ON CERTAIN ZONAL SPHERICAL FUNCTIONS

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§ 0. In this paper, we shall consider certain irreducible unitary representations of the real simple Lie group G_{1n} and zonal spherical functions¹⁾ on M_{1n} for $n=1$ and 2, where M_{pq} is the so-called irreducible bounded symmetric domain of type 1.

Let M be a differentiable manifold, G a transitive Lie transformation group of M and H an isotropic subgroup of G with respect to some fixed point of M . We assume that there is given an irreducible unitary representation $(T, \mathfrak{h})$ of G on a Hilbert space $\mathfrak{h}$. $(T, \mathfrak{h})$ is said to be of *class 1* (with respect to H) if there exists a vector $\xi \neq 0$ in $\mathfrak{h}$ such that $T_h \xi = \xi$ for each element $h \in H$, and such a vector ξ is termed a *linear invariant* of H . In such a case, a function on G defined by

$$(0.1) \quad \varphi(g) = (T_g \eta, \xi), \quad \text{for } g \in G$$

is constant on each right coset mod. H , where $(,)$ denotes an inner product in $\mathfrak{h}$, ξ is a linear invariant of H and η is an arbitrary element of $\mathfrak{h}$ (we assume G operates on M to the right). So the function $\varphi(g)$ is considered as a function on a homogeneous space $M=G/H$ and such a function $\varphi(g)$ is termed a *spherical function* on M . In particular, if both ξ and η are linear invariants, then $\varphi(g)$ is constant on each double coset mod. H (i.e. $\varphi(hgh') = \varphi(g)$ for arbitrary elements $h, h' \in H$) and termed a *zonal spherical function* on M . If $\mathfrak{h}$ is a Hilbert space of square integrable functions on a manifold N and an inner product is defined as usual, then $\varphi(g)$ is represented as follows:

$$(0.2) \quad \varphi(g) = \int_N (T_g \eta)(x) \cdot \overline{\xi(x)} d\mu(x) \quad \text{for } g \in G \text{ and } x \in M.$$

§ 1. In this section, we shall consider a Cartan decomposition of G_{pq} and explain how M_{pq} is identified with a bounded domain in a complex Euclidean

1) For the detailed properties of spherical functions, see [2], [3], [5], [6] and [8].

space.

The irreducible bounded symmetric domain M_{pq} of type 1 classified by E . Cartan is a symmetric space G_{pq}/K_{pq} , where G_{pq} is the following real simple Lie group and K_{pq} is the maximal compact subgroup of G_{pq} :

$$(1.1) \quad M_{pq} = G_{pq}/K_{pq},$$

where $p+q=n$, $1 \leq p \leq [n/2]$, and p, q, n are positive integers,

$$(1.2) \quad G_{pq} = \{X \in SL(n, \mathbb{C}) : X \sum_{pq} X^{*2} = \sum_{pq}\},$$

$$(1.3) \quad \sum_{pq} = \begin{pmatrix} -E_p & 0 \\ 0 & E_q \end{pmatrix},$$

where E_p (res. E_q) is the unit matrix of order p (resp. q),

$$(1.4) \quad \begin{aligned} K_{pq} &= G_{pq} \cap U(n) = \left\{ \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \in SU(n) \right\} \\ &= \left\{ \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} : U \in U(p), V \in U(q), \det(U \cdot V) = 1 \right\} \\ &\cong SU(p) \times SU(q) \times T^1, \end{aligned}$$

where T^1 is a one-dimensional torus.

By its definition, an element X of $SL(n, \mathbb{C})$ belongs to G_{pq} if and only if it preserves an indefinite inner product $[\cdot, \cdot]$ in a n -dimensional complex Euclidean space, where $[\cdot, \cdot]$ is defined by

$$(1.5) \quad [Z, Z] = - \sum_{i=1}^p Z_i \bar{Z}_i + \sum_{j=p+1}^{p+q=n} Z_j \bar{Z}_j$$

for any element $Z = (Z_1, \dots, Z_n) \in \mathbb{C}^n$.

For each element X of $GL(n, \mathbb{C})$, we set

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C and D are matrices of type (p, p) , (p, q) , (q, p) and (q, q) respectively. From (1.2) and

$$X \sum_{pq} X^* = \begin{pmatrix} -AA^* + BB^* & -AC^* + BD^* \\ -CA^* + DB^* & -CC^* + DD^* \end{pmatrix},$$

we have

2) X^* denotes a transposed and complex conjugate matrix of X .

$$(1.6) \quad X \in G_{pq} \iff \begin{cases} \text{a) } \det X = 1, \\ \text{b) } -AA^* + BB^* = -E_p, \\ \text{c) } -AC^* + BD^* = 0, \\ \text{d) } DD^* - CC^* = E_q. \end{cases}$$

From a) of (1.6),

$$\det(\lambda E - AA^*) = \det((\lambda - 1)E - BB^*).$$

Therefore eigen-values of AA^* are all positive real numbers greater than or equal to 1, because eigen-values of an Hermitian matrix of the form BB^* are all non-negative real numbers. This means $\det A \neq 0$. Similarly, $\det D \neq 0$.

We set $Z = D^{-1}C$. From b), c) and d) of (1.6), we have

$$\begin{cases} (A^{-1})(A^{-1})^* = E_p - Z^*Z, \\ AZ^* = B, \\ (D^{-1})(D^{-1})^* = E_q - ZZ^*. \end{cases}$$

Hence, it follows

$$(1.7) \quad X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_{pq} \implies \begin{cases} \text{a) } \det X = 1, \\ \text{b) } (A^{-1})(A^{-1})^* = E_p - Z^*Z, \\ \text{c) } (D^{-1})(D^{-1})^* = E_q - ZZ^*, \\ \text{d) } B = AZ^*, \\ \text{e) } C = DZ. \end{cases}$$

Every regular matrix A may be written uniquely as the product of a unitary matrix U and a positive definite Hermitian matrix H . From b) of (1.7), we have

$$(1.8) \quad H^2 = (E_p - Z^*Z)^{-1}.$$

We denote this H by

$$(1.8)' \quad H = \sqrt[2]{E_p - Z^*Z}.$$

Consequently, X is written as follows:

$$(1.9) \quad \begin{aligned} X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} U \cdot \sqrt[2]{E_p - Z^*Z} & U \cdot \sqrt[2]{E_p - Z^*Z} \cdot Z^* \\ V \cdot \sqrt[2]{E_q - ZZ^*} \cdot Z & V \cdot \sqrt[2]{E_q - ZZ^*} \end{pmatrix} \\ &= \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \cdot \begin{pmatrix} \sqrt[2]{E_p - Z^*Z} & \sqrt[2]{E_p - Z^*Z} \cdot Z^* \\ \sqrt[2]{E_q - ZZ^*} \cdot Z & \sqrt[2]{E_q - ZZ^*} \end{pmatrix}, \end{aligned}$$

where $U \in U(p)$ and $V \in U(q)$. Since $(A^{-1})(A^{-1})^*$ (resp. $(D^{-1})(D^{-1})^*$) is a posi-

tive definite Hermitian matrix, $E_p - Z^*Z$ (resp. $E_q - ZZ^*$) must be positive definite. Since the third and the fourth terms of the right hand side of the following equations are real numbers, $\det(U \cdot V)$ must be equal to 1:

$$(1.10) \quad 1 = \det X = \det U \cdot \det V \cdot \det \begin{pmatrix} \sqrt[n]{E_p - Z^*Z} & 0 \\ 0 & \sqrt[n]{E_q - ZZ^*} \end{pmatrix} \cdot \det \begin{pmatrix} E_p & Z^* \\ Z & E_q \end{pmatrix}.$$

On the other hand, we can prove that $E_p - Z^*Z \succ 0^{(3)}$ is equivalent to $E_q - ZZ^* \succ 0$. For, we have

$$\det(\lambda E - A^*A) = \det(\lambda E - AA^*)$$

for any rectangular matrix A . If we identify Z with a (n, n) -matrix Z' :

$$Z' = \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix},$$

then

$$Z'Z'^* = \begin{pmatrix} ZZ^* & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Z'^*Z' = \begin{pmatrix} Z^*Z & 0 \\ 0 & 0 \end{pmatrix}.$$

Let P_q and P_p (resp. P'_q and P'_p) be characteristic polynomials of ZZ^* and Z^*Z (resp. $Z'Z'^*$ and Z'^*Z'). Then

$$P'_q(\lambda) = \lambda^{n-q} P_q(\lambda) \quad \text{and} \quad P'_p(\lambda) = \lambda^{n-p} P_p(\lambda).$$

So we have

$$P_q(\lambda) = \lambda^{q-p} P_p(\lambda) \quad (p \leq q).$$

Let $\lambda_1, \dots, \lambda_p$ (resp. $\mu_1, \dots, \mu_q$) are roots of $P_p(1-\lambda) = 0$ (resp. $P_q(1-\lambda) = 0$). Then we see

$$\mu_1 = \lambda_1, \dots, \mu_p = \lambda_p, \mu_{p+1} = 1, \dots, \mu_q = 1$$

under a suitable change of indices. Hence it follows

$$E_p - Z^*Z \succ 0 \iff E_q - ZZ^* \succ 0,$$

since $\det((E_p - Z^*Z) - \lambda E_p) = P_p(1-\lambda)$ and $\det((E_q - ZZ^*) - \lambda E_q) = P_q(1-\lambda)$.

Conversely, suppose that there is given a complex (q, p) -matrix Z satisfying the condition $E_p - Z^*Z \succ 0$. Since $(E_p - Z^*Z)^{-1}$ is a positive definite Hermitian matrix, it is easily proved that there exists a unique solution H

3) $X \succ 0$ means that X is a positive definite matrix.

of the matrix equation (1.8), we set

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$\begin{cases} A = U \cdot \sqrt[2]{E_p - Z^* Z}, \\ B = AZ^*, \\ C = DZ, \\ D = V \cdot \sqrt[2]{E_q - ZZ^*} \end{cases}$$

and U (resp. V) is a unitary matrix of order p (resp. q) such that $\det U \cdot \det V = 1$. Then it is easily seen that the relations b), c) and d) of (1.6) hold. Easy computations show that

$$\begin{aligned} \det(E_p - Z^* Z) &= \det(E_q - ZZ^*), \\ \det \sqrt[2]{E_p - Z^* Z} &= \sqrt[2]{\det(E_p - Z^* Z)}, \\ \det \begin{pmatrix} E_p & Z^* \\ Z & E_q \end{pmatrix} &= \det(E_p - Z^* Z). \end{aligned}$$

Hence, from (1.10), we have $\det X = 1$. Thus the relations a)~d) of (1.6) are all satisfied, i.e. $X \in G_{pq}$. The decomposition considered above depends analytically upon each element of G_{pq} .

Summarizing these results, we have the Cartan decomposition of G_{pq} .

G_{pq} is diffeomorphic to the product manifold $K_{pq} \times Z_{pq}$, where $K_{pq} = G_{pq} \cap U(n)$ and Z_{pq} is the set of all complex (q, p) -matrices satisfying the condition $E_p - Z^* Z \succ 0$. Therefore $M_{pq} = G_{pq}/K_{pq}$ is identified with Z_{pq} ([4]).

Hereafter, we shall consider only G_{1n} .

§2. In this section, we shall consider the operation on S^{2n-1} and the transformation of a measure on S^{2n-1} for $n=1$ and 2.

By its definition, G_{12} consists of all elements of $SL(3, \mathbb{C})$ which preserve an indefinite inner product $[\cdot, \cdot]$ in a 3-dimensional complex Euclidean space, where $[\cdot, \cdot]$ is defined by

$$(2.1) \quad [Z, Z] = -|Z_0|^2 + |Z_1|^2 + |Z_2|^2$$

for any element $Z = (Z_0, Z_1, Z_2) \in \mathbb{C}^3$. Therefore the hypersurface in $\mathbb{C}^3$ defined by $[Z, Z] = 0$ is invariant under the action of each element g of G_{12} . Assume that $g \in G_{12}$, $[Z, Z] = 0$ and $Z_0 \neq 0$, and set

$$\zeta_1 = Z_1/Z_0, \quad \zeta_2 = Z_2/Z_0, \quad \zeta'_1 = (gZ)_1/(gZ)_0 \quad \text{and} \quad \zeta'_2 = (gZ)_2/(gZ)_0.$$

Then we have

$$|\zeta_1|^2 + |\zeta_2|^2 = |\zeta'_1|^2 + |\zeta'_2|^2 = 1,$$

and ζ'_1 and ζ'_2 are given as follows:

$$(2.2) \quad \begin{cases} \zeta'_1 = \frac{g_{10} + g_{11}\zeta_1 + g_{12}\zeta_2}{g_{00} + g_{01}\zeta_1 + g_{02}\zeta_2}, \\ \zeta'_2 = \frac{g_{20} + g_{21}\zeta_1 + g_{22}\zeta_2}{g_{00} + g_{01}\zeta_1 + g_{02}\zeta_2}, \end{cases}$$

where $g = (g_{ij})$ ($i, j = 0, 1, 2$) is an element of G_{12} . If we denote (2.2) by $\zeta'_i = g\zeta_i$ ($i = 1, 2$), then G_{12} is considered as a transitive Lie transformation group on

$$S^3 = \{(\zeta_1, \zeta_2) \in C^2 : |\zeta_1|^2 + |\zeta_2|^2 = 1\}.$$

We set

$$(2.3) \quad \begin{cases} \zeta_1 = x_1 + \sqrt{-1}y_1, & \zeta_2 = x_2 + \sqrt{-1}y_2, \\ x_1 = \cos \theta_1, \\ y_1 = \cos \theta_2 \sin \theta_1, \\ x_2 = \cos \theta_3 \sin \theta_2 \sin \theta_1, \\ y_2 = \sin \theta_3 \sin \theta_2 \sin \theta_1, \\ 0 \leq \theta_1 < \pi, \quad 0 \leq \theta_2 < \pi, \quad 0 \leq \theta_3 < 2\pi. \end{cases}$$

and use $(\theta_1, \theta_2, \theta_3)$ as parameters of S^3 . Then the invariant measure $d\mu$ on it (with respect to the rotation group) is given by

$$(2.4) \quad d\mu = c \cdot \sin^2 \theta_1 \sin \theta_2 d\theta_1 d\theta_2 d\theta_3 \quad (c \text{ is a constant}).$$

But in this paper, we shall use $|\zeta_1|^2$, $\arg \zeta_1$ and $\arg \zeta_2$ ($0 \leq |\zeta_1|^2 \leq 1$, $0 \leq \arg \zeta_1 < 2\pi$ and $0 \leq \arg \zeta_2 < 2\pi$) as parameters of S^2 . θ_i ($i = 1, 2, 3$) and ζ_j ($j = 1, 2$) are related as follows:

$$(2.5) \quad \begin{cases} \cos \theta_1 = x_1 = R(\zeta_1) = (\zeta_1 + \bar{\zeta}_1)/2, \\ \cos \theta_2 = -\sqrt{-1}(\zeta_1 - \bar{\zeta}_1)/\sqrt{4 - (\zeta_1 + \bar{\zeta}_1)^2}, \\ \cos \theta_3 = (\zeta_2 + \bar{\zeta}_2)/2|\zeta_2|, \\ \sin \theta_1 = \sqrt{4 - (\zeta_1 + \bar{\zeta}_1)^2}/2, \\ \sin \theta_2 = 2|\zeta_2|/\sqrt{4 - (\zeta_1 + \bar{\zeta}_1)^2}, \\ \sin \theta_3 = -\sqrt{-1}(\zeta_2 - \bar{\zeta}_2)/2|\zeta_2|, \end{cases}$$

From (2.5), we have

$$(2.6) \quad \begin{cases} \sin^2 \theta_1 \sin \theta_2 = |\zeta_2| \sqrt{1 - R(\zeta_1)^2}, \\ \frac{\partial \zeta_1}{\partial |\zeta_1|^2} = \frac{1}{2\bar{\zeta}_1}, \quad \frac{\partial \bar{\zeta}_1}{\partial |\zeta_1|^2} = \frac{1}{2\zeta_1}, \\ \frac{\partial \zeta_2}{\partial |\zeta_1|^2} = \frac{-1}{2\bar{\zeta}_2}, \quad \frac{\partial \bar{\zeta}_2}{\partial |\zeta_1|^2} = \frac{-1}{2\zeta_2}, \\ \frac{\partial \zeta_i}{\partial (\arg \zeta_i)} = \sqrt{-1} \zeta_i, \quad \frac{\partial \bar{\zeta}_i}{\partial (\arg \zeta_i)} = -\sqrt{-1} \bar{\zeta}_i \quad (i=1, 2). \end{cases}$$

Using (2.6), we calculate the Jacobian J . An easy computations shows

$$(2.7) \quad J = \begin{vmatrix} \frac{\partial \theta_1}{\partial |\zeta_1|^2} & \frac{\partial \theta_1}{\partial (\arg \zeta_1)} & \frac{\partial \theta_1}{\partial (\arg \zeta_2)} \\ \frac{\partial \theta_2}{\partial |\zeta_1|^2} & \frac{\partial \theta_2}{\partial (\arg \zeta_1)} & \frac{\partial \theta_2}{\partial (\arg \zeta_2)} \\ \frac{\partial \theta_3}{\partial |\zeta_1|^2} & \frac{\partial \theta_3}{\partial (\arg \zeta_1)} & \frac{\partial \theta_3}{\partial (\arg \zeta_2)} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-(\zeta_1 + \bar{\zeta}_1)}{2|\zeta_1|^2 \sqrt{4 - (\zeta_1 + \bar{\zeta}_1)^2}} & \frac{-\sqrt{-1}(\zeta_1 - \bar{\zeta}_1)}{\sqrt{4 - (\zeta_1 + \bar{\zeta}_1)^2}} & 0 \\ \frac{\sqrt{-1}(\zeta_1 - \bar{\zeta}_1)}{|\zeta_1| 2|\zeta_2| \{4 - (\zeta_1 + \bar{\zeta}_1)^2\}} & \frac{-2(\zeta_1 + \bar{\zeta}_1)|\zeta_2|}{4 - (\zeta_1 + \bar{\zeta}_1)^2} & 0 \\ \dots\dots\dots & \dots\dots\dots & 1 \end{vmatrix}$$

$$= \frac{1}{2|\zeta_2| \sqrt{1 - R(\zeta_1)^2}}.$$

Therefore we have

$$(2.8) \quad d\mu = \sin^2 \theta_1 \sin \theta_2 d\theta_1 d\theta_2 d\theta_3 = \frac{1}{2} d|\zeta_1|^2 d(\arg \zeta_1) d(\arg \zeta_2).$$

Next we examine how the measure (2.8) is transformed under the action of an element g of G_{12} . For each $g \in G_{12}$, we set

$$g_i = (g_{i0}, g_{i1}, g_{i2}) \quad (i=0, 1, 2)$$

and

$$\langle g_i, g_j \rangle = -g_{i0} \bar{g}_{j0} + g_{i1} \bar{g}_{j1} + g_{i2} \bar{g}_{j2}.$$

Then $g \sum_{12} g^* = \sum_{12}$ is equivalent to

$$(2.9) \quad \langle g_0, g_0 \rangle = -1 \quad \text{and} \quad \langle g_i, g_j \rangle = \delta_{ij} \quad (i, j=1, 2).$$

From the fact that $\sum_{i2}^{-1} = \sum_{12}$ and G is a group, it follows

$$g^* \sum_{12} g = \sum_{12}.$$

Then (2.9) is equivalent to

$$(2.9)' \quad \langle {}^t g_0, {}^t g_0 \rangle = -1 \quad \text{and} \quad \langle {}^t g_i, {}^t g_j \rangle = \delta_{ij} \quad (i, j = 1, 2),$$

where ${}^t g_i = (g_{0i}, g_{1i}, g_{2i})$ ($i = 0, 1, 2$).

The relation between $(|\zeta'_1|^2, \arg \zeta'_1, \arg \zeta'_2)$ and $(|\zeta_1|^2, \arg \zeta_1, \arg \zeta_2)$ is given as follows:

$$(2.10) \quad \begin{cases} |\zeta'_1|^2 = \frac{|A|^2}{|C|^2}, \\ \arg \zeta'_1 = \frac{1}{2\sqrt{-1}} \{ \log A - \log C + \log \bar{C} - \log \bar{A} \} + \text{const}, \\ \arg \zeta'_2 = \frac{1}{2\sqrt{-1}} \{ \log B - \log C + \log \bar{C} - \log \bar{B} \} + \text{const}, \end{cases}$$

where

$$\begin{cases} C = g_{00} + g_{01}\zeta_1 + g_{02}\zeta_2, \\ A = g_{10} + g_{11}\zeta_1 + g_{12}\zeta_2, \\ B = g_{20} + g_{21}\zeta_1 + g_{22}\zeta_2. \end{cases}$$

Using (2.10), we have

$$(2.11) \quad \begin{cases} \frac{\partial |\zeta'_1|^2}{\partial |\zeta_1|^2} = \frac{1}{|C|^4} P_{11}, & \frac{\partial |\zeta'_1|^2}{\partial (\arg \zeta_1)} = \frac{\sqrt{-1}}{|C|^4} P_{12}, & \frac{\partial |\zeta'_1|^2}{\partial (\arg \zeta_2)} = \frac{\sqrt{-1}}{|C|^4} P_{13}, \\ \frac{\partial (\arg \zeta'_1)}{\partial |\zeta_1|^2} = \frac{P_{21}}{2\sqrt{-1}|A|^2|C|^2}, & \frac{\partial (\arg \zeta'_1)}{\partial (\arg \zeta_1)} = \frac{P_{22}}{2|A|^2|C|^2}, & \frac{\partial (\arg \zeta'_1)}{\partial (\arg \zeta_2)} = \frac{P_{23}}{2|A|^2|C|^2}, \\ \frac{\partial (\arg \zeta'_2)}{\partial |\zeta_1|^2} = \frac{P_{31}}{2\sqrt{-1}|B|^2|C|^2}, & \frac{\partial (\arg \zeta'_2)}{\partial (\arg \zeta_1)} = \frac{P_{32}}{2|B|^2|C|^2}, & \frac{\partial (\arg \zeta'_2)}{\partial (\arg \zeta_2)} = \frac{P_{33}}{2|B|^2|C|^2}, \end{cases}$$

where

$$\begin{aligned} P_{11} &= \left(\frac{g_{11}}{2\bar{\zeta}_1} - \frac{g_{12}}{2\bar{\zeta}_2} \right) \bar{A}|C|^2 + \left(\frac{\bar{g}_{11}}{2\zeta_1} - \frac{\bar{g}_{12}}{2\zeta_2} \right) A|C|^2 - \left(\frac{g_{01}}{2\bar{\zeta}_1} - \frac{g_{02}}{2\bar{\zeta}_2} \right) |A|^2 \bar{C} \\ &\quad - \left(\frac{\bar{g}_{01}}{2\zeta_1} - \frac{\bar{g}_{02}}{2\zeta_2} \right) |A|^2 C, \\ P_{12} &= g_{11}\zeta_1 \bar{A}|C|^2 - \bar{g}_{11}\bar{\zeta}_1 A|C|^2 - g_{01}\zeta_1 |A|^2 \bar{C} + \bar{g}_{01}\bar{\zeta}_1 |A|^2 C, \\ P_{13} &= g_{12}\zeta_2 \bar{A}|C|^2 - \bar{g}_{12}\bar{\zeta}_2 A|C|^2 - g_{02}\zeta_2 |A|^2 \bar{C} + \bar{g}_{02}\bar{\zeta}_2 |A|^2 C, \end{aligned}$$

$$\begin{aligned}
P_{21} &= \left(\frac{g_{11}}{2\bar{\zeta}_1} - \frac{g_{12}}{2\bar{\zeta}_2} \right) \bar{A}|C|^2 - \left(\frac{g_{01}}{2\bar{\zeta}_1} - \frac{g_{02}}{2\bar{\zeta}_2} \right) |A|^2 \bar{C} + \left(\frac{\bar{g}_{01}}{2\bar{\zeta}_1} - \frac{\bar{g}_{02}}{2\bar{\zeta}_2} \right) |A|^2 C \\
&\quad - \left(\frac{\bar{g}_{11}}{2\bar{\zeta}_1} - \frac{\bar{g}_{12}}{2\bar{\zeta}_2} \right) A|C|^2, \\
P_{22} &= g_{11}\zeta_1 \bar{A}|C|^2 - g_{01}\zeta_1 |A|^2 \bar{C} - \bar{g}_{01}\bar{\zeta}_1 |A|^2 C + \bar{g}_{11}\bar{\zeta}_1 A|C|^2, \\
P_{23} &= g_{12}\zeta_2 \bar{A}|C|^2 - g_{02}\zeta_2 |A|^2 \bar{C} - \bar{g}_{02}\bar{\zeta}_2 |A|^2 C + \bar{g}_{12}\bar{\zeta}_2 A|C|^2, \\
P_{31} &= \left(\frac{g_{21}}{2\bar{\zeta}_1} - \frac{g_{22}}{2\bar{\zeta}_2} \right) \bar{B}|C|^2 - \left(\frac{g_{01}}{2\bar{\zeta}_1} - \frac{g_{02}}{2\bar{\zeta}_2} \right) |B|^2 \bar{C} + \left(\frac{\bar{g}_{01}}{2\bar{\zeta}_1} - \frac{\bar{g}_{02}}{2\bar{\zeta}_2} \right) |B|^2 C \\
&\quad - \left(\frac{\bar{g}_{21}}{2\bar{\zeta}_1} - \frac{\bar{g}_{22}}{2\bar{\zeta}_2} \right) B|C|^2, \\
P_{32} &= g_{21}\zeta_1 \bar{B}|C|^2 - g_{01}\zeta_1 |B|^2 \bar{C} - \bar{g}_{01}\bar{\zeta}_1 |B|^2 C + \bar{g}_{21}\bar{\zeta}_1 B|C|^2, \\
P_{33} &= g_{22}\zeta_2 \bar{B}|C|^2 - g_{02}\zeta_2 |B|^2 \bar{C} - \bar{g}_{02}\bar{\zeta}_2 |B|^2 C + \bar{g}_{22}\bar{\zeta}_2 B|C|^2.
\end{aligned}$$

Therefore the Jacobian J of a transformation g is

$$J = \frac{1}{4|A|^2|B|^2|C|^8} \begin{vmatrix} P_{11} & \sqrt{-1}P_{12} & \sqrt{-1}P_{13} \\ -\sqrt{-1}P_{21} & P_{22} & P_{23} \\ -\sqrt{-1}P_{31} & P_{32} & P_{33} \end{vmatrix}.$$

From (2.9) and (2.9)', we have

$$P_{21} + P_{31} = 0, \quad P_{22} + P_{32} = 2|C|^2|\zeta_1|^2 \quad \text{and} \quad P_{23} + P_{33} = 2|C|^2|\zeta_2|^2.$$

Using these relations, we have

$$\begin{aligned}
J &= \frac{1}{4|A|^2|B|^2|C|^8} \begin{vmatrix} P_{11} & \sqrt{-1}P_{12} & \sqrt{-1}P_{13} \\ -\sqrt{-1}P_{21} & P_{22} & P_{23} \\ 0 & 2|C|^2|\zeta_1|^2 & 2|C|^2|\zeta_2|^2 \end{vmatrix} \\
&= \frac{1}{4|A|^2|B|^2|C|^8} \times 2|C|^2 \{ |\zeta_2|^2 (P_{11}P_{22} - P_{12}P_{21}) - |\zeta_1|^2 (P_{11}P_{23} - P_{21}P_{13}) \} \\
&= \frac{1}{4|A|^2|B|^2|C|^8} \times 2|C|^2 \times 2|A|^2|B|^2|C|^2 = \frac{1}{|C|^4}.
\end{aligned}$$

Thus we have

$$(2.12) \quad J = \frac{1}{|C|^4}.$$

On the other hand, each element k of K_{12} has the following form:

$$k = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{pmatrix}, \quad \text{where } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U(2) \text{ and } \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = e^{-i\theta}.$$

Then it follows that

$$C = |k_{00} + k_{01}\zeta_1 + k_{02}\zeta_2| = |e^{i\theta}| = 1.$$

Therefore the measure (2.8) is invariant under the action of each element k of K_{12} .

Summarizing these results, we have

PROPOSITION 1. *The measure $d\mu$ on S^3 , where $d\mu$ is given by*

$$d\mu = c \cdot d|\zeta_1|^2 d(\arg \zeta_1) d(\arg \zeta_2),$$

is transformed into

$$d\mu' = \frac{1}{|g_{00} + g_{01}\zeta_1 + g_{02}\zeta_2|^4} d\mu.$$

under the action of an element g of G_{12} .

In particular, $d\mu$ is invariant under the action of each element k of a maximal compact subgroup K_{12} .

Next we consider G_{11} . In this case, each element g of G_{11} has the following form:

$$(2.13) \quad g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \quad (|\alpha|^2 - |\beta|^2 = 1)$$

and the action of g on S^1 is given as follows:

$$(2.14) \quad \zeta' = g\zeta = \frac{\bar{\beta} + \bar{\alpha}\zeta}{\alpha + \beta\zeta}.$$

If we write $\zeta = e^{i\theta}$ and use θ as a parameter of S^1 , then θ' corresponding to $\zeta' = g\zeta$ is given by

$$(2.15) \quad \theta' = \frac{1}{\sqrt{-1}} \log \left(\frac{\bar{\beta} + \bar{\alpha}e^{i\theta}}{\alpha + \beta e^{i\theta}} \right) + 2n\pi.$$

An easy computation shows:

$$(2.16) \quad d\theta' = \frac{1}{|\alpha + \beta e^{i\theta}|^2} d\theta.$$

Since each element $k \in K_{11}$ has a form $k = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$, $d\theta'$ is the invariant measure (with respect to K_{11}) on S^1 . Thus the similar proposition holds in this case.

REMARK. The same relation holds for arbitrary n ,

$$\text{i. e.} \quad J = \frac{1}{|g_{00} + g_{01}\zeta_1 + \dots + g_{0n}\zeta_n|^{2n}}.$$

§ 3. In this section, we construct some irreducible unitary representations of class 1 with respect to maximal compact subgroups K_{1n} .

Let $\mathfrak{h}_{2n-1}$ be Hilbert spaces of square integrable functions on S^{2n-1} ($n=1, 2$) with respect to the measures $d\mu$ considered in § 2.

We define operators on $\mathfrak{h}_{n-1}$ as follows:

$$(3.1) \quad T_g f(\zeta) = f(g\zeta) |g_{00} + g_{01}\zeta|^{-1+i\rho},$$

where $g \in G_{11}$, $f \in \mathfrak{h}_1$, $\zeta \in S^1$ and ρ is an arbitrary real number, and

$$(3.2) \quad T_g f(\zeta) = f(g\zeta) |g_{00} + g_{01}\zeta_1 + g_{02}\zeta_2|^{-2+i\rho},$$

where $g \in G_{12}$, $\zeta = (\zeta_1, \zeta_2) \in S^3$, $f \in \mathfrak{h}_3$ and ρ is an arbitrary real number.

Then operators T_g defined above are unitary operators on $\mathfrak{h}_{2n-1}$, and $(T, \mathfrak{h}_{2n-1})$ are unitary representations of G_{1n} . Because, for $n=1$, we have

$$\begin{aligned} (T_g f, T_g f) &= \int_{S^1} f(g\zeta) \overline{f(g\zeta)} |g_{00} + g_{01}\zeta|^{-1+i\rho} \overline{|g_{00} + g_{01}\zeta|^{-1+i\rho}} d\mu \\ &= \int_{S^1} f(g\zeta) \overline{f(g\zeta)} |g_{00} + g_{01}\zeta|^{-2} d\mu(\mu) \\ &= \int_{S^1} f(g\zeta) \overline{f(g\zeta)} d\mu(g\zeta) = (f, f), \end{aligned}$$

and the same is also true for $n=2$. Note that $(T, \mathfrak{h}_3)$ is contained in representations constructed in [7] and irreducible except for $\rho=0$.

In both cases, there exist no invariant vectors with respect to K_{1n} except for constant functions on S^{2n-1} and zonal spherical functions are given as follows:

$$(3.3) \quad \varphi(g) = (T_g 1, 1) = \int_{S^1} |g_{00} + g_{01}\zeta|^{-1+i\rho} d\mu(\zeta),$$

$$(3.4) \quad \varphi(g) = (T_g 1, 1) = \int_{S^3} |g_{00} + g_{01}\zeta_1 + g_{02}\zeta_2|^{-2+i\rho} d\mu(\zeta).$$

Since these functions are constant on each double coset mod K_{1n} , we may

calculate them only on Z_{1n} . We can show easily that each element $z \in Z_{11}$ has the following form:

$$(3.5) \quad z = \begin{pmatrix} \frac{1}{\sqrt{1-|w|^2}} & \frac{\bar{w}}{\sqrt{1-|w|^2}} \\ \frac{w}{\sqrt{1-|w|^2}} & \frac{1}{\sqrt{1-|w|^2}} \end{pmatrix},$$

where $w \in \mathbb{C}$ and $|w| < 1$. If we denote $\zeta \in S^1$ by $\zeta = e^{i\theta}$, then we have

$$(3.6) \quad |g_{00} + g_{01}\zeta|^{-1+i\rho} = \left| \frac{1+|w|^2+2|w|\cos(\theta-\alpha)}{1-|w|^2} \right|^{\frac{-1+i\rho}{2}},$$

where $\tan \alpha = \text{Im}(w)/\text{Re}(w)$. If we set $x = (1+|w|^2)/(1-|w|^2)$, then we have $x^2 - 1 = 4|w|^2/(1-|w|^2)^2$, i. e. $2|w|/(1-|w|^2) = \sqrt{x^2 - 1}$. Thus it follows that

$$(3.7) \quad |g_{00} + g_{01}\zeta|^{-1+i\rho} = (x + \sqrt{x^2 - 1} \cos(\theta - \alpha))^{\frac{-1+i\rho}{2}}.$$

From (3.7), $\varphi(g)$ is given by

$$(3.8) \quad \begin{aligned} \varphi(g) &= \int_{S^1} (x + \sqrt{x^2 - 1} \cos(\theta - \alpha))^{\frac{-1+i\rho}{2}} d\mu(\theta) \\ &= \int_0^{2\pi} (x + \sqrt{x^2 - 1} \cos \theta)^{\frac{-1+i\rho}{2}} d\theta \\ &= P_{-\frac{1}{2}+i\sigma}(x) = P_{-\frac{1}{2}+i\sigma}\left(\frac{1+|w|^2}{1-|w|^2}\right) \quad \left(\sigma = \frac{\rho}{2}\right), \end{aligned}$$

where P_ν is a so-called Legendre function of order ν .

The above formula shows that $\varphi(g)$ depends upon only a radius $|w| = r$ and constant on each circle $\{w : |w| = r, 0 < r < 1\}$.

Thus we have

THEOREM 1. *A series of zonal spherical functions on M_{11} is given by (3.8).*

We can easily find an explicit form of $z \in Z_{12}$. But we exhibit here only the first row of z , because we don't need other rows to compute $\varphi(g)$. The first row of z is given by

$$(3.9) \quad \left(\frac{1}{\sqrt{1-(|w_1|^2+|w_2|^2)}}, \frac{\bar{w}_1}{\sqrt{1-(|w_1|^2+|w_2|^2)}}, \frac{\bar{w}_2}{\sqrt{1-(|w_1|^2+|w_2|^2)}} \right),$$

where $w_1, w_2 \in \mathbb{C}$ and $|w_1|^2 + |w_2|^2 < 1$.

Hence we have

$$(3.10) \quad g_{00} + g_{01}\zeta_1 + g_{02}\zeta_2 = \frac{1}{\sqrt{1-(|w_1|^2 + |w_2|^2)}} (1 + \bar{w}_1\zeta_1 + \bar{w}_2\zeta_2).$$

If we denote w_1 , w_2 , ζ_1 and ζ_2 by $|w_1|e^{i\theta_1}$, $|w_2|e^{i\theta_2}$, $|\zeta_1|e^{i\alpha}$ and $\sqrt{1-|\zeta_1|^2}e^{i\beta}$, then easy but rather long computations show:

$$(3.11) \quad |g_{00} + g_{01}\zeta_1 + g_{02}\zeta_2|^2 = \frac{1}{1-|w_1|^2-|w_2|^2} [1 + |w_1|^2|\zeta_1|^2 + |w_2|^2(1-|\zeta_1|^2) + 2|w_1||\zeta_1| \cos(\theta_1 + \alpha) + 2|w_2|\sqrt{1-|\zeta_1|^2} \cos(\theta_2 + \beta) + 2|w_1||w_2||\zeta_1|\sqrt{1-|\zeta_1|^2} \cos\{(\theta_1 - \theta_2) + (\alpha + \beta)\}].$$

Hence it follows that

$$(3.12) \quad \begin{aligned} \varphi(g) &= \varphi(w_1, w_2) \\ &= \left(\frac{1}{1-|w_1|^2-|w_2|^2} \right)^{-1+i\sigma} \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \{1 + |w_1|^2x + |w_2|^2(1-x) \\ &\quad + 2|w_1|\sqrt{x} \cos \theta_1 + 2|w_2|\sqrt{1-x} \cos \theta_2 \\ &\quad + 2|w_1||w_2|\sqrt{x}\sqrt{1-x} \cos(\theta_1 - \theta_2)\}^{-1+i\sigma} d\theta_1 d\theta_2 dx, \end{aligned}$$

where $\theta_i = \arg \zeta_i$ ($i=1, 2$), $x = |\zeta_1|^2$ and $\sigma = \rho/2$.

Thus we have

THEOREM 2. *A series of zonal spherical functions on M_{12} is given by (3.12).*

The above formula shows that $\varphi(g)$ depends upon only absolute values $|w_1| = r_1$ and $|w_2| = r_2$.

The author does not know whether they can be expressed in terms of classical special functions or not.

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