ON SIMPLE LIE GROUPS OF INFINITE DIMENSION I.

By

Sataru YAMAGUCHI

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§ 0. Introduction.

In 1925, M. Fréchet extended the notion of a differentiable mapping of R^m into R^n to Banach spaces, where R^m and R^n are Euclidean spaces of dimension m and n , respectively [2]. Recently, using this notion, B. Maissen defined Lie groups with Banach spaces as parameter spaces and showed that many fundamental properties and results derived from them in the finite-dimensional case also hold in the infinite-dimensional case [4]*. In the previous paper [7], we defined semi-simplicity and simplicity of infinite-dimensional Lie groups. In this work, we shall show that there exist non-trivial complex simple Lie groups of infinite dimension and such groups are obtained as limits (in certain sense) of the classical complex simple Lie groups of each type. Since simple Lie groups of type B and D define the same group, we can get three different groups and we denote them by $SO(\infty, C)$, $SL(\infty, C)$ and $SP(\infty, C)$. As we can see from the construction of such groups, they are regarded as the smallest simple Lie groups of infinite dimension which contain a series of finite-dimensional complex simple Lie groups of each type.

The present work is motivated by the lecture on the theory of unitary representations of topological groups, especially on multiple Wiener-integrals done by Prof. H. Yoshizawa at Kyushu University in 1962. In the classical spherical harmonics, the Laplacian Δ is characterized as the generator of the algebra of all generalized Laplacians. A differential operator L on the space $\mathfrak{F}$ of C^∞ -functions on the sphere S is said to be a generalized Laplacian if $L \cdot T_g = T_g \cdot L$ for all $g \in O(3)$, where $T_g f$ is defined by $(T_g f)(x) = f(g^{-1}x)$ for any

* In Math. Reviews, vol. 26, no. 1, July (1963), p. 52, R. S. Palais indicated that the notion of Lie groups modelled on Banach spaces and the fundamental properties of them were obtained far early by G. Birkhoff, *Analytical groups*, Trans. Amer. Math. Soc., vol. 43 (1938), pp. 61-101. In his paper some results in [4] are obtained under weaker conditions.

$f \in \mathfrak{F}$ and $x \in S$. Then the set of all generalized Laplacians forms an algebra over $\mathcal{C}$ and it is generated by $\mathcal{A}$. The theory of the classical spherical harmonics is developed systematically with the use of irreducible unitary representations of $0(3)$. In this connection the simplicity and the compactness of $0(3)$ are essential. In his short lecture on multiple Wiener-integrals, he indicated the importance of the group $0(\infty)$ (in his sense) and the Laplacian of the infinite-dimensional type. Let $\mathfrak{V}$ be a nuclear space with an inner product $(\cdot, \cdot)$. By the theorem of Minlos, there corresponds to the characteristic functional

$$\chi(\xi) = \exp \{ -c^2(\xi, \xi)/2 \}$$

the unique measure μ_c in $\mathfrak{V}^*$, the dual of $\mathfrak{V}$. μ_c is called the Gaussian measure. The set of all orthogonal transformations in a Hilbert space $\mathfrak{H}$ ($\mathfrak{V} \subset \mathfrak{H} \subset \mathfrak{V}^*$) which map $\mathfrak{V}$ onto itself homeomorphically becomes a transformation group on $\mathfrak{V}^*$. This group is denoted by $0(\infty)$. It is shown that μ_c is $0(\infty)$ -ergodic. There exists uniquely a symmetric operator on $L^2(\mathfrak{V}^*, \mu_c)$ which is $0(\infty)$ -invariant and is a limit of finite-dimensional differential operators of order 2. This operator is termed a Laplacian $\mathcal{A}_\infty$ on $\mathfrak{V}^*$. There are close connections between $0(\infty)$ and $\mathcal{A}_\infty$ just as in a finite-dimensional case. For example, $0(\infty)$ -invariant symmetric operators on $L^2(\mathfrak{V}^*, \mu_c)$ are generated by $\mathcal{A}_\infty$.

The group $0(\infty)$ considered in this paper is slightly different from his. But it seems to be necessary for us to consider generally the transformation groups in Hilbert spaces which become infinite-dimensional Lie groups. In the study of the group of all orthogonal transformations of a separable Hilbert space onto itself, we found that it is not the smallest group which contains finite-dimensional $0(n)$ as its subgroups and not simple in the sense of our definition. Our study on simple Lie groups of infinite dimension started from $0(\infty, \mathbf{R})$. But we shall treat only the group-theoretical properties, so we shall begin from the study of complex simple Lie groups. We can obtain simple Lie groups of other types as limits of real simple Lie groups of finite dimension. They will be considered in the next paper.

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§ 1. Preliminaries.

In this section we explain the notion of differentiability in the sense of M. Fréchet and quote some definitions and results obtained in [4], which we shall use later on.

Let $\mathfrak{B}$ and $\mathfrak{B}'$ be two Banach spaces and f be a mapping of U into $\mathfrak{B}'$, where U is an open subset of $\mathfrak{B}$.

$f: U \rightarrow \mathfrak{B}'$ is said to be *differentiable* at $x \in U$ in the sense of Fréchet if there exists a bounded linear operator $f'(x): \mathfrak{B} \rightarrow \mathfrak{B}'$ such that

$$(1) \quad f(x+h) - f(x) = f'(x)h + \|h\|(h; x),$$

where $\|(h; x)\| \rightarrow 0$ as $\|h\| \rightarrow 0$.

Let $\{F(x)\}$ be a family of bounded linear operators $F(x): \mathfrak{B} \rightarrow \mathfrak{B}'$ with parameter x in U .

$F(x)$ is said to be *differentiable* at $x \in U$ if for any fixed element $k \in \mathfrak{B}$, a mapping $f_k: U \rightarrow \mathfrak{B}'$ defined by $f_k(x) = F(x)k$ is differentiable in the sense of Fréchet, i.e., there exists a bounded linear operator $F'(x, k): \mathfrak{B} \rightarrow \mathfrak{B}'$ such that

$$(2) \quad F(x+h)k - F(x)k = F'(x, k)h + \|h\|(h, k; x),$$

where $\|(h, k; x)\| \rightarrow 0$ as $\|h\| \rightarrow 0$.

It is proved that $F'(x, k)h$ is continuous and linear in k, h and therefore $F'(x, k)h$ is written as $F'(x, k)h \equiv F'(x)(h, k)$, where $F'(x): \mathfrak{B} \times \mathfrak{B} \rightarrow \mathfrak{B}'$ is a continuous bilinear mapping.

$F'(x)$ (resp. $f'(x)$) is termed a *derivative* of $F(x)$ (resp. $f(x)$) at $x \in U$.

Assume that $f: U \rightarrow \mathfrak{B}'$ is differentiable at $x \in U$ and $f'(x)$ is also differentiable in the above sense. Then we have

$$f'(x+h)k - f'(x)k = f''(x)(h, k) + \|h\|(h; x)k,$$

where $f''(x): \mathfrak{B} \times \mathfrak{B} \rightarrow \mathfrak{B}'$ is a continuous, *symmetric* and bilinear mapping and $\|(h; x)\| \rightarrow 0$ as $\|h\| \rightarrow 0$.

$f''(x)$ is termed the *2nd derivative* of $f(x)$ at $x \in U$.

In the same way, the notion of the *n-th derivative* is defined:

$$(3) \quad \begin{aligned} & f^{(n-1)}(x+h)(h_1, h_2, \dots, h_{n-1}) - f^{(n-1)}(x)(h_1, h_2, \dots, h_{n-1}) \\ &= f^{(n)}(x)(h, h_1, \dots, h_{n-1}) + \|h\|(h; x)(h_1, \dots, h_{n-1}), \end{aligned}$$

where $f^{(n)}(x): \underbrace{\mathfrak{B} \times \mathfrak{B} \times \dots \times \mathfrak{B}}_{n\text{-times}} \rightarrow \mathfrak{B}'$ is a continuous, symmetric and *n-linear* mapping and $\|(h; x)\| \rightarrow 0$ as $\|h\| \rightarrow 0$.

REMARK. If $\mathfrak{B}$ and $\mathfrak{B}'$ are two Euclidean spaces, then the differentiability defined above coincides with the differentiability with respect to usual differentiable structures.

DEFINITION ([4]). A Hausdorff space M is said to be a differentiable manifold of class C^k ($k \geq 1$) over a Banach space $\mathfrak{B}$ if there exists an open covering $\{U_\alpha\}_{\alpha \in A}$ satisfying the following conditions:

(M₁) for each $U_\alpha \in \{U_\alpha\}$, there exists a homeomorphism φ_α of U_α onto an open subset $G_\alpha \subset \mathfrak{B}$,

(M₂) if $U_\alpha \cap U_\beta \neq \emptyset$, then a mapping

$$\varphi_{\beta\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_{\alpha\beta}) \rightarrow \varphi_\beta(U_{\alpha\beta})$$

is a differentiable mapping of class C^k .

A tangent vector X at $p \in M$ is a class $X = \{X_\alpha\}_{\alpha \in A_p}$ such that $X_\alpha \in \mathfrak{B}$ and if $X_\alpha, X_\beta \in X$ then $X_\beta = \varphi'_{\beta\alpha}(x)X_\alpha$ ($x = \varphi_\alpha(p)$), where $A_p = \{\alpha \in A; p \in U_\alpha\}$. A set T_p of all tangent vectors at p forms a vector space. A mapping $\Phi_\alpha : T_p \rightarrow \mathfrak{B}$ defined by $\Phi_\alpha(X) = X_\alpha$ ($X \in T_p$) is an isomorphism (as vector spaces) and using this isomorphism, we can make T_p a Banach space $T_p^{(\alpha)}$. It can be easily seen that if we make T_p a Banach space $T_p^{(\alpha)}$ (resp. $T_p^{(\beta)}$) by Φ_α (resp. Φ_β), they are isomorphic as Banach spaces. The notions of *vector fields* on M and *differentiable vector fields* are also defined. Let X and Y be two differentiable vector fields of class C^r . Then a mapping defined by

$$(4) \quad M \ni p \rightarrow \{X'_\alpha(x)Y_\alpha(x) - Y'_\alpha(x)X_\alpha(x)\}_{\alpha \in A_p} \quad (x = \varphi_\alpha(p))$$

is also a differentiable vector fields of class C^{r-1} and we denote this vector field by $[X, Y]$.

DEFINITION ([4]). A topological space G with a topology T is said to be a k -times differentiable Lie group over $\mathfrak{B}$ if G satisfies the following conditions:

(G₁) G is a topological group with respect to T ,

(G₂) G is a differentiable manifold of class C^k over $\mathfrak{B}$ with respect to T ,

(G₃) mappings $(p, q) \rightarrow pq$ and $p \rightarrow p^{-1}$ are k -times continuously differentiable mappings of $G \times G$ into G and of G into G , respectively.

A right (resp. left) invariant vector field on G is defined as usual and the set $\mathfrak{g}$ of all right (resp. left) invariant vector fields forms a vector space which is isomorphic with T_e . Using this isomorphism we can make $\mathfrak{g}$ a Banach

space. Invariant vector fields are $(k-1)$ -times differentiable and $[X, Y]$ is invariant if X and Y are invariant. Furthermore it is shown that the mapping $[\cdot, \cdot]$ of $\mathfrak{g} \times \mathfrak{g}$ into $\mathfrak{g}$ is *continuous* and bilinear ($\mathfrak{g}$ being considered as a Banach space).

A *one-parameter subgroup* $g(t)$ of G satisfies the following differential equation in a Banach space:

$$(5) \quad \dot{g}(t) = R(g(t))X_e,$$

where $\dot{g}(t) = g'(t) \cdot 1$, $X_e = g'(0) \cdot 1$ and $R(g)$ is a differential of a right translation $R_g: g' \rightarrow g'g$. Conversely, for any $X_e \in T_e$ there exists uniquely a one-parameter subgroup which has X_e as a tangent vector at e . We set $g(t) \equiv g(t, X)$ and $g(1, X) = \exp X$, where $X \in T_e$ is a tangent vector of $g(t)$ at e . Then it is proved that $\exp: T_e \rightarrow G$ is a homeomorphism of certain neighbourhood of $0 \in T_e$ onto that of $e \in G$ and $(k-1)$ -times uniformly continuously differentiable if G is a k -times uniformly continuously differentiable Lie group ($k \geq 2$).

A mapping $f: \mathfrak{B} \rightarrow \mathfrak{B}'$ is said to be *locally uniformly continuously differentiable* if for any $x \in \mathfrak{B}$, there exists a neighbourhood U of x on which $f'(x)$ exists and is uniformly continuous.

Hereafter we consider Lie groups which are at least 3-times uniformly continuously differentiable. In this case, a Lie group has an *analytic* structure and any continuous homomorphism is also analytic. Roughly speaking, $f: \mathfrak{B} \rightarrow \mathfrak{B}'$ is said to be analytic at $x \in \mathfrak{B}$ if f satisfies the following condition:

(A) *for any integer $n \geq 0$, there exist continuous and n -linear mappings $\Phi_n(x): \underbrace{\mathfrak{B} \times \mathfrak{B} \times \dots \times \mathfrak{B}}_{n\text{-times}} \rightarrow \mathfrak{B}'$ such that the series*

$$\sum_{n=0}^{\infty} \Phi_n(x)(h, h, \dots, h) \equiv \sum_{n=0}^{\infty} \Phi_n(x)h^n$$

is absolutely convergent for $\|h\| < \delta$ ($\delta > 0$) and

$$f(x+h) = \sum_{n=0}^{\infty} \Phi_n(x)h^n.$$

A subset G^* of a Lie group G is said to be a *Lie subgroup* of G if G^* satisfies the following conditions:

- 1) G^* is a Lie group,
- 2) the injection $I: G^* \rightarrow G$ is a continuous homomorphism and there exists a neighbourhood U^* of e in G^* such that $I|_{U^*}$ is topological.

The 1-1 correspondence between closed Lie subalgebras and Lie subgroups is given by the following

THEOREM ([4]). *Let G^* be a Lie subgroup of G and $\mathfrak{g}^*$ (resp. $\mathfrak{g}$) be a Lie algebra of G^* (resp. G). Then $I'(e) \cdot \mathfrak{g}^* \equiv \overline{\mathfrak{g}}$ is a closed subalgebra of $\mathfrak{g}$ which is topologically isomorphic with $\mathfrak{g}^*$. Conversely, for any closed subalgebra $\mathfrak{g}^*$ there exists a connected Lie subgroup G^* of G which has $\mathfrak{g}^*$ as its Lie algebra.*

REMARK. A subalgebra of $\mathfrak{g}$ is said to be a *Lie subalgebra* if it is a closed subspace of $\mathfrak{g}$.

§2. Lie group $GL(\infty, C)$

In this section we shall consider $G=GL(\infty, C)$. Let $\mathfrak{H}$ be a separable complex Hilbert space and $\mathfrak{M}=\mathfrak{M}(\infty, C)$ be the set of all bounded linear operators of $\mathfrak{H}$ into $\mathfrak{H}$. $\mathfrak{M}$ is a Banach space. Let $\{e_i\}$ ($i=1, 2, \dots$) be a fixed complete orthonormal basis of $\mathfrak{H}$ and $G=GL(\infty, C)$ be the group of all continuous automorphisms of $\mathfrak{H}$. Then we have

PROPOSITION 1. *G is a Lie group over $\mathfrak{M}$.*

Proof. This follows at once from the following two Lemmas.

LEMMA 1. *G is open in $\mathfrak{M}$.*

Proof. The above fact is well known.

LEMMA 2. *$R_g: \xi \rightarrow \xi g$ (resp. $L_g: \xi \rightarrow g\xi$) and $\psi: \xi \rightarrow \xi^{-1}$ are uniformly continuously differentiable any times with respect to parameters given in Lemma 1.*

Proof. For any g, ξ and sufficiently small $\eta \in G$, we have

$$R_g(\xi + \eta) - R_g(\xi) = \eta g \equiv R(g)\eta + \|\eta\| \cdot 0.$$

$R(g): \mathfrak{M} \rightarrow \mathfrak{M}$ defined by $R(g)\eta = \eta g$ is clearly a bounded linear operator. Thus $R'_g \equiv R(g)$ exists. The first half of Lemma 2 is easily proved from this fact.

For any $\eta \in G$ satisfying $\|\eta\| < \|\xi^{-1}\|^{-1}$, we have

$$1 > \|\xi - (\xi + \eta)\| \cdot \|\xi^{-1}\| \geq \|e - (e + \eta\xi^{-1})\|.$$

Then it is easily seen that $(e + \eta\xi^{-1})$ has an inverse. Since $(e + \eta\xi^{-1})$ and ξ have inverse elements, $(\xi + \eta) = (e + \eta\xi^{-1})\xi$ has an inverse and

$$\begin{aligned}
(\xi + \eta)^{-1} &= \xi^{-1} \{e - (-\eta \xi^{-1})\}^{-1} \\
&= \xi^{-1} \sum_{p=0}^{\infty} (-\eta \xi^{-1})^p.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\psi(\xi + \eta) - \psi(\xi) &= (\xi + \eta)^{-1} - \xi^{-1} \\
&= -\xi^{-1} \eta \xi^{-1} + \|\eta\|(\eta; \xi) \\
&\equiv \psi'(\xi)\eta + \|\eta\|(\eta; \xi),
\end{aligned}$$

where $\psi'(\xi) : \eta \rightarrow -\xi^{-1} \eta \xi^{-1}$ is a bounded linear operator of $\mathfrak{M}$ into $\mathfrak{M}$ and $\|\eta; \xi\| \rightarrow 0$ as $\|\eta\| \rightarrow 0$.

For sufficiently small ζ , we have

$$\begin{aligned}
\psi'(\xi + \zeta)\eta - \psi'(\xi)\eta &= -(\xi + \zeta)^{-1} \eta (\xi + \zeta)^{-1} + \xi^{-1} \eta \xi^{-1} \\
&= -(\xi^{-1} \sum_{p=0}^{\infty} (-\zeta \xi^{-1})^p) \eta (\xi^{-1} \sum_{p=0}^{\infty} (-\zeta \xi^{-1})^p) + \xi^{-1} \eta \xi^{-1} \\
&= \xi^{-1} \zeta \xi^{-1} \eta \xi^{-1} + \xi^{-1} \eta \xi^{-1} \zeta \xi^{-1} + \|\zeta\|(\zeta, \eta; \xi) \\
&\equiv \psi''(\xi)(\zeta, \eta) + \|\zeta\|(\zeta, \eta; \xi),
\end{aligned}$$

where $\psi''(\xi) : (\zeta, \eta) \rightarrow \xi^{-1} \zeta \xi^{-1} \eta \xi^{-1} + \xi^{-1} \eta \xi^{-1} \zeta \xi^{-1}$ is obviously a continuous, symmetric and bilinear mapping of $\mathfrak{M} \times \mathfrak{M}$ into $\mathfrak{M}$ and $\|(\zeta, \eta; \xi)\| \rightarrow 0$ as $\|\zeta\| \rightarrow 0$.

From the explicit forms of $\psi'(\xi)$ and $\psi''(\xi)$, it is easily seen that $\psi(\xi)$ is uniformly continuously differentiable at least twice. The latter half of Lemma 2 is easily proved by the same arguments as above.

PROPOSITION 2. *Let X and Y be two right invariant vector fields. Then we have*

$$[X, Y](g) = (X(e)Y(e) - Y(e)X(e))g,$$

where the product $X(e)Y(e)g$ is a usual one as bounded linear operators of $\mathfrak{G}$ into $\mathfrak{G}$.

Proof. Since X, Y are right invariant, we have $X(g) = R(g)X(e)$ and $Y(g) = R(g)Y(e)$, where $X(g)$ is a value of X at $g \in G$. By definition we have

$$\begin{aligned}
[X, Y](g) &= \{X'_a(g)Y_a(g) - Y'_a(g)X_a(g)\}_{a \in A_j} \\
&= \{(R(g)X(e))'Y(g) - (R(g)Y(e))'X(g)\}.
\end{aligned}$$

Since $R(g + \xi)\eta - R(g)\eta = \eta(g + \xi) - \eta g = \eta\xi$, $R(g)$ is differentiable and $R'(g)$ is given by

$$R'(g)(\xi, \eta) = (R(g)\eta)' \xi = \eta \xi.$$

Hence we have

$$\begin{aligned} [X, Y](g) &= \{X(e)Y(g) - Y(e)X(g)\} \\ &= (X(e)Y(e) - Y(e)X(e))g. \end{aligned}$$

PROPOSITION 3. *A one-parameter subgroup $g(t)$ of G which has $A \in T_e$ as a tangent vector at e is given by*

$$g(t) = e^{tA} = \sum_{p=0}^{\infty} \frac{1}{p!} (tA)^p.$$

Proof. A one-parameter subgroup $g(t)$ which has A as a tangent vector at e is a solution of the following differential equation:

$$(5') \quad \dot{g}(t) = R(g(t))A, \quad g(0) = e.$$

From the theory of differential equations in Banach spaces [3], the solution of (5') is unique. Since $R(g(t))A = Ag(t)$, (5') is written as follows:

$$(5'') \quad \dot{g}(t) = Ag(t), \quad g(0) = e.$$

If we put $g(t) = e^{tA}$, then $g(0) = e$ and

$$\begin{aligned} g(t+h) - g(t) &= \sum_{p=0}^{\infty} \frac{1}{p!} (t+h)^p A^p - \sum_{p=0}^{\infty} \frac{1}{p!} t^p A^p \\ &= \sum_{p=1}^{\infty} \frac{1}{(p-1)!} t^{p-1} A^p h + |h|(h; t) \\ &= Ag(t) + |h|(h; t), \end{aligned}$$

where $\|(h; t)\| \rightarrow 0$ as $|h| \rightarrow 0$.

Therefore $g(t) = e^{tA}$ is uniformly continuously differentiable and

$$g'(t)h = Ag(t)h.$$

This shows that $g(t)$ is a solution of (5'').

REMARK. In general Propositions 1~3 are also true when $\mathfrak{G}$ is a Banach space [4].

Next we consider the matrix representation of $\mathfrak{M}$. For each element $A \in \mathfrak{M}$, we put

$$(6) \quad a_{ij} = \langle Ae_j, e_i \rangle$$

and for any element $x = \sum_{i=1}^{\infty} x_i e_i$ of $\mathfrak{G}$, we put

$$(7) \quad y = \sum_{i=1}^{\infty} y_i e_i, \quad y_i = \sum_{j=1}^{\infty} a_{ij} x_j.$$

Then we have

$$\mathbf{y} = A\mathbf{x}.$$

Hence there corresponds to each element $A \in \mathfrak{M}$ a matrix $A = (a_{ij})$ of infinite order. But the converse is not true. The 1-1 correspondence between $\mathfrak{M}$ and a set of matrices which satisfy certain condition is given by the following Lemma.

LEMMA 3. *A matrix $A = (a_{ij})$ of infinite order corresponds to an element of $\mathfrak{M}$ if and only if for any integer $n > 0$ and $\mathbf{x} = \sum_{j=1}^n x_j \mathbf{e}_j$, $\mathbf{y} = \sum_{i=1}^n y_i \mathbf{e}_i$ of $\mathfrak{H}$ there exists a constant $M \geq 0$ (independent of n , $\mathbf{x}$ and $\mathbf{y}$) such that*

$$(8) \quad \left| \sum_{i,j=1}^n a_{ij} x_j \bar{y}_i \right| \leq M \cdot \|\mathbf{x}\| \cdot \|\mathbf{y}\|.$$

Proof. See for example [6].

REMARK. For any $A \in \mathfrak{M}$, $\|A\|$ is given by

$$(9) \quad \|A\| = \inf \left\{ M; \left| \sum_{i,j=1}^n a_{ij} x_j \bar{y}_i \right| \leq M \cdot \|\mathbf{x}\| \cdot \|\mathbf{y}\| \text{ for all } n, \right. \\ \left. \mathbf{x} = \sum_{j=1}^n x_j \mathbf{e}_j \text{ and } \mathbf{y} = \sum_{i=1}^n y_i \mathbf{e}_i \right\}.$$

Therefore we can identify $\mathfrak{M}$ with the set of all matrices of infinite order which satisfy the condition in Lemma 3.

LEMMA 4. *Let $\{A_m\}$ be a sequence of elements in $\mathfrak{M}$ and assume that A_m converge to A in the norm-topology. Then A_m converge as matrices, i.e., for any $\varepsilon > 0$, there exists an integer $m_\varepsilon > 0$ (independent of (i, j)) such that $|a_{ij} - a_{ij}^{(m)}| < \varepsilon$ for any $m \geq m_\varepsilon$, where $A = (a_{ij})$ and $A_m = (a_{ij}^{(m)})$.*

Proof. In the matrix representation, $\|A - A_m\| < \varepsilon$ implies that

$$\left| \sum_{i,j=1}^n (a_{ij} - a_{ij}^{(m)}) x_j \bar{y}_i \right| \leq \varepsilon \cdot \|\mathbf{x}\| \cdot \|\mathbf{y}\| \text{ for all } n, \mathbf{x} \text{ and } \mathbf{y}.$$

For any pair (i, j) we put $\mathbf{x} = \mathbf{e}_j$ and $\mathbf{y} = \mathbf{e}_i$. Then it follows from the above inequality that $|a_{ij} - a_{ij}^{(m)}| < \varepsilon$.

From Lemma 4, we see that $\exp A = \sum_{p=0}^{\infty} \frac{1}{p!} A^p$ is just the same as in a finite dimensional case.

§ 3. Lie algebra $\mathfrak{o}(\infty, C)$

Lie subalgebras and ideals considered in this paper are all closed subspaces. Let $\mathfrak{g}$ be an infinite-dimensional Lie algebra, i.e. $\mathfrak{g}$ is a Banach space and the bracket operation is a continuous bilinear mapping of $\mathfrak{g} \times \mathfrak{g}$ into $\mathfrak{g}$. Let $\mathfrak{h}_n$ ($n=1, 2, \dots$) be a sequence of subalgebras of $\mathfrak{g}$ such that

$$\mathfrak{h}_1 \subset \mathfrak{h}_2 \subset \dots \subset \mathfrak{h}_n \subset \dots$$

LEMMA 5. We set $\mathfrak{h} = \bigcup_{n=1}^{\infty} \mathfrak{h}_n$. Then $\overline{\mathfrak{h}}$ (the closure of $\mathfrak{h}$ in $\mathfrak{g}$) is a Lie subalgebra of $\mathfrak{g}$.

Proof. Since $\mathfrak{h} = \bigcup_{n=1}^{\infty} \mathfrak{h}_n$ is a linear subspace of $\mathfrak{g}$, so is $\overline{\mathfrak{h}}$. And since $[\cdot, \cdot]$ is a continuous bilinear mapping of $\mathfrak{g} \times \mathfrak{g}$ into $\mathfrak{g}$ and $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\cdot, \cdot]$ is also continuous on $\overline{\mathfrak{h}} \times \overline{\mathfrak{h}}$ and $[\overline{\mathfrak{h}}, \overline{\mathfrak{h}}] \subset \overline{\mathfrak{h}}$.

We define a bounded linear operator $T: \mathfrak{M} \rightarrow \mathfrak{M}$ by $T(A) = {}^t A$, where ${}^t A$ is a transposed matrix of A .

$$\|T(A)\| = \inf_M \left\{ M; \left| \sum_{i,j=1}^n a_{ji} x_i \bar{y}_j \right| \leq M \cdot \|x\| \cdot \|y\| \text{ for all } n, \right. \\ \left. x = \sum_{i=1}^n x_i e_i \text{ and } y = \sum_{j=1}^n y_j e_j \right\} = \|A\|,$$

where $A = (a_{ij})$. Hence for any $A \in \mathfrak{M}$, $T(A) \in \mathfrak{M}$ and $\|T\| = 1$. Therefore T is a bounded linear operator of $\mathfrak{M}$ onto itself. Let $\mathfrak{h}$ be a kernel of $\Phi = T + I: \mathfrak{M} \rightarrow \mathfrak{M}$, where I is the identity mapping of $\mathfrak{M}$. Then Φ is continuous and $\mathfrak{h}$ is a closed linear subspace of $\mathfrak{M}$. $\mathfrak{h}$ is clearly a Lie subalgebra of $\mathfrak{M}$, where $\mathfrak{M}$ is regarded as a Lie algebra of $G = GL(\infty, C)$. Thus we have

LEMMA 6. $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{M}$.

DEFINITION 1. We set $\mathfrak{o}(\infty, C) = \overline{\bigcup_{n=2}^{\infty} \mathfrak{o}(n, C)}$, where $\mathfrak{o}(n, C)$ is a Lie algebra of type B or D which is simple except for $n=2$ and 4 [1].

THEOREM 1. $\mathfrak{o}(\infty, C)$ is a proper Lie subalgebra of $\mathfrak{h}$ and it is the set of all $A \in \mathfrak{h}$ which are completely continuous.

Proof. Each $\mathfrak{o}(n, C)$ is identified with the following subspace of $\mathfrak{h}$:

$$\mathfrak{o}(n, C) \equiv \begin{bmatrix} \mathfrak{o}(n, C) & 0 \\ 0 & 0 \end{bmatrix} \subset \mathfrak{h}.$$

Since $\mathfrak{h}$ is closed, $\mathfrak{o}(\infty, C)$ is contained in $\mathfrak{h}$. If $A \in \mathfrak{o}(n, C)$ then for any

$\mathbf{x} = \sum_{i=1}^{\infty} x_i \mathbf{e}_i \in \mathfrak{H}$ we have

$$(10) \quad A\mathbf{x} = \sum_{i,j=1}^n a_{ij} x_j \mathbf{e}_i = \sum_{i=1}^n \langle A\mathbf{x}, \mathbf{e}_i \rangle \mathbf{e}_i.$$

If $\mathbf{x}_n \rightarrow \mathbf{x}_0$ (weakly), then $\langle \mathbf{x}_n, \mathbf{e}_i \rangle \rightarrow \langle \mathbf{x}_0, \mathbf{e}_i \rangle$. From (10) we see that $\|A\mathbf{x}_0 - A\mathbf{x}_n\| \rightarrow 0$ as $\mathbf{x}_n \rightarrow \mathbf{x}_0$ (weakly). Therefore for any n , $A \in \mathfrak{o}(n, C)$ is completely continuous. It is easily proved that if $A_m \in \bigcup_{n=2}^{\infty} \mathfrak{o}(n, C)$ converge to A in the norm-topology, then A is also completely continuous. Next we shall prove the converse, that is to say, every element $A \in \mathfrak{h}$ which is completely continuous is a limit of $A_m \in \bigcup_{n=2}^{\infty} \mathfrak{o}(n, C)$. Consider the projection P_m (resp $P_m^{\perp}$) of $\mathfrak{H}$ onto $\mathfrak{H}_m$ (resp. $\mathfrak{H}_m^{\perp}$), where $\mathfrak{H}_m$ (resp. $\mathfrak{H}_m^{\perp}$) is a subspace spanned by $\{\mathbf{e}_i\}$ ($i=1, 2, \dots, m$) (resp. orthogonal complement of $\mathfrak{H}_m$ in $\mathfrak{H}$). Set

$$(11) \quad A_m = P_m A \quad \text{and} \quad A_m^{\perp} = P_m^{\perp} A.$$

Then for any $\mathbf{x} = \sum_{i=1}^{\infty} x_i \mathbf{e}_i \in H$, we have

$$A_m \mathbf{x} = P_m A \mathbf{x} = P_m \left(\sum_{i,j=1}^{\infty} a_{ij} x_j \mathbf{e}_i \right) = \sum_{i=1}^m \sum_{j=1}^{\infty} a_{ij} x_j \mathbf{e}_i = \sum_{i=1}^m \langle A\mathbf{x}, \mathbf{e}_i \rangle \mathbf{e}_i$$

and

$$A_m^{\perp} \mathbf{x} = \sum_{i=m+1}^{\infty} \sum_{j=1}^{\infty} a_{ij} x_j \mathbf{e}_i = \sum_{i=m+1}^{\infty} \langle A\mathbf{x}, \mathbf{e}_i \rangle \mathbf{e}_i.$$

Hence, if we split A into the following blocks:

$$A = \begin{bmatrix} A_m^1 & A_m^2 \\ A_m^3 & A_m^4 \end{bmatrix},$$

where A_m^1 is a (m, m) -matrix, then we have

$$(12) \quad A_m = \begin{bmatrix} A_m^1 & A_m^2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_m^{\perp} = \begin{bmatrix} 0 & 0 \\ A_m^3 & A_m^4 \end{bmatrix}.$$

Let S be a set of all $\mathbf{x} \in \mathfrak{H}$ such that $\|\mathbf{x}\|=1$ and $A \in \mathfrak{M}$ be a completely continuous operator. Then S is a bounded set and by the complete continuity of A , $A(S)$ is a compact set in $\mathfrak{H}$. Therefore there exists a constant $C \geq 0$ such that

$$\sum_{i=1}^{\infty} |\langle A\mathbf{x}, \mathbf{e}_i \rangle|^2 \leq C$$

for all $A\mathbf{x} \in A(S)$. From this it is easily seen that for any $\varepsilon > 0$ there exists an integer $m_\varepsilon > 0$ such that

$$(13) \quad \sum_{i=m_\varepsilon+1}^{\infty} |\langle A\mathbf{x}, \mathbf{e}_i \rangle|^2 < \varepsilon^2$$

for all $A\mathbf{x} \in A(S)$. This implies that $\|A_{m_\varepsilon}^\perp\| < \varepsilon$, i.e., $\|A - A_{m_\varepsilon}\| \rightarrow 0$ as $m_\varepsilon \rightarrow \infty$. From (12) we see that $\|A_{m_\varepsilon}^\perp\| < \varepsilon$ implies that

$$(14) \quad \left| \sum_{i=m_\varepsilon+1}^p \sum_{j=1}^n a_{ij} x_j \bar{y}_i \right| < \varepsilon^2 \cdot \|\mathbf{x}\| \cdot \|\mathbf{y}\|$$

for any $p, \mathbf{x} = \sum_{i=1}^n x_i \mathbf{e}_i$ and $\mathbf{y} = \sum_{i=1}^n y_i \mathbf{e}_i$, where $a_{ij} = 0$ if $p \leq m_\varepsilon$. From the skew-symmetry of A , (14) is written as

$$\left| \sum_{i=m_\varepsilon+1}^p \sum_{j=1}^n a_{ji} \bar{y}_i x_j \right| < \varepsilon^2 \cdot \|\mathbf{x}\| \cdot \|\mathbf{y}\|.$$

In particular, the above inequality holds for all $\mathbf{x} = \sum_{i=1}^p x_i \mathbf{e}_i$ such that $x_i = 0$ for $i > m_\varepsilon$. This implies that

$$\|A_{m_\varepsilon} - A_{m_\varepsilon}^1\| = \|A_{m_\varepsilon}^2\| < \varepsilon,$$

where

$$A_{m_\varepsilon}^1 = \begin{bmatrix} A_{m_\varepsilon}^1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{m_\varepsilon}^2 = \begin{bmatrix} 0 & A_{m_\varepsilon}^2 \\ 0 & 0 \end{bmatrix}$$

and $A_{m_\varepsilon}^1 \in \mathfrak{o}(m_\varepsilon, \mathbb{C})$. Hence we have

$$\begin{aligned} \|A - A_{m_\varepsilon}^1\| &= \|A - A_{m_\varepsilon} + A_{m_\varepsilon} - A_{m_\varepsilon}^1\| \\ &\leq \|A - A_{m_\varepsilon}\| + \|A_{m_\varepsilon} - A_{m_\varepsilon}^1\| < 2\varepsilon. \end{aligned}$$

The above inequality shows that each completely continuous operator in $\mathfrak{h}$ is a limit of $A_m \in \bigcup_{n=2}^{\infty} \mathfrak{o}(n, \mathbb{C})$ in the norm-topology.

It is obvious that $\mathfrak{h}$ contains the following element A_0 :

$$A_0 = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & 0 \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & 0 & 1 \\ 0 & & & -1 & 0 \\ & & & & \ddots & \ddots \end{pmatrix}.$$

Then $A_0 e_{2n-1} = -e_{2n}$ and $A_0 e_{2n} = e_{2n-1}$. We set $A_0 e_n = f_n$. Then A_0 maps $\{e_i\}$ ($i=1, 2, \dots$) which is bounded onto a set $\{f_n\}$ ($n=1, 2, \dots$) of orthonormal vectors in a 1-1 fashion. For any (i, j) ($i \neq j$), $\|f_i - f_j\| = \sqrt{2}$. So the set $\{f_n\}$ has no compact closure because any sequence in $\{f_n\}$ is not convergent. This shows that A_0 is not completely continuous. Hence $A_0 \notin \mathfrak{o}(\infty, C)$ and $\mathfrak{o}(\infty, C)$ is a proper subspace of $\mathfrak{h}$. We shall prove later that $\mathfrak{o}(\infty, C)$ is not isomorphic with $\mathfrak{h}$ and it is an ideal of $\mathfrak{h}$.

REMARK. For any $A \in \mathfrak{o}(\infty, C)$, $\exp A$ is an orthogonal matrix, i.e. it leaves invariant the following quadratic form:

$$F(x) = \sum_{i=1}^{\infty} x_i^2,$$

where $x = \sum_{i=1}^{\infty} x_i e_i \in \mathfrak{H}$. On one hand the set of all elements of $GL(\infty, C)$ which leave invariant $F(x)$ forms a group. Its identity component G contains $SO(\infty, C)$ as a proper subgroup, where $SO(\infty, C)$ is the connected Lie subgroup of $GL(\infty, C)$ corresponding to $\mathfrak{o}(\infty, C)$. In fact, as mentioned in §1, there exists a neighbourhood U of $0 \in \mathfrak{M}(\infty, C)$ and a neighbourhood V of $e \in GL(\infty, C)$ which are homeomorphic under the mapping $\exp$. For sufficiently small $\theta_0 \neq 0$, $\theta_0 A_0$ belongs to U and is not completely continuous. Hence it follows that $\theta_0 A_0 \notin \mathfrak{o}(\infty, C)$, $\exp \theta_0 A_0 \notin SO(\infty, C)$ and $\exp \theta_0 A_0 \in G$.

§4. Simplicity of $\mathfrak{o}(\infty, C)$

In this section we shall prove the simplicity of $\mathfrak{o}(\infty, C)$. Let $\mathfrak{g}$ be an infinite-dimensional Lie algebra and $\mathfrak{h}_n$ ($n=1, 2, \dots$) be a sequence of subalgebras of $\mathfrak{g}$ such that

$$\mathfrak{h}_1 \subset \mathfrak{h}_2 \subset \dots \subset \mathfrak{h}_n \subset \dots$$

We set $\mathfrak{h} = \bigcup_{n=1}^{\infty} \mathfrak{h}_n$. Then $\overline{\mathfrak{h}}$ (the closure of $\mathfrak{h}$ in $\mathfrak{g}$) is a Lie subalgebra of $\mathfrak{g}$.

PROPOSITION 4. *Let $\mathfrak{h}_n$ be as above. Assume that for any $n \geq 1$, there exists $m(\geq n)$ such that $\mathfrak{h}_m$ is simple. Then any ideal $\mathfrak{i}$ of $\overline{\mathfrak{h}}$ is contained in $\mathfrak{o} \mathfrak{h} = (\overline{\mathfrak{h}} - \mathfrak{h}) \cup \{0\}$, where the sign “ $-$ ” denotes the difference of sets.*

Proof. Assume there exists n such that $\mathfrak{i} \cap \mathfrak{h}_n \neq (0)$. Then by assumption there exists $\mathfrak{h}_m$ ($m \geq n$) which is simple and $\mathfrak{h}_m \supset \mathfrak{h}_n$. Since $\mathfrak{i} \cap \mathfrak{h}_m \neq (0)$ is an ideal of $\mathfrak{h}_m$ and $\mathfrak{h}_m$ is simple, we have $\mathfrak{i} \cap \mathfrak{h}_m = \mathfrak{h}_m$. In other words, $\mathfrak{i} \supset \mathfrak{h}_m$ and $\mathfrak{i} \supset \mathfrak{h}_m$.

$\supset \cdots \supset \mathfrak{h}_n \supset \cdots \supset \mathfrak{h}_1$. For any $p \geq n$, there exists $q (\geq p)$ such that $\mathfrak{h}_q$ is simple and $\mathfrak{h}_q \cap \mathfrak{i} \supset \mathfrak{h}_n \cap \mathfrak{i} \neq (0)$. By simplicity of $\mathfrak{h}_q$, we have $\mathfrak{h}_q \cap \mathfrak{i} = \mathfrak{h}_q$, i.e., $\mathfrak{i} \supset \mathfrak{h}_q \supset \cdots \supset \mathfrak{h}_p \supset \cdots \supset \mathfrak{h}_1$. From this it follows that $\mathfrak{i} \supset \bigcup_{i=1}^{\infty} \mathfrak{h}_i = \mathfrak{h}$, i.e., $\overline{\mathfrak{h}} \supset \mathfrak{i} \supset \mathfrak{h}$. Since $\mathfrak{i}$ is closed we have $\mathfrak{i} = \overline{\mathfrak{h}}$. Therefore any ideal $\mathfrak{i} \neq \overline{\mathfrak{h}}$ has no element $\neq 0$ in common with each $\mathfrak{h}_n$. This implies that $\mathfrak{i}$ is in $\mathfrak{o} \mathfrak{h}$.

Obviously we have

COROLLARY. $\overline{\mathfrak{h}}$ is simple if $\mathfrak{o} \mathfrak{h}$ contains no ideal except $\{0\}$. In particular, $\mathfrak{h}$ is simple if $\mathfrak{h}$ is a closed subspace of $\mathfrak{g}$.

PROPOSITION 5. There exists no ideal $\mathfrak{i} \neq (0)$ of $\mathfrak{h}$ in $\mathfrak{n}$;

$$\mathfrak{n} = \left\{ \mathfrak{h} - \bigcup_{n=2}^{\infty} \mathfrak{o}(n, C) \right\} \cup \{0\}.$$

We need a few Lemmas to prove Proposition 5. Let $A \neq 0$ be any element of $\mathfrak{n}$. Since $A = (a_{ij}) \in \bigcup_{n=2}^{\infty} \mathfrak{o}(n, C)$ and A is skew-symmetric, for any integer $n \geq 1$ there exists $m \geq n$ such that $a_{pm} = -a_{mp} \neq 0$ for some p . Let p be the number of a row in which a non-zero element appears for the first time and $q = q(A)$ be the number of a column such that $a_{pq} \neq 0$ and $a_{ps} = 0$ for $s < q$. Then A is written as

$$A = \left(\begin{array}{ccccccc} \overbrace{0 \cdots 0}^q 0 0 \cdots 0 & \cdots & 0 & \cdots & \cdots & \cdots & \rightarrow \\ \vdots & & & & & & \\ 0 \cdots 0 0 0 \cdots 0 & \cdots & 0 & \cdots & \cdots & \cdots & \rightarrow \\ 0 \cdots 0 0 0 \cdots 0 & \cdots & -a_{pq} & * & * & * & \rightarrow \\ \vdots & \vdots & \vdots & & & & \\ 0 & & 0 & a_{pq} & * & * & * \\ \vdots & \vdots & \vdots & & & & \\ \vdots & \vdots & \vdots & & & & \end{array} \right) \quad (q \geq 2, p \geq 1).$$

We split A into the following blocks:

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

where A_1 is a square matrix of order k ($k \geq q(A) + 1$). Then for any $X \in \mathfrak{h}$ we have

$$(15) \quad \begin{aligned} [X, A] &= XA - AX \\ &= \begin{bmatrix} X_1A_1 + X_2A_3 - A_1X_1 - A_2X_3 & X_1A_2 + X_2A_4 - A_1X_2 - A_2X_4 \\ X_3A_1 + X_4A_3 - A_3X_1 - A_4X_3 & X_3A_2 + X_4A_4 - A_3X_2 + A_4X_4 \end{bmatrix} \end{aligned}$$

LEMMA 7. Let $\mathfrak{i}$ be an ideal of $\mathfrak{h}$ contained in $\mathfrak{n}$ ($\mathfrak{i} \neq (0)$). Then any element $A \in \mathfrak{h}$ such that $A_2 = A_3 = 0$ and $A_1 \neq 0$ is not contained in $\mathfrak{i}$, where A_1 is a square matrix of order $k \geq q(A) + 1$. In other words, if $A \in \mathfrak{i}$ and $A_1 \neq 0$ then A_2 can not vanish.

Proof. Since $A_1 \in \mathfrak{o}(k, C)$ and $\mathfrak{o}(k, C)$ is semi-simple, there exists an element $B \neq 0$ of $\mathfrak{o}(k, C)$ such that $[A_1, B] \neq 0$ and $[A_1, B] \in \mathfrak{o}(k, C)$. We identify B with

$$\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \in \mathfrak{o}(\infty, C).$$

Then

$$[A, B] = \begin{bmatrix} [A_1, B] & 0 \\ 0 & 0 \end{bmatrix} \in \mathfrak{o}(k, C).$$

Therefore $[A, B] \in \mathfrak{n}$ and this is a contradiction because $\mathfrak{i}$ is an ideal contained in $\mathfrak{n}$.

LEMMA 8. $A \neq 0$ such that $A_4 = 0$ for $k \geq q(A) + 1$ is not contained in $\mathfrak{i}$.

Proof. Assume $A \in \mathfrak{i}$ and $A \neq 0$. Since $A \neq 0$, A_2 can not vanish by Lemma 7. From (15), it follows that

$$[X, A] = \begin{bmatrix} X_1A_1 + X_2A_3 - A_1X_1 - A_2X_3 & X_1A_2 - A_1X_2 - A_2X_4 \\ X_3A_1 + X_4A_3 - A_3X_1 & X_3A_2 - A_3X_2 \end{bmatrix}.$$

Since $A_2 \neq 0$ we can take such a matrix $X_4 \neq 0$ as $A_2X_4 \neq 0$. For the later use, we notice that such a matrix X_4 can be chosen from matrices of finite order. For example, if $t_{pq} \neq 0$ we can take a skew-symmetric matrix X_4 such that $x_{q1} \neq 0$ and $x_{ij} = 0$ for $(i, j) \neq (1, q)$ or $\neq (q, 1)$, where $A_2 = (t_{ij})$ and $X_4 = (x_{ij})$. In the above expression of $[X, A]$ we set $X_1 = 0$, $X_2 = 0$ and $X_3 = 0$. Then we have

$$[X, A] = \begin{bmatrix} 0 & -A_2X_4 \\ X_4A_3 & 0 \end{bmatrix} \neq 0 \text{ and } \in \mathfrak{i}.$$

Therefore there exists in $\mathfrak{i}$ an element B of the following form:

$$B = \begin{bmatrix} 0 & B_2 \\ B_3 & 0 \end{bmatrix} \quad \text{and } B_2 \neq 0.$$

For such a matrix B , we have

$$[X, B] = \begin{bmatrix} X_2 B_3 - B_2 X_3 & X_1 B_2 - B_2 X_4 \\ X_4 B_3 - B_3 X_1 & X_3 B_2 - B_3 X_2 \end{bmatrix}$$

for all $X \in \mathfrak{h}$. If we take such X as $X_1=0$, $X_4=0$ and $X_2 \neq 0$, then

$$[X, B] = \begin{bmatrix} X_2 B_3 - B_2 X_3 & 0 \\ 0 & X_3 B_2 - B_3 X_2 \end{bmatrix}.$$

Choosing $X_2 \neq 0$ suitably, we can make $X_2 B_3 - B_2 X_3$ non-zero. In particular, such X_2 can be chosen from matrices of finite order. For example, if $b_{pq} \neq 0$ ($B_2 = (b_{ij})$) we take as X_2 (resp. X_3) a matrix $X_2 = (x_{ij})$ (resp. $X_3 = (y_{ij})$) such that $x_{1q} = 1$ and $x_{ij} = 0$ for $(i, j) \neq (1, q)$ (resp. $y_{q1} = -1$ and $y_{ij} = 0$ for $(i, j) \neq (q, 1)$). Then $X_2 B_3 - B_2 X_3$ has the following form:

$$X_2 B_3 - B_2 X_3 = \begin{pmatrix} 0 & -b_{2q} & \cdots & -b_{pq} & \cdots & -b_{kq} \\ b_{2q} & & & & & \\ \vdots & & & & & \\ \vdots & & & 0 & & \\ b_{pq} & & & & & \\ \vdots & & & & & \\ \vdots & & & & & \\ b_{kq} & & & & & \end{pmatrix} \neq 0,$$

where $k \geq q(B) + 1$. This contradicts Lemma 7.

Now we shall prove Proposition 5. Let $A \neq 0$ be any element of $\mathfrak{n}$ and $X \neq 0$ be an element of $\mathfrak{h}$ such that $X_2 = X_4 = 0$, i.e., $X \in \mathfrak{o}(k, C)$ ($k \geq q(A) + 1$). Then from (15), we have

$$[X, A] = \begin{bmatrix} X_1 A_1 - A_1 X_1 & X_1 A_2 \\ -A_3 X_1 & 0 \end{bmatrix},$$

where A_1 is a square matrix of order k . Since $A \neq 0$, A_1 is not a zero-matrix by definition of $q(A)$. Therefore there exists $X_1 \in \mathfrak{o}(k, C)$ such that $[X_1, A_1] \neq 0$ because $\mathfrak{o}(k, C)$ is semi-simple for $k \geq 3$. This contradicts Lemma 8 and Proposition 5 is thereby proved.

Finally we shall prove the following Theorem 2 which is the aim of this section,

THEOREM 2. $\mathfrak{p}(\infty, \mathcal{C})$ is a simple Lie algebra and the corresponding connected Lie subgroup $SO(\infty, \mathcal{C})$ of $GL(\infty, \mathcal{C})$ is considered as the smallest infinite-dimensional simple Lie group which contains $SO(n, \mathcal{C})$ as simple subgroups.

Proof. As mentioned in the proof of Lemma 7 and Lemma 8, all the steps used there can be done by means of elements of $\bigcup_{n=2}^{\infty} \mathfrak{p}(n, \mathcal{C})$. It is easily seen from this fact that Proposition 5 holds if we replace $\mathfrak{n}$ by $\mathfrak{p}(\infty, \mathcal{C}) = (\mathfrak{p}(\infty, \mathcal{C}) - \bigcup_{n=2}^{\infty} \mathfrak{p}(n, \mathcal{C})) \cup \{0\}$ and $\mathfrak{h}$ by $\mathfrak{p}(\infty, \mathcal{C})$. Then simplicity of $\mathfrak{p}(\infty, \mathcal{C})$ follows from Corollary to Proposition 4. By definition of $\mathfrak{p}(\infty, \mathcal{C})$, it is the smallest infinite-dimensional Lie algebra containing each $\mathfrak{p}(n, \mathcal{C})$ as its simple subalgebra. It is known that to each subalgebra of the Lie algebra of the given Lie group corresponds uniquely a connected Lie subgroup. From this fact the latter half of Theorem 2 follows.

Theorem 2 assures us of the existence of simple Lie groups of infinite dimension.

REMARK. Obviously $\mathfrak{p}(\infty, \mathcal{C})$ has Lie subalgebras $\mathfrak{p}_n(\infty, \mathcal{C})$ of the following type:

$$\mathfrak{p}_n(\infty, \mathcal{C}) = \left(\begin{array}{c|c} \overbrace{0 \cdots 0}^{n-1} \cdots \rightarrow & \\ \vdots & \\ 0 & 0 \\ \vdots & \\ 0 \cdots 0 \cdots \rightarrow & \\ \vdots & \\ 0 & \mathfrak{p}(\infty, \mathcal{C}) \\ \vdots & \\ \vdots & \end{array} \right).$$

It is easy to prove that $\mathfrak{p}_n(\infty, \mathcal{C})$ is a Lie subalgebra of $\mathfrak{p}(\infty, \mathcal{C})$ and the mapping $\wedge_n: \mathfrak{p}(\infty, \mathcal{C}) = \mathfrak{p}_0(\infty, \mathcal{C}) \rightarrow \mathfrak{p}_n(\infty, \mathcal{C})$ defined by $\wedge_n(A) = A_n$ is an isomorphism (as Banach Lie algebras) of $\mathfrak{p}(\infty, \mathcal{C})$ onto $\mathfrak{p}_n(\infty, \mathcal{C})$, where

$$A_n = \left(\begin{array}{c|c} \overbrace{0 \cdots 0}^{n-1} \cdots \rightarrow & \\ \vdots & \\ 0 & 0 \\ \vdots & \\ 0 \cdots 0 \cdots \rightarrow & \\ \vdots & \\ 0 & A \\ \vdots & \\ \vdots & \end{array} \right) \text{ and } A \in \mathfrak{p}(\infty, \mathcal{C}).$$

So the corresponding Lie subgroups $SO(\infty, C)$ and $SO_n(\infty, C)$ are locally isomorphic. Therefore $\mathfrak{o}(\infty, C)$ has a countable family of proper subalgebras $\mathfrak{o}_n(\infty, C)$ such that $\mathfrak{o}_m(\infty, C)$ is a proper subalgebra of $\mathfrak{o}_n(\infty, C)$ for $m > n$ and they are isomorphic mutually. This is one of the essential differences between Lie groups of finite dimension and those of infinite dimension. From the above consideration, the following questions arise:

1. *Is it always true that any Lie algebra $\mathfrak{g}$ which is not of finite dimension has a proper Lie subalgebra isomorphic with $\mathfrak{g}$ itself?*
2. *Is $\mathfrak{o}(\infty, C)$ isomorphic with $\mathfrak{h}$?*

To conclude this section, we shall answer the question 2. The answer is negative. At first we consider the relation between completely continuous operators and non-completely continuous operators. Let $\mathfrak{B}$ be a Banach space, $\mathfrak{M}(\mathfrak{B})$ be the Lie algebra of all bounded linear operators of $\mathfrak{B}$ and $GL(\mathfrak{B})$ be the Lie group of all continuous automorphisms of $\mathfrak{B}$. It is elementary that for any $A \in \mathfrak{M}$ which is completely continuous and for any $B \in \mathfrak{M}$, $A \cdot B$ and $B \cdot A$ are completely continuous. Therefore we have the following Theorem, which gives us a necessary condition for a Lie subgroups of $GL(\mathfrak{B})$ to be simple.

THEOREM 3. *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{M}$. Then the set of all elements $A \in \mathfrak{h}$ which are completely continuous is an ideal of $\mathfrak{h}$. Hence $\mathfrak{h}$ is not simple if it contains an element $A \neq 0$ which is completely continuous and $B \neq 0$ which is not completely continuous. In other words, if a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{M}$ is simple then $\mathfrak{h}$ must be of one of the following types:*

1. *each element $A \in \mathfrak{h}$ is completely continuous,*
2. *each element $A \in \mathfrak{h}$ is not completely continuous except 0.*

Proof. Let $\mathfrak{g}$ be the set of all completely continuous operators $A \in \mathfrak{M}$. As shown in §3, $\mathfrak{g}$ is a closed set in $\mathfrak{M}$. Obviously $\mathfrak{g}$ is a linear space and as mentioned above $[A, B] = AB - BA$ is completely continuous for all $A, B \in \mathfrak{g}$. Hence $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{M}$. In particular it is an ideal of $\mathfrak{M}$. Therefore $\mathfrak{h} \cap \mathfrak{g}$ is an ideal of $\mathfrak{h}$. Assume that $\mathfrak{h}$ contains an element $A \neq 0$ which is completely continuous and an element which is not completely continuous. Then $\mathfrak{i} = \mathfrak{h} \cap \mathfrak{g} \neq (0)$ and $\mathfrak{i} \neq \mathfrak{g}$. Therefore there exists an ideal $\mathfrak{i} \neq (0)$, $\neq \mathfrak{h}$ of $\mathfrak{h}$ and $\mathfrak{h}$ is not simple.

COROLLARY. *$\mathfrak{o}(\infty, C)$ is not isomorphic with $\mathfrak{h}$ and it is an ideal of $\mathfrak{h}$. Therefore $SO(\infty, C)$ is a distinguished Lie subgroup of the connected Lie*

group corresponding to $\mathfrak{h}$.

Proof. As shown in § 3, $\mathfrak{h}$ contains an element $A_0 \neq 0$ which is not completely continuous. On one hand $\mathfrak{o}(\infty, C)$ is the set of all elements of $\mathfrak{h}$ which are completely continuous. Then Corollary follows at once from the above Theorem.

REMARK. Simple Lie algebras which are constructed in this paper and in the forthcoming one are all of type 1 and contain elements of a finite-dimensional type. In connection with this point, the following problems are interesting:

1. *Is there a simple Lie subalgebra of $\mathfrak{M}$ of type 1 other than what we shall construct?*
2. *Is there a simple Lie algebra of type 1 of which elements are all not of finite dimension except 0?*
3. *Is there a simple Lie subalgebra of $\mathfrak{M}$ of type 2?*

§ 5. Lie algebra $\mathfrak{sl}(\infty, C)$ and its simplicity

In this section we shall consider $\mathfrak{sl}(\infty, C)$ and its simplicity. Let $\mathfrak{sl}(n+1, C)$ ($n \geq 1$) be complex simple Lie algebras of type $A[1]$.

DEFINITION 2. We set $\mathfrak{sl}(\infty, C) = \bigcup_{n=2}^{\infty} \mathfrak{sl}(n, C)$.

We denote by $SL(\infty, C)$ the Lie subgroup of $GL(\infty, C)$ corresponding to it. As shown in § 3, $\mathfrak{sl}(\infty, C)$ is a Lie subalgebra of $\mathfrak{M}(\infty, C)$ and each $A \in \mathfrak{sl}(\infty, C)$ is a completely continuous operator on $\mathfrak{H}$.

PROPOSITION 6. *There exists no ideal $\mathfrak{i} \neq (0)$ of $\mathfrak{sl}(\infty, C)$ in $\mathfrak{o}\mathfrak{sl}(\infty, C) = (\mathfrak{sl}(\infty, C) - \bigcup_{n=2}^{\infty} \mathfrak{sl}(n, C)) \cup \{0\}$.*

Proof. The proof of Proposition 6 is similar as in Proposition 5. But there are slight differences. So we confine ourselves only to mention these differences.

Let $A = (a_{ij}) \neq 0$ be any element of $\mathfrak{o}\mathfrak{sl}(\infty, C)$. In general, the trace of A might be not zero. Concerning this point the following questions are left open:

1. *Is the set of all matrices of $\mathfrak{M}$ of which traces are 0 closed in $\mathfrak{M}$?*
2. *If 1 is true, then is $\mathfrak{sl}(\infty, C)$ the set of all completely continuous operators of $\mathfrak{M}$ of which traces are 0?*

LEMMA 9. *Let $\mathfrak{i} \neq (0)$ be an ideal of $\mathfrak{sl}(\infty, C)$ contained in $\mathfrak{o} \mathfrak{sl}(\infty, C)$. Then any element A of $\mathfrak{sl}(\infty, C)$ such that $A_2 = A_3 = 0$ and $A_1 \neq 0$ (order of $A_1 = k \geq 2$) is not contained in $\mathfrak{i}$. In other words, if $A \in \mathfrak{i}$ and $A_1 \neq 0$ then $A_2 \neq 0$ or $A_3 \neq 0$.*

Proof. Let A be an element of $\mathfrak{sl}(\infty, C)$ such that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix} \quad \text{and} \quad A_1 \neq 0.$$

Then we can find a matrix $B \in \mathfrak{sl}(k, C)$ such that $[A_1, B] \neq 0$. Note that in § 3 we used semi-simplicity of $\mathfrak{o}(k, C)$. In this case A_1 does not belong to $\mathfrak{sl}(k, C)$ in general. Therefore we must find such B directly. But it is quite easy. For such B we have

$$[A, B] = \begin{bmatrix} [A_1, B] & 0 \\ 0 & 0 \end{bmatrix} \neq 0$$

and $[A, B]$ belongs to $\mathfrak{sl}(k, C)$ because $T_r[A, B] = T_r[A_1, B] = 0$.

LEMMA 10. *$A \neq 0$ such that $A_1 \neq 0$ and $A_1 = 0$ is not contained in $\mathfrak{i}$.*

Proof. In the Proof of Lemma 8, we had to choose $X \in \bigcup_{n=2}^{\infty} \mathfrak{o}(n, C)$ such that $X_2 B_3 - B_2 X_3 \neq 0$. In case of $\mathfrak{sl}(\infty, C)$, we may take such a matrix $X \in \bigcup_{n=2}^{\infty} \mathfrak{sl}(n, C)$ that $B_2 X_3 \neq 0$ and $X_2 = 0$ if $B_2 \neq 0$ or $X_2 B_3 \neq 0$ and $X_3 = 0$ if $B_2 \neq 0$.

In the proof of Proposition 5 we took a matrix $X \in \mathfrak{o}(k, C)$ and used semi-simplicity. In case of $\mathfrak{sl}(\infty, C)$, A_1 does not belong to $\mathfrak{sl}(k, C)$ in general. Therefore we must find such a matrix X as $[A_1, X_1] \neq 0$ directly. If $A_1 = (a_{ij})$ has a non-zero coefficient $a_{pq} (p > q)$, then we may take such a matrix $X_1 = (x_{ij})$ that $x_{q1} = 1$ and $x_{ij} = 0$ for $(i, j) \neq (q, 1)$. If A_1 has non-zero coefficients only on the main diagonal (assume $a_{pp} \neq 0$), then (p, q) -coefficient of $[A_1, X_1]$ is $a_{pp} x_{pq} - a_{qq} x_{pq}$. Therefore if there exists $a_{qq} (p \neq q)$ such that $a_{pp} \neq a_{qq}$, then we may take such a matrix $X_1 = (x_{ij})$ that $x_{pq} = 1$ and $x_{ij} = 0$ for $(i, j) \neq (p, q)$. If $a_{pp} = a_{qq}$ for any q ($1 \leq q \leq k$), then $[A_1, X_1] = 0$. Therefore we must prove that for sufficiently large k , the last case can not happen. This is assured by the following

LEMMA 11. *Let A be an element of $\mathfrak{M}$ which is a diagonal matrix. Assume that $A \neq 0$ and $a_{jj} = a_{ii}$ for all i and j . Then A can not belong to $\mathfrak{sl}(\infty, C)$.*

Proof. It is easy to see that such a matrix A is not completely continuous. Since each element of $\mathfrak{sl}(\infty, \mathcal{C})$ is completely continuous, A does not belong to $\mathfrak{sl}(\infty, \mathcal{C})$.

By Lemma 11, the last case is excluded. Therefore Proposition 6 is proved. Obviously we have

THEOREM 4. *$\mathfrak{sl}(\infty, \mathcal{C})$ is a simple Lie algebra and the corresponding connected Lie subgroup $SL(\infty, \mathcal{C})$ of $GL(\infty, \mathcal{C})$ is considered as the smallest simple Lie group of infinite dimension which contains $SL(n, \mathcal{C})$ as simple subgroups.*

§ 6. Lie algebra $\mathfrak{sp}(\infty, \mathcal{C})$

In this section we shall consider $\mathfrak{sp}(\infty, \mathcal{C})$. Let $\mathfrak{H}_{2n}$ be a complex vector space of dimension $2n$ and $\{e_i\}$ ($i=1, 2, \dots, 2n$) be an orthonormal basis of $\mathfrak{H}_{2n}$. Then the group $SP(n, \mathcal{C})$ is the set of all matrices $g \in GL(2n, \mathcal{C})$ which leave invariant the following bilinear form:

$$(16) \quad F(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n (x_i y_{i+n} - x_{i+n} y_i),$$

where $\mathbf{x} = \sum_{i=1}^{2n} x_i e_i$, $\mathbf{y} = \sum_{i=1}^{2n} y_i e_i \in \mathfrak{H}_{2n}$ and its Lie algebra $\mathfrak{sp}(n, \mathcal{C})$ is the set of all matrices $A \in \mathfrak{M}(2n, \mathcal{C})$ of the following type:

$$(17) \quad A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

where A_1, A_2, A_3 and A_4 are matrices of order n , $A_4 = -{}^t A_1$ and A_2, A_3 are symmetric. $\mathfrak{sp}(n, \mathcal{C})$ is said a complex simple Lie algebra of type $\mathcal{C}[1]$.

For the sake of later use, we take such an orthonormal basis $\{e_i; f_i\}$ of $\mathfrak{H}_{2n}$ as

$$(18) \quad e_1, e_2, \dots, e_n; f_1, f_2, \dots, f_n$$

and denote $\mathbf{x} \in \mathfrak{H}_{2n}$ by $\mathbf{x} = \sum_{i=1}^n x_i^{(1)} e_i + \sum_{j=1}^n x_j^{(2)} f_j$. Then (16) is written as

$$(19) \quad F(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n (x_i^{(1)} y_i^{(2)} - x_i^{(2)} y_i^{(1)}).$$

With respect to such a basis, we can identify $SP(n, \mathcal{C})$ with a Lie subgroup

of $SP(n, \mathbb{C})$ ($n > m$) as follows :

$$(20) \quad g \equiv \left(\begin{array}{cc|cc} g_1 & 0 & g_2 & 0 \\ 0 & E_{n-m} & 0 & 0 \\ \hline g_3 & 0 & g_4 & 0 \\ 0 & 0 & 0 & E_{n-m} \end{array} \right) \in SP(n, \mathbb{C}),$$

where E_{n-m} is a unit matrix of order $n-m$ and

$$g = \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix} \in SP(m, \mathbb{C}).$$

In the same way, $\mathfrak{sp}(m, \mathbb{C})$ is identified with a Lie subalgebra of $\mathfrak{sp}(n, \mathbb{C})$ as follows :

$$(21) \quad X \equiv \left(\begin{array}{cc|cc} X_1 & 0 & X_2 & 0 \\ 0 & 0 & 0 & 0 \\ \hline X_3 & 0 & X_4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \in \mathfrak{sp}(n, \mathbb{C}),$$

where

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \in \mathfrak{sp}(m, \mathbb{C}).$$

The above identification leads us to the notion of $\mathfrak{sp}(\infty, \mathbb{C})$. Let $\mathfrak{H}$ be a separable complex Hilbert space and $\tilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathfrak{H}$ be a direct sum. Then $\tilde{\mathfrak{H}}$ is also a separable complex Hilbert space. As a complete orthonormal basis of $\tilde{\mathfrak{H}}$, we take

$$(22) \quad \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n, \dots; \mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n, \dots,$$

where $\{\mathbf{e}_i\}$, $\{\mathbf{f}_j\}$ are the same complete orthonormal bases of $\mathfrak{H}$.

Let $\tilde{\mathfrak{M}}$ be the set of all bounded linear operators of $\tilde{\mathfrak{H}}$ into itself. We put

$$(23) \quad \begin{aligned} \langle A\mathbf{e}_j, \mathbf{e}_i \rangle &= a_{ij}^{(1)}, & \langle A\mathbf{f}_j, \mathbf{e}_i \rangle &= a_{ij}^{(2)} \\ \langle A\mathbf{e}_j, \mathbf{f}_i \rangle &= a_{ij}^{(3)} & \text{and} & \quad \langle A\mathbf{f}_j, \mathbf{f}_i \rangle = a_{ij}^{(4)}. \end{aligned}$$

Then for any $\mathbf{x} = \sum_{i=1}^{\infty} x_i^{(1)} \mathbf{e}_i + \sum_{i=1}^{\infty} x_i^{(2)} \mathbf{f}_i$ of $\tilde{\mathfrak{g}}$, we have

$$(24) \quad \begin{aligned} A\mathbf{x} = & \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{ki}^{(1)} x_i^{(1)} + \sum_{j=1}^{\infty} a_{kj}^{(2)} x_j^{(2)} \right) \mathbf{e}_k + \\ & \sum_{k=1}^{\infty} \left(\sum_{l=1}^{\infty} a_{kl}^{(3)} x_l^{(1)} + \sum_{j=1}^{\infty} a_{kj}^{(4)} x_j^{(2)} \right) \mathbf{f}_k. \end{aligned}$$

Therefore to each $A \in \tilde{\mathfrak{M}}$ there corresponds a matrix of infinite order of the following type:

$$(25) \quad A \equiv \left[\begin{array}{c|c} \begin{array}{ccc} a_{11}^{(1)} & a_{12}^{(1)} & \cdots \rightarrow \\ a_{21}^{(1)} & a_{22}^{(1)} & \cdots \rightarrow \\ \vdots & \vdots & \ddots \\ \downarrow & \downarrow & \searrow \end{array} & \begin{array}{ccc} a_{11}^{(2)} & a_{12}^{(2)} & \cdots \rightarrow \\ a_{21}^{(2)} & a_{22}^{(2)} & \cdots \rightarrow \\ \vdots & \vdots & \ddots \\ \downarrow & \downarrow & \searrow \end{array} \\ \hline \begin{array}{ccc} a_{11}^{(3)} & a_{12}^{(3)} & \cdots \rightarrow \\ a_{21}^{(3)} & a_{22}^{(3)} & \cdots \rightarrow \\ \vdots & \vdots & \ddots \\ \downarrow & \downarrow & \searrow \end{array} & \begin{array}{ccc} a_{11}^{(4)} & a_{12}^{(4)} & \cdots \rightarrow \\ a_{21}^{(4)} & a_{22}^{(4)} & \cdots \rightarrow \\ \vdots & \vdots & \ddots \\ \downarrow & \downarrow & \searrow \end{array} \end{array} \right].$$

A rule to compute coefficients of $A\mathbf{x}$ is given by (24) and (25), where $\mathbf{x} = (x_1^{(1)}, x_2^{(1)}, \dots; x_1^{(2)}, x_2^{(2)}, \dots)$. Let A and B be any elements of $\tilde{\mathfrak{M}}$. Then coefficients of the matrix representation of $C = AB$ are given by

$$c_{ij}^{(1)} = \sum_{k=1}^{\infty} a_{ik}^{(1)} b_{kj}^{(1)} + \sum_{k=1}^{\infty} a_{ik}^{(2)} b_{kj}^{(3)} \quad \text{etc.}$$

Let $\mathbf{x} = \sum_{i=1}^n x_i^{(1)} \mathbf{e}_i + \sum_{i=1}^q x_i^{(2)} \mathbf{f}_i$, $\mathbf{y} = \sum_{i=1}^n y_i^{(1)} \mathbf{e}_i + \sum_{i=1}^q y_i^{(2)} \mathbf{f}_i$ be any elements of $\tilde{\mathfrak{g}}$. Then

$$\begin{aligned} \langle A\mathbf{x}, \mathbf{y} \rangle &= \left\langle \sum_{k=1}^{\infty} \left(\sum_{i=1}^n a_{ki}^{(1)} x_i^{(1)} + \sum_{j=1}^q a_{kj}^{(2)} x_j^{(2)} \right) \mathbf{e}_k + \right. \\ &\quad \left. \sum_{k=1}^{\infty} \left(\sum_{l=1}^n a_{kl}^{(3)} x_l^{(1)} + \sum_{j=1}^q a_{kj}^{(4)} x_j^{(2)} \right) \mathbf{f}_k, \sum_{i=1}^n y_i^{(1)} \mathbf{e}_i + \sum_{i=1}^q y_i^{(2)} \mathbf{f}_i \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n a_{ii}^{(1)} x_i^{(1)} \overline{y_i^{(1)}} + \sum_{i=1}^n \sum_{j=1}^q a_{ij}^{(2)} x_j^{(2)} \overline{y_i^{(1)}} \\ &\quad + \sum_{i=1}^q \sum_{j=1}^n a_{ii}^{(3)} x_i^{(1)} \overline{y_i^{(2)}} + \sum_{i=1}^q \sum_{j=1}^q a_{ij}^{(4)} x_j^{(2)} \overline{y_i^{(2)}}. \end{aligned}$$

The norm $\|A\|$ is given by

$$(26) \quad \|A\|^2 = \inf_M \{M; |\langle A\mathbf{x}, \mathbf{y} \rangle| \leq M \cdot \|\mathbf{x}\| \cdot \|\mathbf{y}\| \text{ for any } p, q, \mathbf{x} \text{ and } \mathbf{y}\}.$$

We identify $\mathfrak{sp}(n, \mathcal{C})$ with a subalgebra of $\widetilde{\mathfrak{M}}$ and denote it by the same symbol $\mathfrak{sp}(n, \mathcal{C})$:

$$X \equiv \left(\begin{array}{cc|cc} X_1 & 0 & X_2 & 0 \\ 0 & 0 & 0 & 0 \\ \hline X_3 & 0 & X_4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right),$$

where

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \in \mathfrak{sp}(n, \mathcal{C}).$$

DEFINITION 3. We set $\mathfrak{sp}(\infty, \mathcal{C}) = \overline{\bigcup_{n=1}^{\infty} \mathfrak{sp}(n, \mathcal{C})}$.

Let $\mathfrak{h}$ be the set of all matrices $A \in \widetilde{\mathfrak{M}}$ of the following type:

$$A = \left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array} \right),$$

where A_2, A_3 are symmetric and $A_4 = -{}^t A_1$.

THEOREM 5. $\mathfrak{h}$ is a Lie subalgebra of $\widetilde{\mathfrak{M}}$, $\mathfrak{sp}(\infty, \mathcal{C})$ is a proper Lie subalgebra of $\mathfrak{h}$ and $\mathfrak{sp}(\infty, \mathcal{C})$ is the set of all completely continuous operators of $\mathfrak{h}$. $\mathfrak{sp}(\infty, \mathcal{C})$ is not isomorphic with $\mathfrak{h}$ and it is an ideal of $\mathfrak{h}$. Therefore $SP(\infty, \mathcal{C})$ is a distinguished Lie subgroup of the connected Lie group corresponding to $\mathfrak{h}$.

Proof. First we shall prove that $\mathfrak{h}$ is a closed linear subspace of $\widetilde{\mathfrak{M}}$ and closed under the bracket operation, i.e., $\mathfrak{h}$ is a Lie subalgebra of $\widetilde{\mathfrak{M}}$.

Let $\{A_m\}$ be a sequence of elements $A_m \in \mathfrak{h}$ which converge to some matrix $A \in \widetilde{\mathfrak{M}}$. We have to show that $A \in \mathfrak{h}$. We denote A_m by

$$A_m = \left(\begin{array}{c|c} (\alpha_{ij}^{(m)}) & (\beta_{ij}^{(m)}) \\ \hline (\gamma_{ij}^{(m)}) & (\delta_{ij}^{(m)}) \end{array} \right).$$

Then by definition of the norm, it is not difficult to see that $A_1^{(m)}, A_2^{(m)}, A_3^{(m)},$

$A_4^{(m)}$ converge to A_1, A_2, A_3, A_4 respectively, where

$$\begin{aligned} A_1^{(m)} &= [\alpha_{ij}^{(m)}], & A_2^{(m)} &= [\beta_{ij}^{(m)}], \\ A_3^{(m)} &= [\gamma_{ij}^{(m)}], & A_4^{(m)} &= [\delta_{ij}^{(m)}]. \end{aligned}$$

To prove that $A \in \mathfrak{h}$, we show that A_2, A_3 are symmetric and $A_4 = -{}^t A_1$. As shown in Lemma 4, for any $\varepsilon > 0$ there exists $M > 0$ such that $|\beta_{ij} - \beta_{ij}^{(m)}| < \varepsilon$ and we have

$$|\beta_{ij} - \beta_{ji}| = |\beta_{ij} - \beta_{ij}^{(m)} + \beta_{ji}^{(m)} - \beta_{ji}| < 2\varepsilon.$$

Therefore A_2 is symmetric and also A_3 . Since $\delta_{ij}^{(m)} + \alpha_{ji}^{(m)} = 0$, we have

$$\begin{aligned} |\delta_{ij} + \alpha_{ji}| &= |\delta_{ij} - \delta_{ij}^{(m)} + \alpha_{ji} - \alpha_{ji}^{(m)}| \\ &\leq |\delta_{ij} - \delta_{ij}^{(m)}| + |\alpha_{ji} - \alpha_{ji}^{(m)}| < 2\varepsilon. \end{aligned}$$

Therefore we have $\delta_{ij} + \alpha_{ji} = 0$ for every (i, j) and also $A_4 = -{}^t A_1$. The above arguments show that $A \in \mathfrak{h}$. Closedness under the bracket operation is easily verified by a direct computation.

Next we shall prove the latter half of Theorem 4. We need a few lemmas.

LEMMA 12. *Any completely continuous operator A which is symmetric is a limit of symmetric matrices A_m of finite order.*

Proof. We split A into the following blocks:

$$A = \left(\begin{array}{c|c} A_1^{(p)} & A_2^{(p)} \\ \hline A_3^{(p)} & A_4^{(p)} \end{array} \right).$$

where $A_1^{(p)}$ is a square matrix of order p . From the complete continuity of A , it follows that A is a limit of $\{A_{12}^{(p)}\}$, where $A_{12}^{(p)}$ is given by

$$A_{12}^{(p)} = \left(\begin{array}{c|c} A_1^{(p)} & A_2^{(p)} \\ \hline 0 & 0 \end{array} \right).$$

Then by symmetricity of A , it is easily proved just as in Theorem 1 that $\|A_2^{(p)}\| \rightarrow 0$ and $\|A - A_1^{(p)}\| \rightarrow 0$ as $p \rightarrow \infty$, where

$$A_2^{(p)} = \left[\begin{array}{c|c} 0 & A_2^{(p)} \\ \hline 0 & 0 \end{array} \right] \quad \text{and} \quad A_1^{(p)} = \left[\begin{array}{c|c} A_1^{(p)} & 0 \\ \hline 0 & 0 \end{array} \right].$$

This proves Lemma 12.

Let $P_1^{(p)}$ (resp. $P_1^{\perp(p)}$) be the projection of $\mathfrak{H}$ onto $\mathfrak{H}_1^{(p)}$ (resp. $\mathfrak{H}_1^{\perp(p)}$), where $\mathfrak{H}_1^{(p)}$ is a subspace spanned by $\{e_1, \dots, e_p\}$ (resp. $\mathfrak{H}_1^{\perp(p)}$ is a subspace spanned by $\{e_{p+1}, e_{p+2}, \dots, e_n, \dots\}$). Let $P_2^{(p)}$ (resp. $P_2^{\perp(p)}$) be the projection of $\mathfrak{H}$ onto $\mathfrak{H}_2^{(p)}$ (resp. $\mathfrak{H}_2^{\perp(p)}$), where $\mathfrak{H}_2^{(p)}$ is a subspace spanned by $\{f_1, \dots, f_q\}$. We set

$$P_{(p,q)} = P_1^{(p)} \oplus P_2^{(p)} \quad \text{and} \quad P_{(p,q)}^{\perp} = P_1^{\perp(p)} \oplus P_2^{\perp(p)}.$$

Then $P_{(p,q)}$ (resp. $P_{(p,q)}^{\perp}$) is the projection of $\tilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathfrak{H}$ onto $\mathfrak{H}_1^{(p)} \oplus \mathfrak{H}_2^{(p)}$ (resp. $\mathfrak{H}_1^{\perp(p)} \oplus \mathfrak{H}_2^{\perp(p)}$). If we set

$$(27) \quad A_{(p,q)} = P_{(p,q)} A \quad \text{and} \quad A_{(p,q)}^{\perp} = P_{(p,q)}^{\perp} A,$$

then an easy computation shows that

$$(28) \quad A_{(p,q)}^{\perp} = \left[\begin{array}{c|c} \begin{matrix} 0 & 0 \\ \vdots & \vdots \\ * & * \end{matrix} & \begin{matrix} 0 & 0 \\ \vdots & \vdots \\ * & * \end{matrix} \\ \hline \begin{matrix} 0 & 0 \\ \vdots & \vdots \\ * & * \end{matrix} & \begin{matrix} 0 & 0 \\ \vdots & \vdots \\ * & * \end{matrix} \end{array} \right] \quad \text{and} \quad A_{(p,q)} = \left[\begin{array}{c|c} \begin{matrix} * & * \\ \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} * & * \\ \vdots & \vdots \\ 0 & 0 \end{matrix} \\ \hline \begin{matrix} * & * \\ \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} * & * \\ \vdots & \vdots \\ 0 & 0 \end{matrix} \end{array} \right],$$

For brevity we set $A_{(p,p)} = {}^p A$

LEMMA 13. *Let $A \in \tilde{\mathfrak{M}}$ be completely continuous. Then A is a limit of $A_{(p,q)}$. In particular, $\lim_{p \rightarrow \infty} \|A - {}^p A\| = 0$.*

Proof. The proof of Lemma 13 is just the same as in Theorem 1. So we omit it.

We split A into the following blocks:

$$(29) \quad A = \left[\begin{array}{cc|cc} A_{11}^{(p)} & A_{12}^{(p)} & A_{21}^{(p)} & A_{22}^{(p)} \\ A_{13}^{(p)} & A_{14}^{(p)} & A_{23}^{(p)} & A_{24}^{(p)} \\ \hline A_{31}^{(p)} & A_{32}^{(p)} & A_{41}^{(p)} & A_{42}^{(p)} \\ A_{33}^{(p)} & A_{34}^{(p)} & A_{43}^{(p)} & A_{44}^{(p)} \end{array} \right] = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array} \right].$$

where $A_{11}^{(p)}, A_{21}^{(p)}, A_{31}^{(p)}, A_{41}^{(p)}$ are square matrices of order p .

Let $A \in \mathfrak{h}$ be completely continuous. Then by Lemma 13 A is a limit of pA . As mentioned in the first place of the proof of Theorem 5, A_1, A_2, A_3 and A_4 are limits of ${}^pA_1, {}^pA_2, {}^pA_3$ and pA_4 respectively, where

$${}^pA = \left(\begin{array}{c|c} {}^pA_1 & {}^pA_2 \\ \hline {}^pA_3 & {}^pA_4 \end{array} \right).$$

Since A_2, A_3 are symmetric, A_2, A_3 are limits of $A_{21}^{(p)}, A_{31}^{(p)}$, respectively by Lemma 12 (note that if A is completely continuous, then A_1, A_2, A_3 and A_4 are also completely continuous). Since A_1, A_4 are limits of ${}^pA_1, {}^pA_4$ respectively, $\|A_{13}^{(p)}\|$ and $\|A_{43}^{(p)}\|$ converge to zero, where

$$A_{13}^{(p)} = \begin{bmatrix} 0 & 0 \\ A_{13}^{(p)} & 0 \end{bmatrix} \quad \text{and} \quad A_{43}^{(p)} = \begin{bmatrix} 0 & 0 \\ A_{43}^{(p)} & 0 \end{bmatrix}.$$

Since $A_4 = -{}^tA_1$, it follows that $\|A_{12}^{(p)}\| = \| -{}^tA_{43}^{(p)} \|$ and $\|A_{42}^{(p)}\| = \| -{}^tA_{13}^{(p)} \|$ converge to zero. From the above arguments, it follows that

$$\begin{aligned} & \left\| \begin{pmatrix} A_{11}^{(p)} & A_{12}^{(p)} & A_{21}^{(p)} & A_{22}^{(p)} \\ A_{13}^{(p)} & A_{14}^{(p)} & A_{23}^{(p)} & A_{24}^{(p)} \\ \hline A_{31}^{(p)} & A_{32}^{(p)} & A_{41}^{(p)} & A_{42}^{(p)} \\ A_{33}^{(p)} & A_{34}^{(p)} & A_{43}^{(p)} & A_{44}^{(p)} \end{pmatrix} - \begin{pmatrix} A_{11}^{(p)} & 0 & A_{21}^{(p)} & 0 \\ 0 & 0 & 0 & 0 \\ \hline A_{31}^{(p)} & 0 & A_{41}^{(p)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\| \leq \\ & \left\| \begin{pmatrix} 0 & 0 & 0 & 0 \\ A_{13}^{(p)} & A_{14}^{(p)} & A_{23}^{(p)} & A_{24}^{(p)} \\ \hline 0 & 0 & 0 & 0 \\ A_{33}^{(p)} & A_{34}^{(p)} & A_{43}^{(p)} & A_{44}^{(p)} \end{pmatrix} \right\| + \|A_{12}^{(p)}\| + \|A_{22}^{(p)}\| + \|A_{32}^{(p)}\| \\ & + \|A_{42}^{(p)}\| \rightarrow 0 \end{aligned}$$

as $p \rightarrow \infty$, where

$$\left(\begin{array}{cc|cc} A_{11}^{(p)} & 0 & A_{21}^{(p)} & 0 \\ 0 & 0 & 0 & 0 \\ \hline A_{31}^{(p)} & 0 & A_{41}^{(p)} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \in \mathfrak{sp}(p, C).$$

This shows that each completely continuous operator $A \in \mathfrak{h}$ is a limit of $A_m \in \bigcup_{n=1}^{\infty} \mathfrak{sp}(n, C)$.

Let $\tilde{E}$ be an element of $\tilde{\mathfrak{M}}$ such that

$$\tilde{E} = \begin{bmatrix} E_{\infty} & 0 \\ 0 & -E_{\infty} \end{bmatrix},$$

where E_{∞} is a unit matrix of infinite order. Then clearly $\tilde{E}$ belongs to $\mathfrak{h}$ but not completely continuous. Therefore $\mathfrak{sp}(\infty, C)$ is a proper subspace of $\mathfrak{h}$. Hence the proof of Theorem 5 is completed.

§ 7. Simplicity of $\mathfrak{sp}(\infty, C)$

In the last section, we shall prove simplicity of $\mathfrak{sp}(\infty, C)$.

PROPOSITION 7. *There exists no ideal $\mathfrak{i} \neq (0)$ of $\mathfrak{sp}(\infty, C)$ in $\mathfrak{sp}(\infty, C)$*
 $= (\mathfrak{sp}(\infty, C) - \bigcup_{n=1}^{\infty} \mathfrak{sp}(n, C)) \cup \{0\}$.

Proof. Let $A \neq 0$ be any element of $\mathfrak{sp}(\infty, C)$. If A has a coefficient $a_{qp} \neq 0$ other than α_{ii}, δ_{ii} ($i=1, 2, \dots$) (we can assume without loss of generality $p > q$), then we take such a matrix $E_{(q)}$ as

$$E_{(q)} = \left(\begin{array}{cc|cc} E_q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -E_q & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \in \mathfrak{sp}(q, C),$$

where E_q is a unit matrix of order q . Then $[A, E_{(q)}]$ is given by

$$(30) \quad [A, E_{(q)}] = \left(\begin{array}{c|c|c|c} \overbrace{\begin{array}{c} 0 \\ \vdots \\ 0 \end{array}}^p & \begin{array}{c} -A_{12} \\ \vdots \\ -A_{qp} \end{array} & -2A_{21} & -A_{22} \\ \hline \overbrace{\begin{array}{c} A_{13} \\ \vdots \\ A_{33} \end{array}}^q & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} & -A_{23} & 0 \\ \hline 2A_{31} & A_{32} & 0 & -A_{42} \\ \hline \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} & \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} & \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} & \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \end{array} \right).$$

Next we take such a matrix $E_{(p)}^1$ as

$$E_{(p)}^1 = \left(\begin{array}{c|c} \overbrace{\begin{array}{c} 0 \\ \vdots \\ 0 \end{array}}^p & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \\ \hline \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \end{array} \right) \in \mathfrak{sp}(p, \mathbb{C}).$$

Then $[[A, E_{(q)}], E_{(p)}^1]$ is given by

$$(31) \quad [[A, E_{(q)}], E_{(p)}^1] = \left(\begin{array}{c|c} \overbrace{\begin{array}{ccc} \alpha_{1p} & & -\beta_{1p} \\ 0 & \alpha_{2p} & -\beta_{2p} \\ \vdots & \vdots & \vdots \\ 0 & \alpha_{qp} & -\beta_{qp} \end{array}}^p & \begin{array}{ccc} 0 & & 0 \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & & 0 \end{array} \\ \hline \begin{array}{ccc} * \cdots * 0 \cdots 0 \cdots \rightarrow \\ \vdots \\ 0 & \downarrow & 0 \end{array} & \begin{array}{ccc} * \cdots * 0 \cdots 0 \cdots \rightarrow \\ \vdots \\ 0 & \downarrow & 0 \end{array} \end{array} \right) \in \mathfrak{sp}(p, C)$$

$$\left(\begin{array}{c|c} \begin{array}{ccc} \gamma_{1p} & & -\delta_{1p} \\ 0 & \gamma_{2p} & -\delta_{2p} \\ \vdots & \vdots & \vdots \\ 0 & \gamma_{qp} & -\delta_{qp} \end{array} & \begin{array}{ccc} 0 & & 0 \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & & 0 \end{array} \\ \hline \begin{array}{ccc} * \cdots * 0 \cdots 0 \cdots \rightarrow \\ \vdots \\ 0 & \downarrow & 0 \end{array} & \begin{array}{ccc} * \cdots * 0 \cdots 0 \cdots \rightarrow \\ \vdots \\ 0 & \downarrow & 0 \end{array} \end{array} \right)$$

and at least one of α_{qp} , β_{qp} , γ_{qp} and δ_{qp} is not zero. This is a contradiction because A is in $\mathfrak{a} \mathfrak{sp}(\infty, C)$. Therefore we assume that A has a coefficient $a_{qq} \neq 0$ and $a_{pq} = 0$ for $p \neq q$. If $\beta_{qq} \neq 0$ or $\gamma_{qq} \neq 0$, it follows from (30) that

$$[A, E_{(q)}] = \left(\begin{array}{cc|cc} 0 & 0 & -2A_{21} & 0 \\ 0 & 0 & 0 & 0 \\ \hline 2A_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \in \mathfrak{sp}(q, C)$$

and $[A, E_{(q)}] \neq 0$. This is a contradiction. Therefore the only possible case is that there exists $\alpha_{pp} \neq 0$, $\alpha_{pq} = \delta_{pq} = 0$ for $p \neq q$ and $A_2 = A_3 = 0$, i.e.,

$$A = \left(\begin{array}{ccc|ccc} * & & & & & \\ & \ddots & & & & \\ & & * & & & \\ 0 & & \alpha_{pp} & & & 0 \\ & & * & & & \\ & & & \ddots & & \\ & & & & * & \\ \hline & & & & & \\ 0 & & & & & \\ & & & * & & \\ & & & \ddots & & \\ & & & & * & \\ & & & & -\alpha_{pp} & \\ & & & 0 & & \end{array} \right).$$

In this case we take such a matrix X as

$$X = \begin{pmatrix} 0 & \overbrace{\begin{pmatrix} 0 & \vdots & 0 \\ \vdots & 0 & \vdots \\ \cdots & 0 & 10 \cdots \rightarrow \\ \vdots & 0 & \vdots \\ 0 & \vdots & 0 \end{pmatrix}}^p & \\ \hline 0 & 0 \end{pmatrix} \in \mathfrak{sp}(p, \mathbb{C}).$$

Then we have

$$[A, X] = \begin{pmatrix} 0 & \overbrace{\begin{pmatrix} 0 & \vdots & 0 \\ \vdots & 0 & \vdots \\ \cdots & 0 & 2\alpha_{pp} 0 \cdots \rightarrow \\ \vdots & 0 & \vdots \\ 0 & \vdots & 0 \end{pmatrix}}^p & \\ \hline 0 & 0 \end{pmatrix} \in \mathfrak{sp}(p, \mathbb{C}).$$

This is a contradiction. Therefore the only possible case is excluded and the proof of Proposition 7 is completed.

THEOREM 6. $\mathfrak{sp}(\infty, \mathbb{C})$ is a simple Lie algebra and the corresponding connected Lie subgroup $SP(\infty, \mathbb{C})$ of $GL(\infty, \mathbb{C})$ is considered as the smallest simple Lie group of infinite dimension which contains $SP(n, \mathbb{C})$ as simple subgroups.

Proof. The proof of Theorem 6 is just the same as in § 4.

REMARK. Clearly $\mathfrak{sp}(\infty, \mathbb{C})$ has proper simple subalgebras $\{\mathfrak{sp}_n(\infty, \mathbb{C})\}$ which are isomorphic with $\mathfrak{sp}(\infty, \mathbb{C})$:

$$\mathfrak{sp}_n(\infty, C) = \left\{ \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & A_1 & 0 & A_2 \\ \hline 0 & 0 & 0 & 0 \\ 0 & A_3 & 0 & A_4 \end{array} \right) ; \left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array} \right) \in \mathfrak{sp}(\infty, C) \right\}.$$

Therefore Lie algebras $\mathfrak{M}(\infty, C)$, $\mathfrak{n}(\infty, C)$, $\mathfrak{sl}(\infty, C)$ and $\mathfrak{sp}(\infty, C)$ have a countable family of proper subalgebras isomorphic with the original algebra (cf. question 1, § 4).

Simple Lie groups other than *type A* constructed in this paper and in the forthcoming one are characterized as the subgroups of the groups of all operators which leave invariant suitable definite or indefinite quadratic forms in $\mathfrak{H}$. On the contrary, the distinctive character of simple Lie groups of type *A*, especially $SL(\infty, C)$ and $SL(\infty, R)$ are vague. In finite-dimensional cases, $SL(n, C)$ and $SL(n, R)$ are characterized as the groups of the (usual) volume-preserving linear transformations of C^n and R^n , respectively. In case of $\mathfrak{H}$, there doesn't exist such a volume as the extension of the usual volume in a Euclidean space. The present author doesn't know whether $SL(\infty, C)$ and $SL(\infty, R)$ are characterized by means of a suitable measure in $\mathfrak{H}$.

The following problem is also left open:

Are $\mathfrak{n}(\infty, C)$, $\mathfrak{sl}(\infty, C)$ and $\mathfrak{sp}(\infty, C)$ not isomorphic mutually?

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