SEMI-SIMPLICITY IN INFINITE-DIMENSIONAL LIE GROUPS

By

Satoru YAMAGUCHI

(Received June 28, 1963)

I. In 1925, M. Fréchet extended the notion of a differentiable mapping of a Euclidean space R^m into R^n to a mapping of a Banach space $\mathfrak{B}$ into $\mathfrak{B}'$ ([2]). Recently, using this notion, B. Maissen defined Lie groups with Banach spaces as parameter spaces and showed that many analogous results hold as in finite-dimensional cases ([4]). To take a very simple case, $GL(\mathfrak{B})$ is such an example, where $\mathfrak{B}$ is a Banach space and $GL(\mathfrak{B})$ is a group of continuous automorphisms of $\mathfrak{B}$. Here we explain briefly the notion of a differentiable mapping $f: \mathfrak{B} \rightarrow \mathfrak{B}'$.

f is said to be differentiable at $x \in \mathfrak{B}$ in the sense of Fréchet if there exists a bounded linear operator $f'(x): \mathfrak{B} \rightarrow \mathfrak{B}'$ such that

$$f(x+h) - f(x) = f'(x)h + \|h\| \cdot \langle h; x \rangle,$$

where $\|\langle h; x \rangle\| \rightarrow 0$ as $\|h\| \rightarrow 0$.

The notion of the differentiability of higher order is also defined ([5]).

A mapping $f: \mathfrak{B} \rightarrow \mathfrak{B}'$ is said to be locally uniformly continuously differentiable if for any $x \in \mathfrak{B}$, there exists a neighbourhood U of x on which $f'(x)$ exists and is uniformly continuous.

In this note and in the forthcoming papers, we shall consider Lie groups which are at least 3-times uniformly continuously differentiable.

Let $L(\mathfrak{B})$ be a space of continuous endomorphisms of $\mathfrak{B}$. Then it can be proved that with respect to the norm-topology in $L(\mathfrak{B})$, $GL(\mathfrak{B})$ is open in $L(\mathfrak{B})$ and becomes a topological group with the following property:

Each element $g \in GL(\mathfrak{B})$ has a neighbourhood such that it is homeomorphic to an open subset of a Banach space $L(\mathfrak{B})$ and the mapping $(g, g') \rightarrow gg'^{-1}$ of $GL(\mathfrak{B}) \times GL(\mathfrak{B})$ into $GL(\mathfrak{B})$ is any times differentiable (in the sense of Fréchet) with respect to such parameters.

This example leads us to the notion of infinite-dimensional Lie groups with Banach spaces as parameter spaces. In what follows a Lie group G

means such an infinite-dimensional Lie group.

In this paper, we shall define the semi-simplicity of infinite dimensional Lie groups and consider some of its properties. Certain examples will be considered in the next paper.

II. At the beginning, we mention some definitions and results obtained in ([4]).

A Hausdorff space M is said to be a differentiable manifold of class C^k over a Banach space $\mathfrak{B}$ if there exists an open covering $\{U_\alpha\}_{\alpha \in A}$ satisfying the following conditions:

M I. each $U_\alpha \in \{U_\alpha\}$ is homeomorphic to an open subset $0_\alpha \subset \mathfrak{B}$ under a mapping φ_α .

M II. if $U_\alpha \cap U_\beta \neq \emptyset$, then mapping

$$\varphi_{\beta\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

is of class C^k .

Notions of tangent vectors at $p \in M$, a tangent space T_p at $p \in M$, vector fields on M , tensor fields on M and differentiability of these fields are also defined as usual.

A differentiable manifold G of class C^k over a Banach space $\mathfrak{B}$ is said to be a differentiable Lie group of class C^k over $\mathfrak{B}$ if the following conditions are satisfied:

LG I. G is a topological group with respect to this topology.

LG II. the mapping $(g, g') \rightarrow gg'^{-1}$ of $G \times G$ into G is of class C^k .

The notion of invariant vector fields on G can be defined and the set of all left (or right) invariant vector fields forms a Lie algebra which is isomorphic to T_e . The Lie algebra $\mathfrak{G}$ of G is a Banach space in this case and a bracket operation is a continuous linear mapping of $\mathfrak{G} \times \mathfrak{G}$ into $\mathfrak{G}$.

Let $\mathfrak{G}$ be a Lie algebra of G . Then for any closed subalgebra $\mathfrak{G}'$ of $\mathfrak{G}$ there exists a Lie subgroup G' of G whose Lie algebra is $\mathfrak{G}'$ and conversely. Note that Lie subalgebra corresponding to Lie subgroups are closed linear subspaces of $\mathfrak{G}$. It is always satisfied in a finite-dimensional case.

If G' is a distinguished Lie subgroup of G , then the Lie subalgebra $\mathfrak{G}'$ corresponding to it is a closed ideal of $\mathfrak{G}$. Conversely a Lie subgroup G' corresponding to a closed ideal $\mathfrak{G}'$ of $\mathfrak{G}$ is distinguished if G is connected.

Let G (resp. G') be a Lie group and $\mathfrak{G}$ (resp. $\mathfrak{G}'$) be a Lie algebra corresponding to it. Let $\varphi : \mathfrak{G} \rightarrow \mathfrak{G}'$ be a continuous homomorphism as Lie

algebras. Then there exists a local homomorphism Φ whose differential at $e \in G$ is φ , i.e.

$$\Phi'(e) = \varphi.$$

If G is simply connected Φ is extended to a homomorphism of G into G' .

The facts described in this section can be proved just as in a finite-dimensional case ([1]) with the help of analysis, especially the theory of differential equations in a Banach space ([3], [5]).

III. We shall define the *semi-simplicity* of infinite-dimensional Lie groups. Let G and $\mathfrak{G}$ as in II.

DEFINITION. $\mathfrak{G}$ is said to be *semi-simple* if for any closed ideal $\mathfrak{G}_1$ of $\mathfrak{G}$ there exists a closed ideal $\mathfrak{G}_2$ such that $\mathfrak{G}$ is a topological direct sum of $\mathfrak{G}_1$ and $\mathfrak{G}_2$. G is said to be *simple* if closed ideals of G are only (0) and G .

From the above definition, we have

THEOREM. Let G be a simply connected semi-simple Lie group and $\mathfrak{G}$ be its Lie algebra. If $\mathfrak{G}$ is decomposed into a direct sum of two closed ideals $\mathfrak{G}_1$ and $\mathfrak{G}_2$, then G is isomorphic to the direct product of $\tilde{G}_1$ and $\tilde{G}_2$, where $\tilde{G}_1$ (resp. $\tilde{G}_2$) is a simply connected Lie group whose Lie algebra is isomorphic to $\mathfrak{G}_1$ (resp. $\mathfrak{G}_2$).

Proof. We use the following lemma.

LEMMA. The universal covering group of a connected Lie group G is also a Lie group and its Lie algebra is isomorphic to that of G .

The above lemma is proved just as in a finite-dimensional case.

Let G_i ($i=1, 2$) be a Lie subgroup of G corresponding to $\mathfrak{G}_i$ and $\tilde{G}_i$ be a universal covering group of G_i . A Lie algebra of $\tilde{G}_1 \times \tilde{G}_2$ is isomorphic to $\mathfrak{G}$. So we have a natural isomorphism $\alpha: \mathfrak{G} \rightarrow \mathfrak{G}_1 \times \mathfrak{G}_2$ and a reciprocal mapping $\beta: \mathfrak{G}_1 \times \mathfrak{G}_2 \rightarrow \mathfrak{G}$, where $\mathfrak{G}_1 \times \mathfrak{G}_2$ is considered as a Lie algebra of $\tilde{G}_1 \times \tilde{G}_2$. Since G and $\tilde{G}_1 \times \tilde{G}_2$ are both simply connected there correspond to these mappings onto-homomorphisms $A: G \rightarrow \tilde{G}_1 \times \tilde{G}_2$ and $B: \tilde{G}_1 \times \tilde{G}_2 \rightarrow G$ such that $\alpha = A'(e)$ and $\beta = B'((e, e))$. Since $\eta_1 = \beta \cdot \alpha$ (resp. $\eta_2 = \alpha \cdot \beta$) is the identity mapping of $\mathfrak{G}$ (resp. $\mathfrak{G}_1 \times \mathfrak{G}_2$), there corresponds to this mapping the identity mapping $I_1: G \rightarrow G$ (resp. $I_2: \tilde{G}_1 \times \tilde{G}_2 \rightarrow \tilde{G}_1 \times \tilde{G}_2$). And $I_1, B \cdot A$ coincide on some neighbourhood of the unit element e of G . Therefore they must coincide on the whole of G . Similarly we have $I_2 = A \cdot B$. In other words, A is an isomorphism of G onto $\tilde{G}_1 \times \tilde{G}_2$. q.e.d.

COROLLARY. *Let $\mathfrak{G}$ be a Lie algebra. Then simply connected Lie groups whose Lie algebras are isomorphic to $\mathfrak{G}$ are, if they exist, mutually isomorphic.*

This follows immediately from the argument in the proof of our Theorem.

REMARK. Our definition has no meaning if there exists no simple Lie group whose dimension is *not finite*. Some examples in a separable complex Hilbert space will be considered in the next paper.

References

- [1] C. CHEVALLEY, Theory of Lie groups. Princeton University Press, (1946).
- [2] M. FRÉCHET, *La notion de différentielle dans l'analyse générale*, Ann. Sci. Ecole Normale Sup., 42 (1925).
- [3] H. KELLER, *Ueber die Differentialgleichung erster Ordnung in normierten linearen Räumen*. Rend. Circ. Math. Palermo II, VIII (1959).
- [4] B. MAISSEN, *Lie-Gruppen mit Banachräume als Parameterräume*. Acta Mathematica, 108 (1962).
- [5] R. u. F. NEVANLINNA, Absolute Analysis. Springer Verlag, 1959.