AN EXTENSION THEOREM FOR SEMI-VALUATIONS OF THE FIRST KIND

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The purpose of the present paper is to show the following extension theorem for semi-valuations of the first kind:

(THE EXTENSION THEOREM.) *If k is a subfield of a field K , then any semi-valuation s of the first kind of k , of order n , can be extended to a semi-valuation of the first kind of K , of the same order n . In particular, if K is a finite algebraic extension of k , then the number of distinct extensions of s in K , of order n , is finite and is not greater than d^n , where d is the degree of separability of K over k .*

1. Introductory concepts.

We shall recall in this section a few definitions and properties of semi-valuations from the previous papers [1] and [2].

1.1. The order of a semi-valuation. Let K be a field and let K^* denote the multiplicative group of K . Let Γ be an additive abelian l -group. A *semi-valuation* of K is a mapping s of K^* onto Γ such that the following conditions are satisfied:

- (i) $s(xy) = s(x) + s(y)$,
- (ii) $s(x+y) \geq \inf \{s(x), s(y)\}$.

For any x in K^* , the corresponding element $s(x)$ of Γ is called the *value* of x in the given semi-valuation. Γ is called the *value group* of s . The subring consisting of all elements x of K with $s(x) \geq 0$ and of 0 is called the *semi-valuation ring* of s .

By the *order* of a semi-valuation s we mean the number of maximal l -closed upper classes of its value group (a set $\mathcal{Q}$ of positive elements in Γ is called an l -closed upper class of Γ , if $\beta \geq \alpha$ with $\alpha \in \mathcal{Q}$ implies $\beta \in \mathcal{Q}$ and $\mathcal{Q}$ is

a sublattice of Γ). The order of s may be infinite. We shall say that a semi-valuation is *of the first kind*, if its order is finite. In particular, if s is of order 1, then it is a valuation.

1.2. Equivalent semi-valuations. Let s_1 and s_2 be two semi-valuations of K , with value groups Γ_1 and Γ_2 respectively. We shall say that s_1 and s_2 are *equivalent semi-valuations* if there exists an order preserving isomorphism φ of Γ_1 onto Γ_2 such that $s_2(x) = \varphi[s_1(x)]$ for all x in K^* . We shall make no distinction between equivalent semi-valuations; we agree in fact to identify any two semi-valuations of K if they are equivalent.

It is obvious that equivalent semi-valuations of K have the same semi-valuation ring. Conversely, if two semi-valuations s_1 and s_2 of K have the same semi-valuation ring R , then they are equivalent. For let Γ_1 and Γ_2 be the value groups of s_1 and s_2 respectively. K^* is preordered by prescribing $x \geq y$ if $xy^{-1} \in R$, and the two semi-valuations s_1 and s_2 are the order preserving homomorphisms of K^* onto Γ_1 and Γ_2 respectively. By assumption, they have the same kernel, namely the set E of units in R . Hence $\Gamma_1 \cong K^*/E \cong \Gamma_2$ as ordered groups. Therefore our assertion is proved.

Let $s_1, \dots, s_m$ be m semi-valuations of K , with semi-valuation rings $R_1, \dots, R_m$ respectively. We shall say that $s_1, \dots, s_m$ are *incomparable*, if $R_i \not\subseteq R_j$ for every pair of different indices i, j .

1.3. Decomposition and composition of semi-valuations of the first kind.

Let s be a semi-valuation of the first kind of K , of order n . By Γ and R we denote its value group and semi-valuation ring respectively. Let $\mathcal{Q}_1, \dots, \mathcal{Q}_n$ denote n maximal l -closed upper classes of Γ . For each $\mathcal{Q}_i$, by A_i we denote the isolated group generated by $\Gamma^+ - \mathcal{Q}_i$, where Γ^+ is the set of positive elements of Γ ; and by Γ_i the factor group Γ/A_i . Γ_i is canonically totally ordered. Then the following statements hold (Cf. Ch. III in [2]):

(i) The mapping $v_i^*: x \rightarrow \overline{s(x)_i}$, the coset of $s(x)$ in Γ_i , is a valuation of K , with value group Γ_i .

(ii) $v_1^*, \dots, v_n^*$ are incomparable.

(iii) The semi-valuation ring of s is the intersection of all valuation rings of these valuations.

(iv) The mapping $s(x) \rightarrow (v_1^*(x), \dots, v_n^*(x))$ is an order preserving isomorphism of Γ onto Γ^* , where $\Gamma^* = \{(v_1^*(x), \dots, v_n^*(x)) | x \in K^*\}$ and Γ^* is ordered

by prescribing $(v_1^*(x), \dots, v_n^*(x)) \geq (v_1^*(y), \dots, v_n^*(y))$ if $v_i^*(x) \geq v_i^*(y)$ for all i .

(v) The mapping $s^*: x \rightarrow (v_1^*(x), \dots, v_n^*(x))$ is a semi-valuation of K , with value group Γ^* , and s and s^* are equivalent.

By these properties we agree to call each v_i^* the *component* of s .

Conversely, we have

LEMMA 1. *If $v_1, \dots, v_n$ are incomparable valuations of a field K , and we set $\Gamma = \{(v_1(x), \dots, v_n(x)) | x \in K^*\}$, then Γ is an l -group, and the mapping $x \rightarrow s(x) = (v_1(x), \dots, v_n(x))$ of K^* onto Γ is a semi-valuation of K , of order n .*

PROOF. It was proved in [2] (Cf. Ch. IV) that Γ is an l -group, and s is a semi-valuation of K . We shall show that s is of order n . We set $\Omega_i = \{(\alpha_1, \dots, \alpha_n) \in \Gamma^+ | \alpha_i > 0\}$ for each i . Then it is clear that these Ω_i are different l -closed upper classes of Γ .

If there is an l -closed upper class Ω of Γ such that $\Omega \supsetneq \Omega_i$, then in Ω there exists an element $(\alpha_1, \dots, \alpha_n)$ with $\alpha_i = 0$. On the other hand, in Ω_i there exists an element $(\beta_1, \dots, \beta_n)$ such that $\beta_i > 0$ and $\beta_j = 0$ for $j \neq i$ (Cf. Corollary 1 to Lemma 10 in [2]). Hence $\inf\{(\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n)\} = (0, \dots, 0) \in \Omega$ and $\Omega = \Gamma^+$; therefore Ω_i is maximal.

Finally, if there is a maximal l -closed upper class Ω' of Γ besides $\Omega_1, \dots, \Omega_n$, then in Ω' there exist n elements $\alpha^i = (\alpha_1^i, \dots, \alpha_n^i)$ with $\alpha_i^i = 0$, for $i = 1, \dots, n$. Hence $\inf\{\alpha^1, \dots, \alpha^n\} = (0, \dots, 0) \in \Omega'$; and $\Omega' = \Gamma^+$ contrary to the assumption on Ω' . Therefore there is no maximal l -closed upper class of Γ besides $\Omega_1, \dots, \Omega_n$.

Thus the lemma is proved.

We remark that, in Lemma 1, each v_i and component v_i^* of s , associated with Ω_i , are equivalent. For let A_i be the isolated group associated with Ω_i . Then we have $A_i = \{(\alpha_1, \dots, \alpha_n) \in \Gamma | \alpha_i = 0\}$. For each i , Γ is preordered by prescribing $(\alpha_1, \dots, \alpha_n) \geq (\beta_1, \dots, \beta_n)$ if $\alpha_i \geq \beta_i$, and the mapping $\alpha_i \rightarrow (\beta_1, \dots, \beta_n)$ with $\beta_i = \alpha_i$ is an order preserving homomorphism of the value group of v_i onto Γ . It has A_i as the kernel. Hence the value group and Γ/A_i are isomorphic as ordered groups. Therefore v_i^* and v_i are equivalent.

2. Proof of the extension theorem.

In this section we shall prepare a few lemmas and give the proof of the extension theorem.

2.1. Extension of semi-valuations. Let k be a field and K be an overfield of k . If s' is a semi-valuation of K , the restriction s of s' to k is clearly a semi-valuation of k . The semi-valuation s' is said to be an *extension* of s .

We first have

LEMMA 2. *Let k be a subfield of a field K . Let s and s' be semi-valuations of k and K , with valuation rings R and R' , respectively. Then s' is an extension of s if and only if $R' \cap k = R$.*

Proof. The necessity condition is clear.

Assume now that $R' \cap k = R$, and let s_1 be the restriction of s' to k . Then we have $R = R_1$, where R_1 is the semi-valuation ring of s_1 . Hence s and s_1 are equivalent semi-valuations of k . Therefore $s = \varphi s_1$, where φ is an order preserving isomorphism of the value group of s_1 onto the value group of s . Extend φ to an order preserving isomorphism φ' of the value group of s' , and set $s'_1 = \varphi' s'$. Then s' and s'_1 are equivalent semi-valuations, and s'_1 is an extension of s . Therefore s' is an extension of s .

COROLLARY. *Let k be a subfield of a field K . Let s_1 and s_2 be semi-valuations of k . If s'_1 and s'_2 are extensions of s_1 and s_2 in K respectively, and if s_1 and s_2 are incomparable, then s'_1 and s'_2 are incomparable, and s'_i is not an extension of s_j with $j \neq i$ for $i, j = 1, 2$.*

This is an immediate consequence from Lemma 2 and definition of incomparability.

2.2. Relation between components. Let s' be a semi-valuation of the first kind of a field K , of order n . Let $v'_1, \dots, v'_n$ denote components of s' , with valuation rings $R'_1, \dots, R'_n$ respectively. For each i , let v_i be the restriction of v'_i to k , a subfield of K , with valuation ring R_i , and set $\Gamma = \{(v_1(x), \dots, v_n(x)) | x \in k^*\}$. Then it is clear that the mapping $s: x \rightarrow (v_1(x), \dots, v_n(x))$ of k^* onto Γ is the restriction of s' to k .

Now, we consider the set M of valuation rings $R_1, \dots, R_n$. We can determine uniquely the subset M_0 of M such that the following conditions are satisfied:

- (i) $R_{i'} \not\supseteq R_{j'}$, for every pair of $R_{i'}$ and $R_{j'}$ of M_0 , with $i' \neq j'$,
- (ii) R_i is an overring of some ring in M_0 , for each $R_i \in M - M_0$.

In fact, it follows from the finiteness of M that M can be inductively ordered by prescribing $R_i \supseteq R_j$ if $R_i \supseteq R_j$. Hence there exist minimal valuation rings

in M . Let M_0 be the set of all minimal valuation rings of M , then M_0 has the required properties.

Then we have

LEMMA 3. *In the above hypothesis, the order of s is equal to the number of elements of M_0 .*

PROOF. Suppose that M_0 consists of m valuation rings $R_1, \dots, R_m$, and set $\Gamma^* = \{(v_1(x), \dots, v_m(x)) | x \in k^*\}$. Then the mapping $s^*: x \rightarrow (v_1(x), \dots, v_m(x))$ of k^* onto Γ^* is a semi-valuation of k , of order m by Lemma 1, because $v_1, \dots, v_m$ are incomparable by the property (i). Next, s^* and s are equivalent by the remark in § 1.2, because two semi-valuation rings of s and s^* are equal by the above property (ii). Hence s is of order m . Therefore our assertion is proved.

As immediate consequence of Lemma 3, Corollary to Lemma 2 and the remark in § 1.3, we have

COROLLARY. *Let k be a subfield of a field K . Let s and s' be semi-valuations of the first kind of k and K , respectively, of the same order. Let M and M' be sets of all components of s and s' , respectively. Then s' is an extension of s , if and only if there is only one 1-1 correspondence between M and M' such that each valuation of M' is an extension of the corresponding valuation of M .*

2.3. **Proof of the extension theorem.** Let k be a field and K be an overfield of k . Let s be any semi-valuation of the first kind of k , of order n , and $v_1, \dots, v_n$ be components of s . Then each v_i can be extended to a valuation v'_i of K (Cf. Theorem 5', § 4, Ch. VI in [3]). Since $v_1, \dots, v_n$ are incomparable, $v'_1, \dots, v'_n$ are also incomparable by Corollary to Lemma 2. Hence $v'_1, \dots, v'_n$ compose a semi-valuation s' of K , of order n , by Lemma 1. Furthermore, s' is an extension of s by Lemma 2. Therefore, the first part of the extension theorem is proved.

Next, it follows from Corollary to Lemma 3 that any extension of s in K , of order n , must be obtained in the above manner. On the other hand, it is well known that, if K is a finite algebraic extension of k and v is a valuation of k , then the number of distinct extensions of v in K is not greater than the degree d of separability of K over k (Cf. Corollary 4 to Theorem

12, § 7, Ch. VI in [3]). Hence the number of distinct extensions of s in K , of the same order n , is not greater than d^n by Corollary to Lemma 3. Therefore the second part of the extension theorem is proved.

References

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