ON SOME RELATIONS BETWEEN BERNOULLI NUMBERS AND CLASS NUMBERS OF CYCLOTOMIC FIELDS

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1. Introduction.

Let p be an odd prime number and Q the field of rational numbers. We denote by $\zeta_{p^{n+1}}$ a fixed primitive p^{n+1} -th root of unity. The first factor of the class number of cyclotomic field $k_n = Q(\zeta_{p^{n+1}})$, the relative class number of k_n , is given by the formula as follows [1], [5].

$$h_n^- = 2p^{n+1} \prod_{\chi} \left(-\frac{1}{2p^{n+1}} \sum_m m \chi^{-1}(m) \right),$$

where m in the sum runs over all integers satisfying $0 \leq m < p^{n+1}$, $(m, p) = 1$ and χ runs over all odd characters of the reduced residue class group modulo p^{n+1} .

Let $-S(\chi)$ denote the factor having χ in the above product, then $h_n^- = 2p^{n+1} \prod_{\chi} (-S(\chi))$. If the conductor $\mathfrak{f}(\chi)$ of χ is equal to p^{l+1} , $0 \leq l \leq n$, it is easily seen that $S(\chi) = \frac{1}{2p^{l+1}} \sum_{0 \leq m_1 < p^{l+1}, (m_1, p) = 1} m_1 \chi^{-1}(m_1)$.

If p^{e_n} denotes the highest power of p dividing the class number h_n of k_n , there exist such rational integers λ, μ, ν that we have $e_n = \lambda n + \mu p^n + \nu$ for all sufficiently large numbers n . The numbers λ, μ, ν are called Iwasawa's invariants in the cyclotomic fields. Similarly there exist integers λ^-, μ^-, ν^- and $e_n^- = \lambda^- n + \mu^- p^n + \nu^-$ holds for all sufficiently large numbers n , where e_n^- denotes the exponent of the highest power of p dividing the first factor h_n^- [3], [4].

The several conditions for $\mu > 0$ were given in [3]. In this paper we regard $S(\chi)$ as a number in the local field $Q_p(\zeta_{p^n})$, where Q_p means the rational p -adic number field, and give an explicit expression of $S(\chi)$ modulo $\mathfrak{p}_n^2$, where $\mathfrak{p}_n$ means the prime ideal in $Q_p(\zeta_{p^n})$, by using the ordinary Bernoulli numbers and therefrom clarify a part of relations between class numbers and Bernoulli numbers.

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2. Congruences for Bernoulli numbers.

We define a linear difference operator Δ_{p-1} on a sequence f_n ($n \geq a$) of rational p -adic numbers by $\Delta_{p-1} f_n = f_{n+(p-1)} - f_n$. Also we define Bernoulli numbers B_m by

$$\frac{xe^x}{e^x - 1} = x + \frac{x}{e^x - 1} = 1 + \frac{x}{2} - \sum_{m=1}^{\infty} \frac{(-1)^m B_m}{(2m)!} x^{2m}.$$

We select in Q_p a fixed primitive $(p-1)$ -th root of unity ζ_{p-1} and a number $1+p$ as a base of the group of reduced residue classes modulo p^{n+1} . For rational integers x, y , we have then a canonical p -adic expansion

$$\zeta_{p-1}^x (1+p)^y = \sum_{i=0}^{\infty} t_i(x, y) p^i, \quad 0 \leq t_i(x, y) \leq p-1.$$

Set $S_n(x, y) = \sum_{i=0}^n t_i(x, y) p^i$. For an odd integer v , we obtain

$$S_n(x, y)^{v+1} \equiv \zeta_{p-1}^{(v+1)x} (1+p)^{(v+1)y} - (v+1) \zeta_{p-1}^{vx} (1+p)^{vy} \sum_{j=n+1}^{\infty} t_j(x, y) p^j \pmod{p^{2n+2}}.$$

Therefore if $v+1 \not\equiv 0 \pmod{p-1}$, then we see

$$\sum_{\substack{x \bmod p-1 \\ y \bmod p^n}} S_n(x, y)^{v+1} \equiv -(v+1) \sum_{r=n+1}^{\infty} \sum_{\substack{j+k=r \\ j \geq n+1}} \left\{ \sum_{x, y} \binom{vy}{k} t_j(x, y) \zeta_{p-1}^{vx} \right\} p^r \pmod{p^{2n+2}}.$$

As is well known, we have

$$\begin{aligned} \sum_{m=1}^{p^{n+1}} m^{v+1} &= \frac{1}{v+1} p^{(n+1)(v+2)} + \frac{1}{2} p^{(n+1)(v+1)} + \binom{v+1}{1} \frac{B_1}{2} p^{(n+1)v} - \binom{v+1}{3} \frac{B_2}{4} p^{(n+1)(v-2)} + \\ &+ \cdots + (-1)^{\frac{v+1}{2}-1} B_{\frac{v+1}{2}} p^{n+1}, \end{aligned}$$

from which follows by making use of the von Staudt-Clausen theorem

$$\sum_{m=1}^{p^{n+1}} m^2 \equiv B_1 p^{n+1} \pmod{p^{2n+2}} \quad \text{and} \quad \sum_{m=1}^{p^{n+1}} m^{v+1} \equiv (-1)^{\frac{v+1}{2}-1} B_{\frac{v+1}{2}} p^{n+1} \pmod{p^{3n+3-v}}$$

for $v \geq 3$.

Therefore we have

$$\sum_{\substack{1 \leq m \leq p^{n+1} \\ (m, p) = 1}} m^{v+1} = \sum_{m=1}^{p^{n+1}} m^{v+1} - p^{v+1} \sum_{m=1}^{p^n} m^{v+1} \equiv (-1)^{\frac{v+1}{2}+1} B_{\frac{v+1}{2}} p^{n+1} (1-p^v) \pmod{p^{2n+2-v}}.$$

If $n \geq l+v$, then we see

$$p^{n+1} \frac{(-1)^{\frac{v+1}{2}}}{v+1} B_{\frac{v+1}{2}} (1-p^v) \equiv \sum_{r=n+1}^{n+l+1} \sum_{\substack{j+k=r \\ j \geq n+1}} \left\{ \sum_{x,y} \binom{vy}{k} t_j(x, y) \zeta_{p-1}^{vx} \right\} p^r \pmod{p^{n+l+2}}.$$

By operating Δ_{p-1}^l ($\Delta_{p-1}^0 = 1$) with respect to v of this congruence, we see

$$p^{n+1} \Delta_{p-1}^l \frac{(-1)^{\frac{v+1}{2}}}{v+1} B_{\frac{v+1}{2}} (1-p^v) \equiv \sum_{r=n+1}^{n+l+1} \sum_{\substack{j+k=r \\ j \geq n+1}} \left\{ \sum_{x,y} t_j(x, y) \zeta_{p-1}^{vx} \Delta_{p-1}^l \binom{vy}{k} \right\} p^r \pmod{p^{n+l+2}},$$

for $n \geq l+v+(p-1)l = pl+v$.

Because we have $\Delta_{p-1}^l \binom{vy}{k} = 0$ for $0 \leq k \leq l-1$ and $\Delta_{p-1}^l \binom{vy}{l} = (p-1)^l y^l$ which are easily verified, we obtain the following

THEOREM 1. *For such an odd integer v that $v+1 \not\equiv 0 \pmod{p-1}$ and for $n \geq pl+v$, we have the congruence*

$$\sum_{\substack{x \bmod p-1 \\ y \bmod p^n}} y^l t_{n+1}(x, y) \zeta_{p-1}^{vx} \equiv \frac{(-1)^l}{p^l} \Delta_{p-1}^l \frac{(-1)^{\frac{v+1}{2}}}{v+1} B_{\frac{v+1}{2}} (1-p^v) \pmod{p}.$$

This theorem can be regarded as a generalization of Kummer's congruence which states $\Delta_{p-1}^l \frac{(-1)^{\frac{v+1}{2}}}{v+1} B_{\frac{v+1}{2}} \equiv 0 \pmod{p^l}$ if $v \geq 1$ and $v+1 \not\equiv 0 \pmod{p-1}$. Therefore we know that if $v \geq 1$, then $\frac{(-1)^{\frac{v+1+(p-1)}{2}}}{v+1+(p-1)} B_{\frac{v+1+(p-1)}{2}} \equiv \frac{(-1)^{\frac{v+1}{2}}}{v+1} B_{\frac{v+1}{2}} \pmod{p}$. Hence $\frac{(-1)^m}{2m} B_m$ is a p -adic integer for $2m \not\equiv 0 \pmod{p-1}$.

COROLLARY 1. *Under the conditions that $v+1 \not\equiv 0 \pmod{p-1}$, v odd, $v \geq l+1$ and $n \geq pl+v$, we have*

$$\sum_{\substack{x \bmod p-1 \\ y \bmod p^n}} y^l t_{n+1}(x, y) \zeta_{p-1}^{vx} \equiv \frac{(-1)^l}{p^l} \Delta_{p-1}^l \frac{(-1)^{\frac{v+1}{2}}}{v+1} B_{\frac{v+1}{2}} \pmod{p}.$$

3. $S(\chi) \bmod \mathfrak{p}_n^p$.

We regard χ as a residue class character of p -adic integers. We may assume that $\zeta_{p^{n+1}} = \zeta_{p^n}$ for any n . Let χ_1 be a character satisfying $\chi_1(\zeta_{p-1}) = \zeta_{p-1}$, $\chi_1(1+p) = 1$ and χ_2 a character satisfying $\chi_2(\zeta_{p-1}) = 1$, $\chi_2(1+p) = \zeta_{p^n}$, then we obtain the component decomposition of χ as follows:

$$\chi = \chi_1^{v_1} \chi_2^{v_2} \quad \text{with } (v_1, 2) = 1, \quad 1 \leq v_1 \leq p-2, \quad 0 \leq v_2 < p^n.$$

Furthermore we see

$$2S(\chi) = \frac{1}{p^{n+1}} \sum_{\substack{x \bmod p-1 \\ y \bmod p^n}} S_n(x, y) \zeta_{p-1}^{-v_1 x} \zeta_{p^n}^{-v_2 y}.$$

PROPOSITION 1. *The number $p \Pi' S(\chi)$, where χ runs over all characters with $v_1 = 1$, $\mathfrak{f}(\chi) = p^{n+1}$, is a unit in $\mathbb{Q}_p(\zeta_{p^n})$. If $v_1 \neq 1$, then*

$$2S(\chi) \equiv - \sum_{\substack{x \bmod p-1 \\ y \bmod p^n}} t_{n+1}(x, y) \zeta_{p-1}^{-v_1 x} \zeta_{p^n}^{-v_2 y} \pmod{p}.$$

PROOF. Because we have $S_{n+1}(x, y) \equiv \zeta_{p-1}^x (1+p)^y \pmod{p^{n+2}}$, it follows that

$$\begin{aligned} \sum_{\substack{x \bmod p-1 \\ 0 \leq y < p^n}} S_{n+1}(x, y) \zeta_{p-1}^{-v_1 x} \zeta_{p^n}^{-v_2 y} &\equiv \sum_{\substack{x \bmod p-1 \\ 0 \leq y < p^n}} \zeta_{p-1}^{x(1-v_1)} (1+p)^y \zeta_{p^n}^{-v_2 y} \pmod{p^{n+2}}, \\ &\equiv \begin{cases} 0 & \pmod{p^{n+2}} \text{ if } v_1 \neq 1, \\ (p-1) \frac{(1+p)^{p^n} - 1}{(1+p)\zeta_{p^n}^{-v_2} - 1} & \pmod{p^{n+2}} \text{ if } v_1 = 1. \end{cases} \end{aligned}$$

From this congruence with $v_1 = 1$ we conclude $2S(\chi) \equiv \frac{1}{p^{n+1}} \frac{(1+p)^{p^n} - 1}{(1+p)\zeta_{p^n}^{-v_2} - 1} (p-1) \pmod{1}$. Because we have $\prod_{(v_2, p)=1} ((1+p)\zeta_{p^n}^{-v_2} - 1) = (p)$, this implies $2^{p^{n-1}(p-1)} p \Pi' S(\chi) \equiv 1 \pmod{\mathfrak{p}_n}$. When $v_1 \neq 1$, we have easily the formula in our proposition.

For convenience sake set $B'_{p-v_1} = \frac{(-1)^{\frac{p-v_1}{2}}}{p-v_1} B_{\frac{p-v_1}{2}}$ and denote by $S(j, l)$ a

Stirling number of the first kind, then we obtain the following

THEOREM 2. *When $n \geq p^2 - 1$ and $v_1 \neq 1$, we set $2S(\chi) \equiv \sum_{j=0}^{p-1} A_j(\chi) \lambda_n^j \pmod{\mathfrak{p}_n^p}$, $\lambda_n = 1 - \zeta_{p^n}$. Then the coefficients $A_j(\chi)$ are given by the following congruences.*

$$A_j(\chi) \equiv (-1)^{j-1} \sum_{l=0}^j \frac{1}{j!} S(j, l) \frac{1}{p^l} (v_2 - p^n)^l \Delta_{p-1}^l B'_{p-v_1} (1 - p^{p-v_1-1}) \pmod{p}.$$

PROOF. When $v_1 \neq 1$, it follows from Proposition 1 in the field $\mathbb{Q}_p(\zeta_{p^n})$ that

$$2S(\chi) \equiv \sum_{j=0}^{n-1} (-1)^{j-1} \lambda_n^j \sum_{x,y} \binom{(p^n - v_2)y}{j} t_{n+1}(x, y) \zeta_{p-1}^{-v_1 x} \pmod{p_n^j}.$$

On the other hand,

$$\begin{aligned} \sum_{x,y} \binom{(p^n - v_2)y}{j} t_{n+1}(x, y) \zeta_{p-1}^{-v_1 x} &= \frac{1}{j!} \sum_{l=0}^j S(j, l) (p^n - v_2)^l \sum_{x,y} y^l t_{n+1}(x, y) \zeta_{p-1}^{-v_1 x}, \\ &\equiv \frac{1}{j!} \sum_{l=0}^j S(j, l) (v_2 - p^n)^l \frac{1}{p^l} \mathcal{A}_{p-1}^l B'_{p-v_1} (1 - p^{p-v_1-1}) \pmod{p}. \end{aligned}$$

Theorem 2 is equivalent to the following

THEOREM 2'. Assume that $n \geq p^2 - 1$ and $0 \leq j \leq p-1$. For a character χ with $v_1 \neq 1$ and $\mathfrak{f}(\chi) = p^{n+1}$, it is necessary and sufficient for $S(\chi) \equiv 0 \pmod{p_n^{j+1}}$ that we have

$$\mathcal{A}_{p-1}^l B'_{p-v_1} (1 - p^{p-v_1-1}) \equiv 0 \pmod{p^{l+1}} \quad \text{for all } l, 0 \leq l \leq j.$$

PROOF. $S(\chi) \equiv 0 \pmod{p_n^{j+1}}$ is equivalent to $A_l(\chi) \equiv 0 \pmod{p}$ for all l , $0 \leq l \leq j$. From the orthogonality relations of Stirling numbers $\sum_t S(l, t) S(t, m) = \delta_{l,m}$, follows

$$\sum_{t=0}^l S(l, t) t! (-1)^{l-t} A_t(\chi) \equiv (p^n - v_2)^l \frac{(-1)^l}{p^l} \mathcal{A}_{p-1}^l B'_{p-v_1} (1 - p^{p-v_1-1}) \pmod{p}.$$

Therefore our assertion is valid for $(v_2, p) = 1$ and $j \leq p-1$.

Theorem 2' gives an arithmetic meaning of Kummer's congruences because we have from Theorem 1 for $v_1 \neq 1$, $v+1 \not\equiv 0 \pmod{p-1}$, that

$$\begin{aligned} \frac{(-1)^l}{p^l} \mathcal{A}_{p-1}^l B'_{p-v_1} (1 - p^{p-v_1-1}) &\equiv \frac{(-1)^m}{p^m} \mathcal{A}_{p-1}^m B'_{p-v_1} (1 - p^{p-v_1-1}) \pmod{p} \\ &\quad \text{if } l \equiv m \pmod{p-1}, \end{aligned}$$

and

$$\begin{aligned} \frac{(-1)^l}{p^l} \mathcal{A}_{p-1}^l B'_{v+1} (1 - p^v) &\equiv \frac{(-1)^l}{p^l} \mathcal{A}_{p-1}^l B'_{w+1} (1 - p^w) \pmod{p} \\ &\quad \text{if } v \equiv w \pmod{p-1}. \end{aligned}$$

REMARK. Theorem 2' is concerned only with irregular prime numbers. The condition for $l=0$ is equal to $B'_{p-v_1} \equiv 0 \pmod{p}$. Furthermore if $n=0$, we see easily by a similar method as before that $2S(\chi) \equiv -B'_{p-v_1} \pmod{p}$ for $v_1 \neq 1$ [1].

We can also see that $v_1=5$ for $p=37$, $v_1=15$ for $p=59$, $v_1=9$ for $p=67$

satisfy the condition for $l=0$ in Theorem 2' [7].

Because we assume for v_1 in Theorem 2 that $3 \leq v_1 \leq p-2$, the conditions in Theorem 2' are equivalent with that

$$\Delta_{p-1}^l B'_{p-v_1} \equiv (-1)^l B'_{p-v_1} p^{p-v_1-1} \pmod{p^{l+1}} \quad \text{for } l, 0 \leq l \leq j.$$

Now the assumptions being the same as Theorem 2', $S(\chi) \equiv 0 \pmod{p^{j+1}}$ is equivalent to $\Pi' S(\chi) \equiv 0 \pmod{p^{j+1}}$, where χ runs over all characters with fixed v_1 and $\mathfrak{f}(\chi) = p^{n+1}$, and so we conclude the following corollaries.

COROLLARY 1. *Let $p^{e_n^-}$ denote the highest power of p dividing h_n^-/h_{n-1}^- $= p \Pi' S(\chi)$. If we have $E_n^- < \frac{p-3}{2}p$ for a number n ($n \geq p^2-1$), there exist an odd integer v , $3 \leq v \leq p-2$, and an integer l , $0 \leq l \leq p-1$ so that it holds*

$$\Delta_{p-1}^l B'_{p-v}(1-p^{p-v-1}) \equiv 0 \pmod{p^{l+1}}.$$

Particularly we see

COROLLARY 2. *Let p^{e_n} have the same meaning as in the introduction. If $e_n < \frac{p-3}{2}p$ holds for a number n ($n \geq p^2-1$), the same conclusion as Corollary 1 is valid.*

Furthermore because we have $\lambda^- \geq \lambda^+$, $\mu^- \geq \mu^+$, $\lambda = \lambda^- + \lambda^+$, $\mu = \mu^- + \mu^+$, the fact that $\lambda = \mu = 0$ is equivalent to that p is a regular prime, as is easily seen from Theorem 2'.

4. A condition for $S(\chi) \equiv 0 \pmod{p}$.

The following lemma is due to Iwasawa [3] and easily verified.

LEMMA. *If we have $\sum_{i \bmod p^n} a_i \zeta_{p^n}^i \equiv 0 \pmod{p}$ with all p -adic integers a_i , then we conclude $\sum_{i \equiv \omega \pmod{p^{n-1}}} a_i \equiv 0 \pmod{p}$ for arbitrary p -adic integer ω .*

Using this lemma we can prove the following

THEOREM 3. *Assume that $S(\chi) \equiv 0 \pmod{p}$ for $v_1 \neq 1$ and $\mathfrak{f}(\chi) = p^{n+1}$. Then we have $S(\chi) \equiv 0 \pmod{p}$ for any such odd character χ that belongs to the same v_1 and has $\mathfrak{f}(\chi)$ dividing p^{n+1} .*

PROOF. The assumption is equal to $\sum_{x,y} S_n(x,y) \zeta_{p-1}^{-v_1 x} \zeta_{p^n}^{-v_2 y} \equiv 0 \pmod{p^{n+2}}$. Because we have $\sum_{x \bmod p-1} S_n(x,y) \zeta_{p-1}^{-v_1 x} \equiv \sum_{x \bmod p-1} \zeta_{p-1}^{(1-v_1)x} \zeta_{p^n}^{-v_2 x^2} \equiv 0 \pmod{p^{n+1}}$, it follows from Lemma that,

$$\sum_{\substack{x \bmod p-1 \\ y \equiv \omega(p^{n-1})}} S_n(x, y) \zeta_{p-1}^{-v_1 x} \equiv 0 \pmod{p^{n+2}} \quad \text{for any integer } \omega.$$

Now set $S_n(x, y) = S_{n-1}(x, \omega) + jp^n$, $0 \leq j \leq p-1$, then j is determined by the congruence $j \equiv t_n(x, \omega) + \zeta_{p-1}^x \eta \pmod{p}$. Herein η means an integer satisfying $y \equiv \omega + \eta p^{n-1} \pmod{p}$, $0 \leq \eta \leq p-1$. When η runs over all integers from 0 to $p-1$, j also runs over from 0 to $p-1$. Therefore the above condition is equal to that

$$p \sum_{x \bmod p-1} S_{n-1}(x, \omega) \zeta_{p-1}^{-v_1 x} + \sum_{x \bmod p-1} \sum_j jp^n \zeta_{p-1}^{-v_1 x} \equiv 0 \pmod{p^{n+2}},$$

that is,

$$\sum_{x \bmod p-1} S_{n-1}(x, \omega) \zeta_{p-1}^{-v_1 x} \equiv 0 \pmod{p^{n+1}}.$$

Thus in particular we have

$$\sum_{x, y} S_{n-1}(x, y) \zeta_{p-1}^{-v_1 x} \zeta_{p^{n-1}}^{-v_2 y} \equiv 0 \pmod{p^{n+1}} \quad \text{for any } v_2.$$

This completes the proof of Theorem 3, since in the case of $\mathfrak{f}(\chi) = p^{n+1}$ our assertion holds trivially.

Next we give a condition for $\mu > 0$, which is equivalent to that there exists some odd integer $v_1 \neq 1$ such that $S(\chi) \equiv 0 \pmod{p}$ holds for all sufficiently large numbers n .

THEOREM 4. *In order that $S(\chi) \equiv 0 \pmod{p}$ for all $n \geq n_0$, where all χ have a fixed $v_1 \neq 1$, and n_0 means a fixed number, it is necessary and sufficient that we have $\sum_{x \bmod p-1} S_n(x, y)^{p-v_1} \equiv 0 \pmod{p^{n+1}}$ for all $n \geq 0$ and for any integer y .*

PROOF. $S(\chi) \equiv 0 \pmod{p}$ for all $n \geq n_0$, $\chi = \chi_1^{v_1} \chi_2^{v_2}$, $v_1 \neq 1$, yields, as can be seen in the proof of Theorem 3, that $\sum_{x \bmod p-1} S_n(x, y) \zeta_{p-1}^{-v_1 x} \equiv 0 \pmod{p^{n+2}}$ holds for all $n \geq 0$ and for any y . On the other hand, by the same method as before, we have

$$\sum_{x \bmod p-1} t_{n+1}(x, y) \zeta_{p-1}^{-v_1 x} \equiv -\frac{1}{p^{n+1}} \frac{1}{p-v_1} \sum_{x \bmod p-1} S_n(x, y)^{p-v_1} \pmod{p}.$$

Therefore the above statement is equivalent to that we have $\sum_{x \bmod p-1} S_n(x, y)^{p-v_1} \equiv 0 \pmod{p^{n+2}}$ for any integer y .

Now we give a short remark to Iwasawa's result [3] which is stated as follows: It is necessary and sufficient for $\mu > 0$ that there exists an odd

integer v , $3 \leq v \leq p-2$, so that we have

$$\sum_{\gamma} \gamma^v / (1 - (1-x)^\gamma) \equiv 0 \pmod{p},$$

where γ means a $(p-1)$ -th root of unity and x denotes an indeterminate.

Set $\phi_\gamma(x) = \frac{\gamma^v}{1 - (1-x)^\gamma} - \frac{\gamma^{v-1}}{x}$ and then it follows easily that

$$\phi_\gamma(1-e^t) = -\gamma^{v-1} \sum_{m=1}^{\infty} (-1)^m B_m (\gamma^{2m} - 1) \frac{t^{2m-1}}{(2m)!}.$$

Therefore we have

$$\sum_{\gamma} \phi_\gamma(x) = - \sum_{2m+v-1 \equiv 0 \pmod{p-1}} (-1)^m (p-1) B_m \frac{t^{2m-1}}{(2m)!}.$$

Because we have $\frac{(\log(1-x))^i}{i!} = \sum_{j=i}^{\infty} \frac{(-1)^j}{j!} S(j, i) x^j$, we obtain

$$\sum_{\gamma} \phi_\gamma(x) = \sum_{j=p-v-1}^{\infty} \frac{(-1)^j}{j!} (1-p) x^j \sum_{i=0}^{\infty} (1 + \Delta_{p-1})^i B'_{p-v} S(j, p-v-1).$$

Thus we obtain the following

THEOREM 5. *It is necessary and sufficient for $\mu > 0$ that there exists an odd integer v , $3 \leq v \leq p-2$ such that we have*

$$\Delta_{p-1}^{-1} \frac{1}{j!} S(j, p-v-1) B'_{p-v} \equiv 0 \pmod{p},$$

where Δ_{p-1}^{-1} means the linear operator defined by $\Delta_{p-1}^{-1} = \sum_{i=0}^{\infty} (1 + \Delta_{p-1})^i$.

Furthermore by regarding the fact mentioned above in Theorem 4, and by using Theorem 2', we obtain the following proposition.

PROPOSITION 2. *In order that $\mu > 0$, it is necessary that there exists an odd integer v , $3 \leq v \leq p-2$, so that it holds*

$$\Delta_{p-1}^l B'_{p-v} (1 - p^{p-v-1}) \equiv 0 \pmod{p^{l+1}} \quad \text{for all } l,$$

where l means an arbitrary non-negative integer.

This proposition can be also obtained from $\sum_{\gamma} \phi_\gamma(x)$ by making use of classical Kummer's congruences [6].

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