A BIVARIATE NORMAL TEST WITH TWO SIDED ALTERNATIVE

By

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§ 1. Introduction.

Suppose X, Y be distributed as bivariate normal with known-variance matrix, which we assume without loss of generality, to have unit variance. Let us consider a problem of testing the hypothesis that means are zero $H_0: \theta_1 = \theta_2 = 0$ against the alternative that means are either in the 1st or the 3rd quadrant namely, $H_1: (\theta_1 \geq 0 \text{ and } \theta_2 \geq 0) \text{ or } (\theta_1 \leq 0 \text{ and } \theta_2 \leq 0)$ where at least one of the inequalities are strict in both cases.

This is a kind of modification of a multivariate analogue of the one sided test by one of the authors [7] in bivariate case. The section 2 is devoted to the derivations of maximum likelihood estimate of the mean vector under this restriction and the likelihood ratio test.

The section 3 is devoted to the generalisation to the situation where variances are unknown but know to be common and correlation coefficient is also known. The relation between the multivariate analogue of the one sided tests and the test of homogeneity for ordered alternatives treated by Bartholomew ([1]~[4]) was discussed in [7]. It may be pointed out that our result furnishes the exact distribution of a two sided test when $k=3$ in [3].

The investigations on the power function is now proceeding and the authors wish to publish their findings in a near future.

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2. The case of known variance.

At first we shall derive the maximum likelihood estimates (MLE) of the mean vector under the alternative hypotheses. Let $(X_1, Y_2), \dots, (X_n, Y_n)$ be a random sample and $(\bar{X}, \bar{Y})$ be sample mean vector. If both of the components are of the same sign, it is MLE, and if not the result due to Brunk [5] tells us that the MLE should be on the boundary of the region restricted by the inequalities in the alternative hypothesis, and hence at least one of the components of MLE should be zero.

To find the MLE it is convenient to consider the following factorization of the likelihood:

$$\begin{aligned}
 (1) \quad L(\theta_1, \theta_2) &= \left(\frac{1}{\sqrt{2\pi}} \right)^n \exp \left[-\frac{1}{2} Q(x, y) \right] \exp \left[-\frac{n}{2} (\bar{x} - \theta_1)^2 \right] \times \\
 &\quad \times \exp \left[-\frac{n}{2(1-\rho^2)} (\bar{y} - \rho\bar{x} - (\theta_2 - \rho\theta_1))^2 \right] \\
 &= \left(\frac{1}{\sqrt{2\pi}} \right)^n \exp \left[-\frac{1}{2} Q(x, y) \right] \exp \left[-\frac{n}{2} (\bar{y} - \theta_2)^2 \right] \times \\
 &\quad \times \exp \left[-\frac{n}{2(1-\rho^2)} (\bar{x} - \rho\bar{y} - (\theta_1 - \rho\theta_2))^2 \right]
 \end{aligned}$$

where

$$Q(x, y) = \sum \{ (x_i - \bar{x})^2 - 2\rho(x_i - \bar{x})(y_i - \bar{y}) + (y_i - \bar{y})^2 \} / (1 - \rho^2).$$

We have

$$(2) \quad \text{Max}_{\theta_1, \theta_2=0} L(\theta_1, \theta_2) = \left(\frac{1}{\sqrt{2\pi}} \right)^n \exp \left[-\frac{1}{2} Q(x, y) \right],$$

$$(3) \quad \text{Max}_{\theta_1=0} L(\theta_1, \theta_2) = \left(\frac{1}{\sqrt{2\pi}} \right)^n \exp \left[-\frac{1}{2} Q(x, y) \right] \exp \left[-\frac{n}{2} \bar{x}^2 \right],$$

$$(4) \quad \text{Max}_{\theta_2=0} L(\theta_1, \theta_2) = \left(\frac{1}{\sqrt{2\pi}} \right)^n \exp \left[-\frac{1}{2} Q(x, y) \right] \exp \left[-\frac{n}{2} \bar{y}^2 \right].$$

Therefore we have the MLE as follows

$$(5) \quad \begin{cases} \hat{\theta}_1 = \bar{X}, \hat{\theta}_2 = \bar{Y} & \text{if } \bar{X} \text{ and } \bar{Y} \text{ are of the same sign} \\ \hat{\theta}_1 = \bar{X} - \rho\bar{Y}, \hat{\theta}_2 = 0 & \text{if } \bar{X} \text{ and } \bar{Y} \text{ are of the different sign and} \\ & |\bar{X}| > |\bar{Y}|, \\ \hat{\theta}_1 = 0, \hat{\theta}_2 = \bar{Y} - \rho\bar{X} & \text{if } \bar{X} \text{ and } \bar{Y} \text{ are of the different sign and} \\ & |\bar{X}| < |\bar{Y}|. \end{cases}$$

The likelihood ratio test can be derived easily by taking the ratios of $L(0,0)$

to (2), (3) and (4).

Let

$$(6) \quad \begin{aligned} Z_1 &= (\bar{X} - \rho \bar{Y}) / \sqrt{1 - \rho^2} \\ Z_2 &= (\bar{Y} - \rho \bar{X}) / \sqrt{1 - \rho^2} \end{aligned}$$

then we have easily

$$(7) \quad Z_1^2 + \bar{Y}^2 = \bar{X}^2 + Z_2^2 = \frac{1}{1 - \rho^2} (\bar{Y}^2 - 2\rho \bar{X} \bar{Y} + \bar{X}^2) \equiv \bar{Z}^2$$

and the region of rejection is given by the following.

$$(8) \quad \left\{ \begin{array}{ll} \text{(I)} & n\bar{Z}^2 \geq \bar{Z}_0^2 \quad \text{if} \quad \bar{X} \geq 0, \bar{Y} \geq 0 \\ \text{(II, 1)} & \sqrt{n} Z_1 < -\bar{Z}_0 \quad \text{if} \quad \bar{X} < 0, \bar{Y} > 0, |\bar{X}| > |\bar{Y}| \\ \text{(II, 2)} & \sqrt{n} Z_2 > \bar{Z}_0 \quad \text{if} \quad \bar{X} < 0, \bar{Y} > 0, |\bar{X}| < |\bar{Y}| \\ \text{(III)} & n\bar{Z}^2 \geq \bar{Z}_0^2 \quad \text{if} \quad \bar{X} < 0, \bar{Y} < 0 \\ \text{(IV, 1)} & \sqrt{n} Z_2 < -\bar{Z}_0 \quad \text{if} \quad \bar{X} > 0, \bar{Y} < 0, |\bar{X}| < |\bar{Y}| \\ \text{(IV, 2)} & \sqrt{n} Z_1 > \bar{Z}_0 \quad \text{if} \quad \bar{X} > 0, \bar{Y} < 0, |\bar{X}| > |\bar{Y}|, \end{array} \right.$$

where $\bar{Z}_0$ is a constant depending on ρ and the significance level α , and hence

$$n\bar{Z}^2 \geq \bar{Z}_0^2 \quad \text{if} \quad (\bar{X}, \bar{Y}) \text{ is in the 1st or 3rd quadrant}$$

and $\sqrt{n} Z_1 > \bar{Z}_0$ or $\sqrt{n} Z_2 > \bar{Z}_0$ if $(\bar{X}, \bar{Y})$ is in the 2nd or 4th orthand.

It would be interesting and helpful for understanding this test to see how the rejection region appear by drawing figures for positive and negative correlation. It is clear that $\bar{Z}_0$ is to be determined by

$$(10) \quad \begin{aligned} \alpha &= 2\{P_r(n\bar{Z}^2 \geq \bar{Z}_0^2, \bar{X} > 0, \bar{Y} > 0) + P_r(\bar{X} < 0, \sqrt{n} Z_2 > \bar{Z}_0) \\ &\quad + P_r(\bar{Y} > 0, -\sqrt{n} Z_1 > \bar{Z}_0) - P_r(-\sqrt{n} Z_1 > \bar{Z}_0, \sqrt{n} Z_2 > \bar{Z}_0)\}. \end{aligned}$$

The first term can be evaluated by polar transform, and others can be done by noticing that $\sqrt{n} Z_1, \sqrt{n} Z_2$ have unit variances and covariance ρ and $\bar{X}$ and $\bar{Y}$ respectively are independent to Z_2 and Z_1 . Therefore we have

$$(11) \quad \begin{aligned} \alpha &= 2\left\{ (e^{-\bar{Z}_0^2/2}) \frac{1}{2\pi} \cos^{-1}(-\rho) + \int_{\bar{Z}_0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right] dx \right. \\ &\quad \left. - \int_{\bar{Z}_0}^{\infty} \int_{\bar{Z}_0}^{\infty} \frac{1}{2(1-\rho^2)} \left[-\frac{1}{2(1-\rho^2)} (x^2 - 2\rho xy + y^2) \right] dx dy \right\}. \end{aligned}$$

Each term can be computed easily from the tables available. The values of $\bar{Z}_0$ have been tabulated for $\alpha=0.05, 0.01$ and $\rho=-0.95(0.5)0.95$, as

is shown in the table.

Table of $\bar{z}_0$

ρ	5%	1%	ρ	5%	1%
-.95	2.07	2.67	.00	2.35	2.94
-.90	2.11	2.72	.05	2.36	2.94
-.85	2.14	2.75	.10	2.37	2.95
-.80	2.17	2.77	.15	2.37	2.96
-.75	2.19	2.79	.20	2.38	2.96
-.70	2.20	2.80	.25	2.39	2.97
-.65	2.22	2.82	.30	2.39	2.98
-.60	2.23	2.83	.35	2.40	2.98
-.55	2.25	2.84	.40	2.40	2.99
-.50	2.26	2.85	.45	2.41	3.00
-.45	2.27	2.86	.50	2.41	3.00
-.40	2.28	2.87	.55	2.42	3.01
-.35	2.29	2.88	.60	2.42	3.02
-.30	2.30	2.89	.65	2.43	3.02
-.25	2.31	2.90	.70	2.43	3.03
-.20	2.32	2.91	.75	2.44	3.04
-.15	2.33	2.92	.80	2.44	3.05
-.10	2.34	2.92	.85	2.44	3.05
-.05	2.34	2.93	.90	2.44	3.06
			.95	2.45	3.08

3. The case of unknown but common variance.

The assumption that X, Y have common unit variance can be replaced by the assumption that they have a common unknown variance. In this case the factrization of the likelihood comes out to be

$$\begin{aligned}
 (12) \quad L(\theta_1, \theta_2, \sigma^2) &= \left(\frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \right)^2 \exp \left[-\frac{1}{2\sigma^2} Q(x, y) \right] \exp \left[-\frac{n}{2\sigma^2} (\bar{x} - \theta_1)^2 \right] \times \\
 &\quad \times \exp \left[-\frac{n}{2\sigma^2(1-\rho^2)} (\bar{y} - \rho\bar{x} - (\theta_2 - \rho\theta_1))^2 \right] \\
 &= \left(\frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \right)^2 \exp \left[-\frac{1}{2\sigma^2} Q(x, y) \right] \exp \left[-\frac{n}{2\sigma^2} (\bar{y} - \theta_2)^2 \right] \times \\
 &\quad \times \exp \left[-\frac{n}{2\sigma^2(1-\rho^2)} (\bar{x} - \rho\bar{y} - (\theta_1 - \rho\theta_2))^2 \right],
 \end{aligned}$$

where $Q(x, y)$ is as defined in (1).

Let

$$(13) \quad u_i = (x_i - \rho y_i) / \sqrt{1 - \rho^2},$$

then we can easily find the corresponding random variables U_i and Y_i ($i=1, \dots, n$) are all mutually independent. And we have

$$(14) \quad Q(X, Y) = \sum_{i=1}^n (u_i - \bar{u})^2 + (y_i - \bar{y})^2,$$

and it is distributed as $\sigma^2 \chi^2$ with the degrees of freedom $2(n-1)$. We are perhaps justified to apply the restricted maximum likelihood principle in this situation. (c.f. Thompson [8]). According to this, the MLE of the mean vector remains the same whereas the MLE $\hat{\sigma}^2$ of σ^2 is found to be $\frac{1}{2n} Q(X, Y)$, and the likelihood ratio test is given by replacing $\bar{z}_0^2$ with $\hat{\sigma}^2 \bar{F}$, in (8), where $\bar{F}$ is a constant depending on ρ , the sample size n , and the significance level.

More generally when we have an unbiased estimate $\hat{\sigma}^2$ independent to $\bar{X}$ and $\bar{Y}$ and distributed proportional to χ^2 distribution with some degrees of freedom, say r , we have the same rejection region except for that $\bar{F}$ depends on the degree of freedom r rather than sample size n .

$$(15) \quad \alpha = 2 \left\{ Pr(2F_{2,r} > \bar{F}) - \frac{1}{2\pi} \cos^{-1}(-\rho) + Pr(F_{1,r} > \bar{F}) - T(-\bar{F}, -\bar{F}; \rho) \right\},$$

where $F_{1,r}$ and $F_{2,r}$ are random variables subject to F distributions with $(1, r)$ and $(2, r)$ degrees of freedom respectively, and $T(*, *; \sigma)$ is bivariate analogue of Student's t -distribution function defined by Dunnett and Sobel [6].

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