiplication with respect to the homotopy or the homology structure $\text{mod } \mathcal{C}$ of the space. And we try to construct such a multiplication on Postnikov complexes of a simply connected CW-complex. In appendix the results are applied for the Thom complex $MSO(6)$ and hence for the realizability of sum of 6-dimensional realizable classes with coefficients in $\mathbb{Z}_3$ in compact orientable differentiable manifolds of dimension ≤ 14 . Arguments are quite similar to those of H. Suzuki [1].

1. Definitions.

Let A be a topological space. Let $\mathcal{C}$ be a class (a non-empty collection) of abelian groups which satisfies the following conditions: (1) every group consisting of only one element belongs to $\mathcal{C}$, (2) every group isomorphic to a subgroup of a group in $\mathcal{C}$, quotient group in $\mathcal{C}$ also belong to $\mathcal{C}$, (3) every extension of two groups of $\mathcal{C}$ again belongs to $\mathcal{C}$. The definition of $\mathcal{C}$ is due to J.-P. Serre [2].

DEFINITION (1,1) A is said to be a *Hopf space mod $\mathcal{C}$* , if there is a continuous map

$$\mu: A \times A \rightarrow A, \quad (1)$$

and a unit element $\text{mod } \mathcal{C}$, $e \in A$, such that restriction maps

$$A \times e \rightarrow A, \quad e \times A \rightarrow A$$

induce $\mathcal{C}$ -isomorphisms of homotopy groups of A . μ is called a *multiplication mod $\mathcal{C}$* .

In the definition, the condition that $\mu|A \times e$ and $\mu|e \times A$ induce $\mathcal{C}$ -isomorphisms of homotopy groups of A is equivalent to that they induce $\mathcal{C}$ -isomorphisms of homology groups of A , by the well known theorem of J.H.C. Whitehead for the $\mathcal{C}$ -theory (See J.-P. Serre [2]). This definition in the latter form is originally given by J.C. Moore (See for instance, R. Thom [3]).

If $\mathcal{C}$ is the class of abelian groups of finite orders, k -invariants of Hopf spaces $\text{mod } \mathcal{C}$ are zero $\text{mod } \mathcal{C}$ (See R. Thom [3]).

We shall be more interested in the class $\mathcal{C} = \mathcal{C}_p$ of finite abelian groups of orders not divisible by a given prime p .

DEFINITION (1, 2) A Hopf space $\text{mod } \mathcal{C}_p$ is called briefly *p-Hopf space*. We refer μ to the *p-multiplication* and e to the *p-unit element*.

REMARK. When we have no ambiguity, we call the multiplication $\text{mod } \mathcal{C}$ and the unit element $\text{mod } \mathcal{C}$ simply the multiplication and the unit element.

2. Fiber spaces with the structure of the Hopf space $\text{mod } \mathcal{C}$.

We assume that the Hopf space $\text{mod } \mathcal{C}$, A is connected. Let A' be another connected simply connected Hopf space $\text{mod } \mathcal{C}$ with the multiplication

$$\mu': A' \times A' \rightarrow A',$$

and the unit element $e' \in A'$. A map $f: A \rightarrow A'$ is said to be *homotopy multiplicative* with respect to μ and μ' , if $f(e) = e'$ and we have for any $x, y \in A$

$$f \cdot \mu(x, y) \simeq \mu'(f(x), f(y)), \quad (2)$$

i.e., there is a homotopy $F(x, y, t): A \times A \times I \rightarrow A'$ ($t \in I = [0, 1]$) such that

$$F(x, y, 0) = \mu'(f(x), f(y)),$$

$$F(x, y, 1) = f \cdot \mu(x, y).$$

Let us denote by (E', p', A') , the fiber space over A' of paths starting from e' . E' has a natural multiplication ν' defined by [1]. Let (E, p, A) the fiber space over A induced from E' by means of the map $f: A \rightarrow A'$.

PROPOSITION (2, 1) If $f: A \rightarrow A'$ is homotopy multiplicative with respect to μ and μ' , then the induced fiber space E over A becomes a Hopf space $\text{mod } \mathcal{C}$. The multiplication ν in E is defined by

$$\nu((x, u), (y, v)) = (\mu(x, y), \nu'(u, v) \cdot l_{x, y}), \quad (3)$$

where $x, y \in A$, $u, v \in E'$ (paths) such that $f(x) = p'(u)$, $f(y) = p'(v)$ and $l_{x, y}$ is a path given by $F(x, y, t)$ for fixed x and y , the dot $\cdot$ means the composition of paths.

PROOF. One can see that the multiplication $\nu: E \times E \rightarrow E$ is well defined by the same manner as [1, Prop. 2]. We denote by e' again the constant path $[0, 1] \rightarrow e'$. Now consider a map $\nu_{(e, e')}: E \rightarrow E$ defined by $\nu((x, u), (e, e'))$ for any $(x, u) \in E$. We see that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\nu_{(e, e')}} & E \\ p \downarrow & & p \downarrow \\ A & \xrightarrow{\mu|A \times e} & A \end{array}$$

is commutative. $\nu_{(e, e')}$ carries the fiber $F = \mathcal{Q}(A')$ to the fiber over $f(\mu(e, e))$ which has the same homotopy type as the loop space over $\mu'(e', e')$. By the exact homotopy sequence of the fiber space (E', p', A') and the five-lemma mod $\mathcal{C}^{1)}$ noted in J.-P. Serre [2, Chap. 1, 2], one can check that the map $\nu_{(e, e')}$ restricted on the fiber F induces $\mathcal{C}$ -isomorphisms of homotopy groups of fibers.

Now we have the exact homotopy sequence of the fiber space (E, p, A) and commutative diagram

$$\begin{array}{ccccccc} \cdots \rightarrow \pi_i(F) & \xrightarrow{i} & \pi_i(E) & \xrightarrow{p} & \pi_i(A) & \xrightarrow{\delta} & \pi_i(F) \rightarrow \cdots \\ & \Downarrow \mathcal{C} & \downarrow \nu_{(e, e')} & & \Downarrow \mathcal{C} & & \Downarrow \mathcal{C} \\ \cdots \rightarrow \pi_i(F) & \xrightarrow{i} & \pi_i(E) & \xrightarrow{p} & \pi_i(A) & \xrightarrow{\delta} & \pi_i(F) \rightarrow \cdots \end{array}$$

Again by the five-lemma mod $\mathcal{C}$, $\nu_{(e, e')}$ induces $\mathcal{C}$ -isomorphisms of $\pi_i(E)$ for $i \geq 0$.

By the same manner, we can show the map $E \rightarrow E$ defined by $\nu((e, e'), (x, u))$ gives $\mathcal{C}$ -isomorphisms of homotopy groups $\pi_i(E)$. Thus we have proved E is a Hopf space mod $\mathcal{C}$ and (e, e') is the unit element mod $\mathcal{C}$.

1) Vertical isomorphisms in the five-lemma can be replaced all by $\mathcal{C}$ -isomorphisms. A proof is given by a slight modification of that of [4, Chap. I, Lemmas 4.3, 4.4 and 4.5].

3. p -multiplication in $K(\pi, n)$.

Let π be a finitely generated abelian group and n be an integer ≥ 1 . By the well known theorem of abelian groups (See, for instance, S. Lefschetz [5]), π is isomorphic to the direct sum of a finitely generated free abelian group and a finite number of cyclic groups. More precisely, we have

$$\pi \approx \underbrace{Z + \cdots + Z + Z_q + \cdots + Z_r}_{\text{finite}},$$

where $q, \dots, r$ are powers of prime numbers.

One can take as $K(\pi, n)$, the product space

$$K(Z, n) \times \cdots \times \underbrace{K(Z, n) \times K(Z_q, n) \times \cdots \times K(Z_r, n)}_{\text{finite}}. \quad (4)$$

On the other hand, $K(\pi, n)$ is regarded as the loop space $\mathcal{Q}(K(\pi, n+1))$ over $K(\pi, n+1)$. The natural Hopf multiplication in $K(\pi, n)$ is defined by the composition of loops which is denoted by σ . Obviously, the decomposition (4) is natural for the loop space construction, i.e.,

$$\begin{aligned} \mathcal{Q}(K(\pi, n+1)) &= \mathcal{Q}(K(Z, n+1)) \times \cdots \times \mathcal{Q}(K(Z_r, n+1)) \\ &\quad \times \mathcal{Q}(K(Z_q, n+1)) \times \cdots \times \mathcal{Q}(K(Z_r, n+1)). \end{aligned}$$

By taking a geometrical realization of a minimal subcomplex, all space $K(\pi, n)$ are assumed to be countable CW-complexes with Hopf multiplications.

PROPOSITION (3, 1) *$K(\pi, n)$ has a p -multiplication which is not usual Hopf multiplication, if the order of π has two different prime factors or it is infinite.*

PROOF. In general, one can define a multiplication on the loop space of a connected space X as follows. For any two loops u, v and for any two non-negative integers α, β , we put

$$\begin{aligned} \sigma_{(\alpha, \beta)}(u, v) &= \sigma(\sigma^\alpha(u), \sigma^\beta(v)), \\ \sigma^\alpha(u) &= \underbrace{\sigma(\cdots \sigma(\sigma(u, u), u), \cdots, u)}_{\alpha-1}. \end{aligned}$$

If $X = K(Z, n)$ or $K(Z_q, n)$ and $b \in H^n(Z, n; Z)$ or $H^n(Z_q, n; Z_p)$ (q is a power of p) is a basic class, then we have clearly

$$\sigma_{(\alpha, \beta), r}^*(b) = \alpha b, \quad \sigma_{(\alpha, \beta), l}^*(b) = \beta b$$

where $\sigma_{(\alpha, \beta), r} = \sigma_{(\alpha, \beta)}|K(Z, n) \times e$ or $= \sigma_{(\alpha, \beta)}|K(Z_q, n) \times e$ and $\sigma_{(\alpha, \beta), l} = \sigma_{(\alpha, \beta)}|e \times K(Z, n)$ or $= \sigma_{(\alpha, \beta)}|e \times K(Z_q, n)$ (e is the base point). We also have

$$\sigma_{(\alpha, \beta)}^*(b) = \alpha(b \otimes \omega) + \beta(\omega \otimes b), \quad (5)$$

where ω is the generator of the zero dimensional cohomology group. If α and β are prime to p , then $\sigma_{(\alpha, \beta)}$ gives a p -multiplication and moreover if α or $\beta \neq 1$ in $K(Z, n)$, then it is not usual Hopf multiplication.

By taking the cartesian product of $K(Z, n), \dots, K(Z, n), K(Z_q, n), \dots, K(Z_r, n)$ in (4), we obtain a p -multiplication which is not usual Hopf multiplication. We denote it again by $\sigma_{(\alpha, \beta)}$ ²⁾. Thus Proposition (3, 1) is proved.

4. Hopf complexes mod $\mathcal{C}$.

We consider the notion of the Hopf multiplication mod $\mathcal{C}$ for semi-simplicial complexes.

DEFINITION (4, 1) Let $K = \bigcup_{q \in \mathbb{Z}^+} K_q$ be a semi-simplicial complex where $\mathbb{Z}^+$ denotes the set of non-negative integers. K is called a *Hopf complex mod $\mathcal{C}$* if there is a map

$$K_q \times K_q \rightarrow K_q, \quad (6)$$

and an element $e_0 \in K_0$ such that for e_0 and $e_p = (s_0)^q e_0$ ($q \geq 0$) maps $K_q \times e_q \rightarrow K_p$ and $e_q \times K_q \rightarrow K_q$ induce $\mathcal{C}$ -isomorphisms of homology groups of K , where s_0 denotes degeneracy operator. We call e_q a *unit element mod $\mathcal{C}$* of K_q .

Further more, it is natural to introduce the notion of p -multiplication for semi-simplicial complexes.

DEFINITION (4, 2) A Hopf complex mod $\mathcal{C}_p$ is called a *p -Hopf complex*. e_q is called a *p -unit element* of K_q .

REMARK. When we have no ambiguity, we call the multiplication mod $\mathcal{C}_p$ (6) and the unit element mod $\mathcal{C}_p$ of the p -Hopf complex simply the multiplication and the unit element.

Let X be a Hopf space mod $\mathcal{C}$ with the multiplication μ and the unit element e . The singular complex $S(X) = \bigcup_{q \in \mathbb{Z}^+} S_q(X)$ is a Hopf complex mod $\mathcal{C}$ with a multiplication defined by

$$S(\mu)(s_1^q, s_2^q)(x) = \mu(s_1^q(x), s_2^q(x)) \quad (7)$$

2) If π is finite, then one can take α, β such that α or β is prime to p and $\alpha \equiv \beta \equiv 0 \pmod{q}$, where q is any prime factor in the order of π , different from p .

for $s_1^q, s_2^q \in S_q(X)$ and $x \in \Delta_q$ $q \geq 0$ where Δ_q is the standard geometric q -simplex. The unit element of $S_q(X)$ is the constant singular simplex $e_q: \Delta^q \rightarrow e$ for each integer $q \geq 0$.

In particular, if X is a p -Hopf space, the singular complex $S(X)$ is a p -Hopf complex with the multiplication defined by (7) and with a unit element $e: \Delta_0 \rightarrow e$ for each $q \geq 0$.

Now one introduces the Hopf multiplication mod $\mathcal{C}$ and hence p -multiplication in a minimal subcomplex $M(X) = \bigcup_{q \in \mathbb{Z}^+} M_q(X)$ with the base point e in the singular complex $S(X)$ by means of a similar argument to [1, Lemma 1] as follows.

Suppose X is the Hopf space mod $\mathcal{C}$. Let i be the inclusion map $M(X) \subset S(X)$ and let $\lambda: S(X) \rightarrow M(X)$ be the chain homotopy inverse of i . The multiplication mod $\mathcal{C}$, $M(\mu)$ in $M(X)$ is defined by

$$M_q(\mu) = \lambda \cdot S_q(\mu) \cdot (i \times i), \quad (q \geq 0). \quad (8)$$

The unit element is obviously e_q ($q \geq 0$). $M(X)$ is a Hopf complex mod $\mathcal{C}$.

If X is simply connected and its homotopy groups are countable in each dimensions, then $M(X)$ is countable. And hence, by J. Milnor [6] the geometric realization $|M(X)|$ has a multiplication³⁾ $|M(X)| \times |M(X)| \rightarrow |M(X)|$ such that maps

$$|M(X)| \times e \rightarrow |M(X)|, \quad e \times |M(X)| \rightarrow |M(X)|$$

are realizations of semi-simplicial maps

$$M_q(X) \times e_q \rightarrow M_q(X), \quad e_q \times M_q(X) \rightarrow M_q(X), \quad (q \geq 0).$$

Therefore, $|M(X)|$ is again the Hopf complex mod $\mathcal{C}$. The natural map $|M(X)| \rightarrow X$ is a weak homotopy equivalence.

5. p -multiplications in Postnikov complexes.

Let X be an $(m-1)$ -connected CW -complex ($m > 2$) and suppose that its homotopy groups are finitely generated in each dimensions. Let the Postnikov system of X be

3) This is given by the geometric realization of the multiplication mod $\mathcal{C}$, $M(\mu)$ in $M(X)$ which is a semi-simplicial map: $M(X) \times M(X) \rightarrow M(X)$.

$$\dots \rightarrow X^{(n)} \xrightarrow{p^{(n)}} X^{(n-1)} \rightarrow \dots \xrightarrow{p^{(m+1)}} X^{(m+1)} \xrightarrow{p^{(m)}} X^{(m)}, \quad (9)$$

with k -invariants k_n . In particular, $X^{(m)} = K(\pi_m(X), m)$. $X^{(n)}$ is a CW-complex of the same weak homotopy type as the fiber space $E^{(n)}$ induced from the fiber space of paths over $K(\pi_n(X), n+1)$ by means of the map $f: X^{(n-1)} \rightarrow K(\pi_n(X), n+1)$ such that

$$f_n^*(b) = k_{n+1}, \quad (10)$$

where b is the basic class of $H^{n+1}(\pi_n(X), n+1; \pi_n(X))$. We may identify $X^{(n)}$ and $|M(E^{(n)})|$. It is of the same homotopy type as a cell complex in which the number of cells is finite for each dimension.

THEOREM (5, 1) *Suppose $\pi_m(X) = \pi_n(X) = Z$ and $X^{(n-1)}$ has a p -multiplication $\mu^{(n-1)}$, $n < 2m-1$, and also suppose the p -component of $H^{n+1}(X^{(n-1)}; \pi_n(X))$ is cyclic. If k_{n+1} is its generator, then $X^{(n)}$ has a p -multiplication.*

PROOF. By the assumption that the p -component of $H^{n+1}(X^{(n-1)}; \pi_n(X))$ is cyclic, $\pi_m(X) = Z$ and $n < 2m-1$, using the Künneth formula for the cohomology group of a product complex, one can deduce

$$\mu^{(n-1)*} k_{n+1} = \alpha(k_{n+1} \otimes \omega) + \beta(\omega \otimes k_{n+1}), \quad (11)$$

where α, β are integers. But from the universal coefficient formula and by Proposition 2 of [2], α, β must be prime to p , for $\mu^{(n-1)}$ is the p -multiplication.

Now we take a multiplication $\sigma_{(\alpha, \beta)}$ in $K(\pi_n(X), n+1)$, given by Proposition (3, 1). Then we obtain

$$\begin{aligned} (\sigma_{(\alpha, \beta)} \cdot (f_n \times f_n))^*(b) &= (f_n \times f_n)^* \cdot \sigma_{(\alpha, \beta)}^*(b) \\ &= (f_n \times f_n)^*(\alpha(b \otimes \omega) + \beta(\omega \otimes b)) \\ &= \alpha(f_n^* b \otimes \omega) + \beta(\omega \otimes f_n^* b) \\ &= \alpha(k_{n+1} \otimes \omega) + \beta(\omega \otimes k_{n+1}). \end{aligned}$$

Combining together the result with (11), we have

$$\begin{aligned} (\sigma_{(\alpha, \beta)} \cdot (f_n \times f_n))^*(b) &= \mu^{(n-1)*} k_{n+1} \\ &= \mu^{(n-1)*} \cdot f_n^* b \\ &= (f_n \cdot \mu^{(n-1)})^*(b). \end{aligned}$$

From the obstruction theory, f_n is homotopy multiplicative⁴⁾ with respect to $\mu^{(n-1)}$ and $\sigma_{(\alpha, \beta)}$.

From Proposition (2, 1) and the construction of $X^{(n)} = |M(E^n)|$, $X^{(n)}$ is a p -Hopf space.

6. Appendix.

Let $X = MSO(6)$, be the Thom complex for the rotation group. As for the Postnikov complex of X , following results are obtained by slight modifications of R. Thom's computations [7]. X is 5-connected,

$$\begin{aligned} X^{(6)} &= X^{(7)} = X^{(8)} = X^{(9)} = K(Z, 6), \\ X^{(10)} &= K = K(Z, 6; Z, 10; St_3^5 \iota), \end{aligned}$$

where ι is the basic class of $H^6(Z, 6; Z)$, i.e., $\pi_{10}(X) \approx Z$, $k_{11} = St_3^5 \iota$, and

$$X^{(11)} = X^{(12)} = X^{(13)} = Y = K \times K(Z_2, 11).$$

That is, X has the same (homotopy) 14-type as the complex Y . There exists, therefore, a map φ of the 14-skeleton Y_{14} of Y to X such that it induces isomorphisms $\varphi^*: H^i(X; G) \xrightarrow{\approx} H^i(Y_{14}; G)$ for $0 \leq i < 13$, where G is any coefficient group.

LEMMA (6, 1) *The complex Y is a 3-Hopf space.*

PROOF. $X^{(9)}$ has clearly a 3-multiplication⁵⁾ $\mu^{(9)}$. $k_{11} = St_3^5 \iota \in H^{11}(Z, 6; Z)$ generates the 3-component of $H^{11}(Z, 6; Z)$ (See H. Cartan [8]). From Theorem (5, 1), it follows immediately that $X^{(10)} = K$ has a 3-multiplication $\mu^{(10)}$.

$K(Z_2, 11)$ has also a 3-multiplication⁵⁾ and hence the product space $K \times K(Z_2, 11) = Y = X^{(13)}$ has a 3-multiplication $\mu^{(13)}$. Thus Lemma (6, 1) is proved.

We apply the result to the realizability of a sum of realizable cohomology classes with coefficients in Z_3 as follows.

THEOREM (6, 2) *Let M be a compact orientable differentiable manifold of dimension ≤ 14 . If u_1, u_2 are realizable⁶⁾ classes in $H^6(M; Z_3)$, then*

4) One may assume that f_n carries the unit element of $X^{(n-1)}$ to the base point of $K(\pi_n(X), n+1)$.

5) One may take usual multiplications in the Eilenberg-Mac Lane complexes.

6) "Realizable" means "realizable for the rotation group $SO(6)$ by a C^∞ -submanifold" in the sense of R. Thom [7].

$u_1 + u_2$ is also realizable.

PROOF. By the assumption of the theorem, we have continuous maps $f_i: M \rightarrow MSO(6)$ such that $f_i^*U = u_i$ ($i=1, 2$), where U is the basic class of $H^6(MSO(6); \mathbb{Z}_3)$.

Let us denote by q the natural map of $MSO(6)=X$ to Y which induce isomorphisms $H^i(Y; G) \approx H^i(X; G)$, $0 \leq i < 13$. Let $d: M \rightarrow M \times M$ be the diagonal map. The composition map $\mu^{(13)} \cdot (q \cdot f_1 \times q \cdot f_2) \cdot d$;

$$M \xrightarrow{d} M \times M \xrightarrow{q \cdot f_1 \times q \cdot f_2} Y \times Y \xrightarrow{\mu^{(13)}} Y$$

is homotopic to a map $f_0: M \rightarrow Y_{14}$ by the well known cellular approximation theorem, because $\dim M \leq 14$.

We take again the composition map $f = \varphi \cdot f_0$; $M \xrightarrow{f_0} Y_{14} \xrightarrow{\varphi} X$. By the construction of f_0 and the property of φ , one sees immediately that

$$f^*U = u_1 + u_2.$$

And hence $u_1 + u_2$ is also realizable.

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