A REMARK ON THE STRONG MARKOV PROPERTY

By

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1. The purpose of this note is to prove a theorem on the strong Markov property which has been essentially known but whose explicit proof could not be found in any references.

Let $(E, \mathcal{B})$ be a measurable space such that each single point is measurable, and let $X=(x_t, \zeta, \mathcal{M}_t, P_x, \theta_t)$ be a measurable stationary Markov process with the phase space E . (We shall use the notations and terminologies of [1] with no special reference.) The Borel field $\mathcal{M}_t$ which is given in the space $\Omega_t = \{\zeta > t\}$ for each $t \geq 0$ is *relatively increasing with t* in the sense that, if $t \geq s$ and $A \in \mathcal{M}_s$, then $A \cap \Omega_t \in \mathcal{M}_t$. For each fixed x , $P_x(\cdot)$ defines a probability measure on $(\Omega, \mathcal{M})$, where $\mathcal{M}$ is a Borel field in Ω including all $\mathcal{M}_t$ for $t \geq 0$. Let $\mathcal{N}^{(x)}$ be the completed Borel field of $\mathcal{M}$ relative to P_x , and $\mathcal{N}_t^{(x)}$ the Borel field formed by all the sets of $\mathcal{N}^{(x)}$ which are contained in the space Ω_t and differ by at most a set of P_x -measure 0 from some set of $\mathcal{M}_t$. Since the transition function

$$P(t, x, \Gamma) = P_x\{x_t \in \Gamma\}^{1)}, \quad \Gamma \in \mathcal{B}$$

is measurable in (t, x) for each Γ , the function $P(\gamma, x_\tau, \Gamma)$ is $\mathcal{N}^{(x)}$ -measurable for any $\mathcal{N}^{(x)}$ -measurable random times²⁾ $\tau(\omega)$ and $\gamma(\omega)$. In general, consider a family of Borel fields $(\mathcal{N}) = \{\mathcal{N}_t, \mathcal{N}\}$ which satisfies the same conditions as the family $(\mathcal{M}) = \{\mathcal{M}_t, \mathcal{M}\}$: (N.1) For each $t \geq 0$, $\mathcal{N}_t$ is a Borel field in Ω_t . (N.2) $\mathcal{N}_t$ is relatively increasing with t . (N.3) $\mathcal{N}$ is a Borel field in Ω containing all $\mathcal{N}_t$ for $t \geq 0$. A random time $\tau(\omega)$ is said to be an $(\mathcal{N})$ *stopping time* if $\{\omega; \tau(\omega) < t < \zeta(\omega)\} \in \mathcal{N}_t$ for all $t \geq 0$. For an $(\mathcal{N})$ stopping time τ , $\mathcal{N}_{\tau+}$ denotes the Borel field formed by all sets $A \in \mathcal{N}$ such that $A \subset \Omega_\tau = \{\tau < \zeta\}$ and $A \cap \{\tau < t < \zeta\} \in \mathcal{N}_t$ for $t \geq 0$. If τ is a constant time s , then $\mathcal{N}_{s+}$ is the greatest Borel field contained in $\mathcal{N}_t$ for all $t > s$ in the relative sense.³⁾ Now let us

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1) Of course, it is assumed that $\{x_t \in \Gamma\} \in \mathcal{M}_t$ for each t and $\Gamma \in \mathcal{B}$.

2) A non-negative function defined on Ω .

3) If $A \in \mathcal{N}_{s+}$ and $t > s$, then $A \cap \Omega_t \in \mathcal{N}_t$.

fix the point x and assume that our family $(\mathcal{N})$ satisfies one more condition: (N.4) $\mathcal{N}_t \subset \mathcal{N}_t^{(x)}$, $\mathcal{N} \subset \mathcal{N}^{(x)}$. We shall say that the process X is *strongly Markovian relative to the family $(\mathcal{N})$ at the point x* if, for any $(\mathcal{N})$ stopping time τ , $\mathcal{N}_{\tau+}$ -measurable random time η , $A \in \mathcal{N}_{\tau+}$ and $\Gamma \in \mathcal{B}$.

$$(1) \quad P_x(A, x_{\eta+\tau} \in \Gamma) = E_x(P(\eta, x_\tau, \Gamma); A).^{4)}$$

THEOREM. *If the process X is strongly Markovian relative to the family $(\mathcal{M})$ at x , then it is so relative to any family $(\mathcal{N})$ satisfying the conditions (N.1)–(N.4).*

We shall divide the proof into several steps.

1°) For an $(\mathcal{M})$ stopping time τ , let $\mathcal{M}_{\tau+}^{(x)}$ be the Borel field of all the sets of $\mathcal{N}^{(x)}$ which are contained in $\mathcal{Q}_\tau$ and which differ by at most a set of P_x -measure 0 from some set of $\mathcal{M}_{\tau+}$. It is easy to see that the assumption in the theorem implies that the relation (1) is true for any $(\mathcal{M})$ stopping time τ , $\mathcal{M}_{\tau+}^{(x)}$ -measurable random time η , $A \in \mathcal{M}_{\tau+}^{(x)}$ and $\Gamma \in \mathcal{B}$.

2°) We shall show that, if τ is an $(\mathcal{M})$ stopping time, then $\mathcal{M}_{\tau+}^{(x)} = \mathcal{N}_{\tau+}^{(x)}$. If $A \in \mathcal{M}_{\tau+}^{(x)}$, there is a set A' of $\mathcal{M}_{\tau+}$ such that $P_x(A' \ominus A) = 0$, where the symbol $\ominus$ means the symmetric difference. Since

$$A' \cap \{\tau < t < \zeta\} \in \mathcal{M}_t \quad \text{and} \quad P_x(\{A \cap (\tau < t < \zeta)\} \ominus \{A' \cap (\tau < t < \zeta)\}) = 0,$$

we have $A \cap \{\tau < t < \zeta\} \in \mathcal{N}_t^{(x)}$. By the assumption $A \subset \mathcal{Q}_\tau$ and $A \in \mathcal{N}^{(x)}$. Therefore $A \in \mathcal{N}_{\tau+}^{(x)}$. Conversely, suppose that $A \in \mathcal{N}_{\tau+}^{(x)}$. We have to show that there is a set A' of $\mathcal{M}_{\tau+}$ such that $P_x(A' \ominus A) = 0$. Since $A \cap \{\tau < t < \zeta\} \in \mathcal{N}_t^{(x)}$, there is a set A_t of $\mathcal{M}_t$ such that $P_x(A_t \ominus (A \cap \{\tau < t < \zeta\})) = 0$. We may assume that A_t is contained in the set $\{\tau < t < \zeta\}$. Define the following sets:

$$\begin{aligned} B_r &= \bigcup_{\text{all } r' \leq r} (A_{r'} \cap \mathcal{Q}_r) \\ C_r &= B_r - \bigcup_{\text{all } r' < r} (B_r \cap \tilde{B}_{r'} \cap \{\tau < r' < \zeta\}), \\ A' &= \bigcup_{\text{all } r} C_r, \end{aligned}$$

where r, r' are non-negative rationals and the symbol $\sim$ means the complement of a set in $\mathcal{Q}$. B_r is contained in the set $\{\tau < r < \zeta\}$, is in $\mathcal{M}_r$, increases in the relative sense (if $r' < r$, then $B_{r'} \cap \mathcal{Q}_r \subset B_r$) and differs by at most a

4) In general, $E_x(f(\omega); A) = \int_A f(\omega) P_x(d\omega)$.

set of P_x -measure 0 from the set $A \cap \{\tau < r < \zeta\}$. C_r also has the same properties as B_r , so that $P_x(A' \ominus A) = 0$. Obviously $A' \subset \mathcal{Q}_\tau$ and $A' \in \mathcal{M}$. It follows that $A' \cap \{\tau < r < \zeta\} = C_r$ for all rationals r . Indeed, it is clear that $A' \cap \{\tau < r < \zeta\} \supset C_r$. On the other hand, if $r' \leq r$, then $C_{r'} \cap \{\tau < r < \zeta\} \subset C_r$ (by the relatively increasing property) and if $r' > r$, then $C_{r'} \cap \{\tau < r < \zeta\} = C_r \cap \mathcal{Q}_{r'} \subset C_r$, so that $A' \cap \{\tau < r < \zeta\} \subset C_r$. For any real $t \geq 0$,

$$\begin{aligned} A' \cap \{\tau < t < \zeta\} &= \bigcup_{r < t} (A' \cap \{\tau < r < \zeta\} \cap \mathcal{Q}_t) \\ &= \bigcup_{r < t} (C_r \cap \mathcal{Q}_t) \in \mathcal{M}_t. \end{aligned}$$

3°) For any $(\mathcal{H}^{(x)})$ stopping time τ , there is an $(\mathcal{M})$ stopping time τ_0 which coincides with τ almost everywhere (P_x). To show this, take the sequence of sets $A_{n,k} = \left\{ \frac{k-1}{n} \leq \tau < \frac{k}{n} < \zeta \right\}$. Since $A_{n,k} \in \mathcal{H}_{k/n}^{(x)}$, there is $A'_{n,k} \in \mathcal{M}_{k/n}$ such that $P_x(A'_{n,k} \ominus A_{n,k}) = 0$. One can assume that the sets $A'_{n,k}$, $k=1, 2, \dots$, are disjoint for each n . Define

$$\begin{aligned} \tau_n(\omega) &= \frac{k}{n} \quad \text{if } \omega \in A'_{n,k}, \\ &= \zeta \quad \text{otherwise,} \\ \tau_0(\omega) &= \limsup_{n \rightarrow \infty} \tau_n(\omega). \end{aligned}$$

This τ_0 is the desired $(\mathcal{M})$ stopping time.

4°) To prove the theorem, it is enough to show the case of $\mathcal{H} = \mathcal{H}^{(x)}$, $\mathcal{H}_t = \mathcal{H}_t^{(x)}$. Let τ be any $(\mathcal{H}^{(x)})$ stopping time and τ_0 , the $(\mathcal{M})$ stopping time constructed just above. By definition, $\mathcal{H}_{\tau_0+}^{(x)} = \mathcal{H}_{\tau_0+}^{(x)}$. Since $\mathcal{H}_{\tau_0+}^{(x)} = \mathcal{M}_{\tau_0+}^{(x)}$ by 2°, any $\mathcal{H}_{\tau_0+}^{(x)}$ -measurable random time γ and set A are also $\mathcal{M}_{\tau_0+}^{(x)}$ -measurable. Therefore, according to 1°,

$$\begin{aligned} E_x(P(\gamma, x_\tau, \Gamma); A) &= E_x(P(\gamma, x_{\tau_0}, \Gamma); A) \\ &= P_x(A, x_{\gamma+\tau_0} \in \Gamma) = P_x(A, x_{\gamma+\tau} \in \Gamma), \end{aligned}$$

which proves the theorem.

2. We shall give an application of the theorem to a stationary Markov process with right continuous paths on a metric space. Let E be a metric space, $\mathcal{B}$ the set of all Borel subsets of E and $\mathcal{A}$ the Borel field consisting of all the subsets of E which, for any finite measure μ defined on $(E, \mathcal{B})$, are in the completed field of $\mathcal{B}$ relative to μ . We denote, by w , an E -valued

and right continuous function defined for $t \in [0, \zeta(w))$, $0 < \zeta(w) \leq \infty$. The value at t is written by $w(t) = x_t(w) = x_t$. Let W be a set of such w 's which is closed under the shift transformation θ_t , where θ_t is defined by $\theta_t w(s) = w(t+s)$ for any w and any $0 \leq t < t+s < \zeta(w)$. Let $\mathcal{F}_t$ be the Borel field in $W_t = \{\zeta > t\}$ which is generated by the sets $\{x_s \in \Gamma, \zeta > t\}$ for all $\Gamma \in \mathcal{B}$ and $s \leq t$. $\mathcal{F}$ is the smallest Borel field including all $\mathcal{F}_t$, $t \geq 0$. Suppose there is given a system $\{P_x; x \in E\}$ of probability measures on $(W, \mathcal{F})$ such that, for each $A \in \mathcal{F}$, $P_x(A)$ is $\mathcal{A}$ -measurable and, for each $x \in E$, $P_x(x_0 = x) = 1$. For any finite measure μ on $(E, \mathcal{B})$, define

$$P_\mu(\cdot) = \int_E P_x(\cdot) \mu(dx),$$

which gives again a measure on $(W, \mathcal{F})$. We shall denote, by $\mathcal{G}$ (respectively, $\mathcal{G}_t$), the set of all subsets of W (resp. W_t) which are in the completion of $\mathcal{F}$ (resp. $\mathcal{F}_t$) relative to each P_μ . Clearly each P_x can be extended to $\mathcal{G}$ in the natural way and x_t is a measurable stochastic process on $(W, \mathcal{G}, P_\mu)$ for each μ . It follows that, for any bounded $\mathcal{G}$ -measurable function g , $E_x(g)$ is $\mathcal{A}$ -measurable in x . Combining the previous theorem and several results in [1], we have the following.

COROLLARY. *Suppose that the process $X = (x_t, \zeta, \mathcal{F}_t, P_x, \theta_t)$ satisfies the condition:*

(α) *For any ($\mathcal{F}$) stopping time τ , $\mathcal{F}_{\tau+}$ -measurable set A , $t \geq 0$, $x \in E$ and any bounded continuous function f on E*

$$(2) \quad E_x(f(\tau+t); A) = E_{x_\tau}(f(x_t); A).$$

Then the following assertion is true:

(β) *For any ($\mathcal{G}$) stopping time τ , $\mathcal{G}_{\tau+}$ -measurable set A , $x \in E$ and any $\mathcal{G}$ -measurable bounded function g*

$$E_x(g(\theta_\tau w); A) = E_x(E_{x_\tau}(g(w)); A).$$

According to Theorem 5.7 in [1], (α) implies that X is strongly Markovian relative to the family $(\mathcal{F})$ for all x . For each fixed x , $(\mathcal{G})$ is an $(\mathcal{N})$ family in the theorem, taking $(\mathcal{M}) = (\mathcal{F})$. Therefore X is also strongly Markovian relative to the family $(\mathcal{G})$ for all x , which implies (β) according to Theorem 5.4' in [1].

3. It is well-known [2] that the property (β) , more than (α) , is needed for the potential theoretic study of (roughly speaking) right continuous stationary Markov processes. So far the only case in which the condition (β) was reduced to another simpler condition is the following. Suppose that the process $X=(x_t, \zeta, \mathcal{F}_t, P_x, \theta_t)$ of the previous section has the ordinary Markov property and the Feller property. (By the Feller property it is meant that $E_x(f(x_t))$ is continuous in x for each bounded continuous function f on E and $t \geq 0$.) In this case we need not go by way of the property (α) to get (β) . Directly we can prove the property:

(β_0) For any $(\mathcal{G})$ stopping time τ , $\mathcal{G}_{\tau+}$ -measurable set A , $t \geq 0$, $x \in E$ and any bounded continuous function f on E , the relation (2) is true. (For the proof, for instance, see [3].)

Again using Theorems 5.7 and 5.4' in [1], it follows that (β_0) implies (β) . The same argument holds if the Feller property is replaced by a little weaker property that $\int_0^\infty e^{-\alpha t} E_x(f(x_t)) dt$ is continuous in x for every bounded continuous function f on E and $\alpha > 0$.

However, we need sometimes the property (β) for those processes obtained by various transformations of a given Markov process, such as subprocesses, so-called h -path processes and time changed processes. In general, the Feller property is not invariant under these transformations. Moreover, the family $(\mathcal{G})$ for a transformed process is different from that for the original process. But we can prove that the property (α) is kept under these transformations in most cases. Thus, according to our corollary, the property (β) holds for such transformed processes.

References

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