SUPPLEMENT TO THE PAPER "SEMI-SIMPLICITY IN INFINITE-DIMENSIONAL LIE GROUPS"

By

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In the previous paper [3], we defined the notions of simplicity and semi-simplicity of infinite-dimensional Lie groups G by means of their Banach Lie algebras $\mathfrak{g}$. As is shown in [1, 2], there is a one-one correspondence between the set of all *closed* Lie subalgebras of $\mathfrak{g}$ and the set of all Lie subgroups of G . Therefore, if we speak of Lie subalgebras of a Banach Lie algebra $\mathfrak{g}$ of an infinite-dimensional Lie group G , it is quite natural for us to impose on them the additional condition that they are *closed* subspaces of $\mathfrak{g}$. This additional condition gives rise to some differences between the finite-dimensional case and the infinite-dimensional one. The cause is based on the fact that for a closed subspace $\mathfrak{g}_1$ of a Banach space $\mathfrak{g}$ there doesn't always exist a closed subspace $\mathfrak{g}_2$ such that $\mathfrak{g}$ is a direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ and a sum (as vector spaces) of two closed subspaces of $\mathfrak{g}$ is not necessarily closed.

In [3], simplicity and semi-simplicity of $\mathfrak{g}$ are defined as follows:

$\mathfrak{g}$ is said to be *simple* if closed ideals of $\mathfrak{g}$ are only (0) and $\mathfrak{g}$. $\mathfrak{g}$ is said to be *semi-simple* if it is *non-abelian* (in [3], this condition is missing) and for any closed ideal $\mathfrak{g}_1$ of $\mathfrak{g}$ there exists a closed ideal $\mathfrak{g}_2$ such that $\mathfrak{g}$ is a topological direct sum of them. G is said to be *simple* or *semi-simple* if its Lie algebra $\mathfrak{g}$ is so.

In the above definition, a topological direct sum means that $\mathfrak{g}$ is (homeomorphically) isomorphic to a direct product Banach Lie algebra of $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

In case of finite dimension, simplicity defined above coincides with the usual one and semi-simplicity includes the usual one. Although the notion of semi-simplicity defined above is larger than the usual one, it is too restrictive in case of infinite dimension on account of the fact mentioned before. For example, it is not necessarily evident that a product of two simple Lie groups of infinite dimension is semi-simple. It is at least desirable that a finite direct product of simple Lie groups is semi-simple. So we shall define again the

semi-simplicity of infinite-dimensional Lie groups and add some remarks.

Let I be an arbitrary set of indices ν and suppose a Banach space $\mathfrak{g}_\nu$ is assigned to each index ν . If we assign an element x_ν of the space $\mathfrak{g}_\nu$ to each index ν , we obtain the complex $x = \{x_\nu\}$. We consider the set $\mathfrak{g}$ of all such complexes $x = \{x_\nu\}$ satisfying the following conditions:

1° $\{x_\nu\}$ contains at most a countable number of elements which are different from zero,

2° the series $\sum_\nu \|x_\nu\|$ converges.

For any element $x = \{x_\nu\}$ of $\mathfrak{g}$, we define the norm of x by

$$\|x\| = \sum_\nu \|x_\nu\|.$$

With this definition of the norm, $\mathfrak{g}$ forms a Banach space which is called the (*topological*) *direct sum* of the space $\mathfrak{g}_\nu$ and denoted by

$$\mathfrak{g} \equiv \sum_{\nu \in I} \oplus \mathfrak{g}_\nu.$$

In particular, if each $\mathfrak{g}_\nu$ is a Banach Lie algebra, we define the bracket operation $[x, y]$ of $x = \{x_\nu\}$ and $y = \{y_\nu\}$ in $\mathfrak{g}$ by

$$[x, y] = \{[x_\nu, y_\nu]\}.$$

Note that $[x, y]$ doesn't necessarily belong to $\mathfrak{g} = \sum_\nu \oplus \mathfrak{g}_\nu$ because the series $\sum_\nu \|[x_\nu, y_\nu]\|$ may be divergent.

DEFINITION. If $[x, y]$ belongs to $\mathfrak{g} = \sum_\nu \oplus \mathfrak{g}_\nu$ for any elements x and y of $\mathfrak{g}$ and the bracket operation is a continuous mapping of $\mathfrak{g} \times \mathfrak{g}$ into $\mathfrak{g}$, then we call $\mathfrak{g}$ a *direct product Banach Lie algebra* of $\mathfrak{g}_\nu$.

Examples of direct product Banach Lie algebras:

1° Let $\mathfrak{h}$ be an arbitrary Banach Lie algebra. Put $\mathfrak{g}_\nu = \mathfrak{h}$ for all indices ν . Since $\mathfrak{h}$ is a Banach Lie algebra, there exists a constant K such that $\|[x, y]\| \leq K \cdot \|x\| \cdot \|y\|$ for any elements x and y of $\mathfrak{h}$ (continuity of the bracket operation). Then, for any elements $x = \{x_\nu\}$ and $y = \{y_\nu\}$ of $\mathfrak{g} = \sum_\nu \oplus \mathfrak{g}_\nu$, we have

$$\begin{aligned} \|[x, y]\| &= \sum_\nu \|[x_\nu, y_\nu]\| \leq \sum_\nu K \cdot \|x_\nu\| \cdot \|y_\nu\| \leq K \left(\sum_\nu \|x_\nu\| \right) \left(\sum_\nu \|y_\nu\| \right) \\ &= K \cdot \|x\| \cdot \|y\|. \end{aligned}$$

Therefore $[x, y]$ belongs to $\mathfrak{g}$ and the bracket operation is continuous. So $\mathfrak{g}$ is a direct product Banach Lie algebra.

2° If $\mathfrak{g}_\nu$ is a Banach Lie algebra, then there exists a constant $K_{\mathfrak{g}_\nu}$ such that $\|[x_\nu, y_\nu]\| \leq K_{\mathfrak{g}_\nu} \cdot \|x_\nu\| \cdot \|y_\nu\|$ holds for any elements x_ν and y_ν of $\mathfrak{g}_\nu$. Then we have

$$\|[x, y]\| = \sum_\nu \|[x_\nu, y_\nu]\| \leq \sum_\nu K_{\mathfrak{g}_\nu} \cdot \|x_\nu\| \cdot \|y_\nu\| \leq +\infty.$$

By the suitable choice of the norm in $\mathfrak{g}_\nu$ (for example, by multiplying a suitable constant), we may assume $K_{\mathfrak{g}_\nu}$ is equal to 1, that is, $\mathfrak{g}_\nu$ is isomorphic to a Banach Lie algebra $\mathfrak{g}'_\nu$ with $K_{\mathfrak{g}'_\nu}=1$. Then we have $\|[x', y']\| \leq \|x'\| \cdot \|y'\|$ for any elements $x'=\{x'_\nu\}$ and $y'=\{y'_\nu\}$ of $\mathfrak{g}'=\sum_\nu \oplus \mathfrak{g}'_\nu$. So $\mathfrak{g}'$ is a direct product Banach Lie algebra. It follows from this example that though we can not always form a direct product Banach Lie algebra $\mathfrak{g}$ from an arbitrarily given family $\{\mathfrak{g}_\nu\}$, *there exists always a direct product Banach Lie algebra whose direct factors are isomorphic to the given ones, $\{\mathfrak{g}_\nu\}$.*

Now we shall define again the semi-simplicity of infinite-dimensional Lie groups which coincides with the usual one in case of finite dimension.

DEFINITION. *A Banach Lie algebra $\mathfrak{g}$ is said to be semi-simple if it is (homeomorphically) isomorphic to a direct product Banach Lie algebra whose direct factors are all simple Lie algebras. G is said to be semi-simple if its Lie algebra $\mathfrak{g}$ is semi-simple.*

In the above definition and the foregoing one, the requirement that a direct sum must be topological is based on the fact that G is locally decomposed into a finite direct product if and only if its Lie algebra $\mathfrak{g}$ is isomorphic to a direct product Banach Lie algebra with finite direct factors and it is desirable that a decomposition of $\mathfrak{g}$ reflects a (local) decomposition of G and *vice versa*.

In [4, 5], we showed the existence of simple Lie algebras which are not of finite dimension. So there exists a semi-simple Banach Lie algebra whose direct factors are not of finite dimension (cf. example 1°). As is shown in [1, 2], there corresponds to each Banach Lie algebra $\mathfrak{g}$ a local Lie group whose Lie algebra is $\mathfrak{g}$. So there exists always a local semi-simple Lie group for each semi-simple Banach Lie algebra. By definition, a finite direct product of simple Lie groups is semi-simple.

By the same argument used in [3], we can prove the following

THEOREM. *Let G be a simply connected Lie group whose Lie algebra is a direct product Banach Lie algebra with finite direct factors $\mathfrak{g}_i$ ($i=1, 2, \dots$),*

n). Then G is decomposed into the direct product $G=G_1 \times G_2 \times \dots \times G_n$, where G_i is the Lie subgroup of G corresponding to $\mathfrak{g}_i$. In particular, if G is a simply connected semi-simple Lie group whose Lie algebra has finite simple direct factors, then G is decomposed into a direct product of simple Lie subgroups.

In case of finite-dimensional semi-simple Lie algebras $\mathfrak{g}$, it is well known that for any ideal $\mathfrak{g}_1$ of $\mathfrak{g}$, there exists an ideal $\mathfrak{g}_2$ such that $\mathfrak{g}$ is isomorphic to the direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. In [3], we adopted this property as a definition of semi-simplicity in infinite-dimensional Lie groups. If we define the semi-simplicity of $\mathfrak{g}$ as described in this paper, which coincides with the usual one in case of finite dimension, the above property doesn't necessarily hold contrary to the finite-dimensional case (even if a number of simple direct factors of $\mathfrak{g}$ is finite).

References.

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