NOTE ON THE CORRESPONDENCE BETWEEN MEASURES AND ADDITIVE FUNCTIONALS OF MULTI- DIMENSIONAL WIENER PROCESS

By

Toshio YAMADA

(Received June 30, 1965)

§ 1. Introduction.

The purpose of this paper is to give a short note on the continuity of the correspondence between measures and positive continuous additive functionals of N -dimensional Wiener process ($N \geq 3$). § 2 is devoted to definitions and notations. In § 3, we shall discuss the continuity of the correspondence. In § 4, we shall give a few pathological examples.

The author wishes to express his hearty thanks to Prof. H. Tanaka and Prof. S. Watanabe for their kind suggestions and discussions.

§ 2. Definitions and notations.

Let $R^N (N \geq 3)$ be N -dimensional Euclidean space, and $\mathfrak{B}(R^N)$ its topological Borel field. The path space Ω is the set of all continuous mappings ω from $[0, \infty)$ into R^N . The value of ω at t is denoted by $\omega(t) = x(t, \omega) = x_t$. The shifted path $\omega_t^+(\cdot) = \omega(\cdot + t)$. Let $\mathfrak{B}_t$ be the Borel field generated by the cylinder sets $\{\omega; x(s, \omega) \in I\}$ ($s \leq t$, $I \in \mathfrak{B}(R^N)$), and $\mathfrak{B} = \bigvee_{t \geq 0} \mathfrak{B}_t$. Let $P_x(B)$ ($x \in R^N$, $B \in \mathfrak{B}$) be the probability measure (Wiener measure) determined by the following transition function; $P(t, x, I) = \int_I (2\pi t)^{-\frac{N}{2}} e^{-\frac{|y-x|^2}{2t}} dy$ ($t \geq 0$, $x \in R^N$, $I \in \mathfrak{B}(R^N)$). The system $X = (x_t, \Omega, \mathfrak{B}_t, P_x)$ is called an N -dimensional Wiener process. Let $\bar{\mathfrak{B}}_t = \bigcap_{x \in R^N} \{\text{completion of } \mathfrak{B}_t \text{ w.r.t. } P_x\}$ and $\bar{\mathfrak{B}} = \bigvee \bar{\mathfrak{B}}_t$, then each $P_x(\cdot)$ is extended to a probability measure (still we use the same notation) over $\bar{\mathfrak{B}}$.

We use the notation $E_x[f; B]$ for the integral over $B \in \bar{\mathfrak{B}}$ of a $\bar{\mathfrak{B}}$ -measurable function f w.r.t. P_x , and write $E_x[f; B] = E_x[f]$ when B is a set of full measure.

We will adopt several terminologies of Dynkin [1]. A functional $\varphi = \varphi_t^s(\omega)$ ($\omega \in \Omega$, $0 \leq s \leq t < \infty$), taking real values, is called an S -functional of the Wiener

process X , if there exists a full set $\tilde{\mathcal{Q}} \in \bar{\mathfrak{B}}$ ($P_x(\tilde{\mathcal{Q}})=1$, $x \in R^N$) with the following properties (A.1)~(A.5):

$$(A.1) \quad \{\omega; \varphi_i^s(\omega) \in \Gamma\} \in \bar{\mathfrak{B}}_t \quad (0 \leq s \leq t, \Gamma \in \mathfrak{B}(R^N))$$

$$(A.2) \quad \omega \in \tilde{\mathcal{Q}}, \quad 0 \leq \varphi_i^s(\omega) < \infty \quad (0 \leq s \leq t < \infty)$$

$$(A.3) \quad \omega \in \tilde{\mathcal{Q}}, \quad \varphi_i^s(\omega) = \varphi_u^s(\omega) + \varphi_l^u(\omega) \quad (0 \leq s \leq u \leq t < \infty)$$

$$(A.4) \quad \omega \in \tilde{\mathcal{Q}}, \quad \theta_h \varphi_i^s(\omega) = \varphi_{t+h}^{s+h}(\omega) \quad (0 \leq s \leq t < \infty, 0 \leq h < \infty)$$

where $\theta_h \varphi$ is defined by $\theta_h \varphi_i^s(\omega) = \varphi_i^s(\omega_h^+)$.

$$(A.5) \quad \omega \in \tilde{\mathcal{Q}}, \quad \varphi_i^s(\omega) \text{ is continuous in } t \quad (0 \leq s \leq t < \infty).$$

An S -functional is called a W -functional if, in addition, it satisfies

$$(A.6) \quad \sup_{x \in R^N} E_x(\varphi_t^0) < \infty \text{ for every fixed } t \quad (0 \leq t < \infty).$$

We call a measure μ on $(R^N, \mathfrak{B}(R^N))$ a W -measure, if $\sup_{x \in R^N} \int \frac{\mu(dy)}{|x-y|^{N-2}} < \infty$.

We call a measure μ on $(R^N, \mathfrak{B}(R^N))$ an S -measure, if there exists a sequence of compact sets $\Gamma_n \uparrow R^N$ such that,

$$(S.1) \quad \text{Each measure } \mu_{\Gamma_n}(\cdot) = \mu(\cdot \cap \Gamma_n) \text{ is an } W\text{-measure.}$$

$$(S.2) \quad \tau_{\Gamma_n}(\omega) \uparrow \infty \text{ (a.s.) w.r.t. each } P_x,$$

where $\tau_{\Gamma_n}(\omega) = \inf\{t; x_t(\omega) \in \Gamma_n\}$.

For multi-dimensional case, the correspondence between measures and additive functionals was investigated by H. P. McKean- H. Tanaka, A. D. Wentzel, and by E. B. Dynkin. In the following of this article, we adopt the correspondence established in theorem 8.4 and 8.5 of Dynkin's book [1].

We denote by $\varphi(\mu)$ the S -functional corresponding to a given S -measure μ , and introduce the notation $\varphi_\infty^0(\mu) = \lim_{t \rightarrow \infty} \varphi_t^0(\mu)$, then, if μ is a W -measure, we have

$$E_x[\varphi_\infty^0(\mu)] = \int_{R^N} U(x, y) \mu(dy) = U^\mu(x), \quad (2.1)$$

$$\text{where} \quad U(x, y) = \frac{1}{2\pi^{\frac{N}{2}}} \Gamma\left(\frac{N}{2} - 1\right) \frac{1}{|x-y|^{N-2}}. \quad (2.2)$$

Given a sequence $\{\mu^n, n=1, 2, \dots\}$ of measures on $(R^N, \mathfrak{B}(R^N))$, we say that (i) $\{\mu^n\}$ converges weakly to μ , if for every continuous f with compact support $\lim_{n \rightarrow \infty} \int_{R^N} f(x) \mu^n(dx) = \int_{R^N} f(x) \mu(dx)$; (ii) $\{\mu^n\}$ converges strictly to μ , if $\lim_{n \rightarrow \infty} \|\mu^n - \mu\| = 0$, where $\|\cdot\|$ is the energy of a signed measure. [Refer to [2]]

§ 3. Continuity of the correspondence between measures and additive functionals.

Condition (3.A); We say that a sequence of measures $\{\mu^n\}$ satisfies (3.A) at x ,

if, for any $\varepsilon > 0$ there exists $\delta > 0$ independent of n such that
$$\int_{|x-y| < \delta} \frac{\mu^n(dy)}{|x-y|^{N-2}} < \varepsilon.$$

Condition (3·B); We say that a sequence of measures $\{\mu^n\}$ satisfies (3·B) on compact set K , if, for any $\varepsilon > 0$ there exists $\delta > 0$ independent of n and of $x \in K$ such that

$$\int_{|x-y| < \delta} \frac{\mu^n(dy)}{|x-y|^{N-2}} < \varepsilon.$$

PROPOSITION 1; Let $\{\mu^n\}$ be a sequence of W -measures, supported by a compact set F , and suppose the corresponding potentials U^{μ^n} are uniformly bounded. If μ^n converges strictly to some W -measure μ , then, $\lim_{n \rightarrow \infty} E_x[\varphi_i^0(\mu^n) - \varphi_i^0(\mu)]^2 = 0$, for each x at which (3·A) is satisfied.

PROPOSITION 2; Let $\{\mu^n\}$ be a sequence of W -measures, and suppose there exists a W -measure ν , supported by a compact set F , such that $\mu^n(E) \leq \nu(E)$ for every $E \in \mathfrak{B}(R^N)$. If μ^n converges weakly to some W -measures μ , then, $\lim_{n \rightarrow \infty} E_x[\varphi_i^0(\mu^n) - \varphi_i^0(\mu)]^2 = 0$ ($x \in R^N$).

PROPOSITION 3; Let $\{\mu^n\}$ be a sequence of W -measures, supported by a compact set F , and suppose that the condition (3·B) is satisfied on F . If μ^n converges weakly to some W -measure μ , then $\lim_{n \rightarrow \infty} E_x[\varphi_i^0(\mu^n) - \varphi_i^0(\mu)]^2 = 0$ ($x \in R^N$).

For the proof of these propositions, we state the following simple facts (1), (2), and (3) and Lemma 1.

(1); Let $\{\mu^n\}$ be a sequence of measures on $(R^N, \mathfrak{B}(R^N))$ and μ^n converges weakly to some W -measure μ with compact support. Then, the condition (3·A) is satisfied at x if and only if $\lim_{n \rightarrow \infty} U^{\mu^n}(x) = U^\mu(x)$.

(2); Let $\{\mu^n\}$ be a sequence of measures on $(R^N, \mathfrak{B}(R^N))$, supported by a compact set F , and $\{\mu^n(F), n=1, 2, \dots\}$ is bounded. If the condition (3·B) is satisfied on K , then the potentials $U^{\mu^n}(x)$ are equicontinuous on K .

(3); Let $\{\mu^n\}$ be a sequence of measures on $(R^N, \mathfrak{B}(R^N))$, supported by a compact set F . Suppose μ^n converges weakly to some W -measure μ , and $\lim_{n \rightarrow \infty} U^{\mu^n}(x) = U^\mu(x)$ on a compact set K . If the potentials $U^{\mu^n}(x)$ are equicontinuous on K , then the condition (3·B) is satisfied on K .

Lemma 1 (J. Deny's theorem [3]); Let $\{\mu^n\}$ be a sequence of measures on $(R^N, \mathfrak{B}(R^N))$. If μ^n converges weakly to μ , then there exists a subsequence $\{\mu^{n_k}\} \subset \{\mu^n\}$ such that the potentials $U^{\mu^{n_k}}(x)$ converge to $U^\mu(x)$, except on a set of Lebesgue measure 0.

(Proof of Proposition 1)

We have

$$\begin{aligned} E_x[\varphi_\infty^0(\mu^n) - \varphi_\infty^0(\mu)]^2 &= 2 \int_{R^N} U(x, y)(U^{\mu^n}(y) - U^\mu(y))\mu^n(dy) \\ &\quad + 2 \int_{R^N} U(x, y)(U^\mu(y) - U^{\mu^n}(y))\mu(dy) \end{aligned} \quad (3.1)$$

From condition (3.A) at x and uniform boundedness of potentials $U^{\mu^n}(x)$ and $U^\mu(x)$, it follows immediately that, for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\begin{aligned} \int_{R^N} U(x, y)(U^{\mu^n}(y) - U^\mu(y))\mu^n(dy) \\ = \int_{R^N} U_\delta(x, y)(U^{\mu^n}(y) - U^\mu(y))\mu^n(dy) + \eta_n(\varepsilon), \end{aligned} \quad (3.2)$$

where $|\eta_n(\varepsilon)| < \varepsilon$, and $U_\delta(x, y) = \min\left(\frac{1}{\delta N - 2}, U(x, y)\right)$.

Introduce $\nu^n(dy) = U_\delta(x, y)\mu^n(dy)$. The energies $\|\nu^n\|$ are bounded. We have

$$\begin{aligned} \left| \int_{R^N} U_\delta(x, y)(U^{\mu^n}(y) - U^\mu(y))\mu^n(dy) \right| \\ = \left| \int_{R^N} (U^{\mu^n}(y) - U^\mu(y))\nu^n(dy) \right| \leq C\|\mu^n - \mu\|\|\nu^n\|, \end{aligned} \quad (3.3)$$

where C is a positive constant.

Therefore, $\lim_{n \rightarrow \infty} \int_{R^N} U(x, y)(U^{\mu^n}(y) - U^\mu(y))\mu^n(dy) \leq \varepsilon$.

We can easily prove that the second term of the right side in (3.1) tends to zero as $n \uparrow \infty$.

Consequently $\lim_{n \rightarrow \infty} E_x[\varphi_\infty^0(\mu^n) - \varphi_\infty^0(\mu)]^2 = 0$.

Next, we shall prove $\lim_{n \rightarrow \infty} E_x[\varphi_i^0(\mu^n) - \varphi_i^0(\mu)]^2 = 0$ for every $0 \leq i < \infty$.

$$\begin{aligned} \text{We have } E_x[\varphi_\infty^0(\mu^n) - \varphi_\infty^0(\mu)]^2 &= E_x[\varphi_i^0(\mu^n) - \varphi_i^0(\mu)]^2 \\ &\quad + 2E_x[\varphi_i^0(\mu^n) - \varphi_i^0(\mu)]E_{x_i}[\varphi_\infty^0(\mu^n) - \varphi_\infty^0(\mu)] \\ &\quad + E_x[E_{x_i}[\varphi_\infty^0(\mu^n) - \varphi_\infty^0(\mu)]^2]. \end{aligned} \quad (3.4)$$

Using Lemma 1 and the fact (1), we can extract a subsequence $\{\mu^{n_k}\} \subset \{\mu^n\}$ such that μ^{n_k} satisfies the condition (3.A) at every $x \in R^N$, except on a set of Lebesgue measure zero.

Therefore, $E_{x_i}[\varphi_\infty^0(\mu^{n_k}) - \varphi_\infty^0(\mu)]^2$ and $E_{x_i}[\varphi_\infty^0(\mu^{n_k}) - \varphi_\infty^0(\mu)]$ tend to zero (a.s. P_x). Noticing that $\sup_{y \in R^N} E_y[\varphi_\infty^0(\mu^n)]^2$ and $\sup_{y \in R^N} E_y[\varphi_\infty^0(\mu^n)]$ are bounded in n , we can see easily that the second (using Schwarz's inequality) and the third term of the right side in (3.4) tend to zero, as $n_k \uparrow \infty$.

Consequently $\lim_{n \rightarrow \infty} E_x[\varphi_i^0(\mu^{n_k}) - \varphi_i^0(\mu)]^2 = 0$.

From this we get immediately $\lim_{n \rightarrow \infty} E_x[\varphi_i^0(\mu^n) - \varphi_i^0(\mu)]^2 = 0$. (Q.E.D.)

(Proof of Proposition 2)

In this case, it is seen easily that μ^n satisfies the condition (3.A) at every $x \in R^N$. Now, we deal with the first term of the right side in (3.1). For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} & \int_{R^N} U(x, y)(U^{\mu^n}(y) - U^\mu(y))\mu^n(dy) \\ &= \int_{R^N} U_\delta(x, y)(U^{\mu^n}(y) - U^\mu(y))\mu^n(dy) + \xi_n(\varepsilon), \end{aligned} \quad (3.5)$$

where $|\xi_n(\varepsilon)| < \varepsilon$.

$$\begin{aligned} & \int_{R^N} U_\delta(x, y)(U^{\mu^n}(y) - U^\mu(y))\mu^n(dy) \\ & \leq \int_{R^N} U_\delta(x, y)|U^{\mu^n}(y) - U^\mu(y)|\nu(dy). \end{aligned} \quad (3.6)$$

From the fact (1), $\lim_{n \rightarrow \infty} U^{\mu^n}(y) = U^\mu(y)$, $y \in R^N$ is satisfied. And using the fact that potentials $U^{\mu^n}(y)$ and $U^\mu(y)$ are uniformly bounded, it follows from (3.5) and (3.6),

$$\overline{\lim}_{n \rightarrow \infty} \int_{R^N} U(x, y)(U^{\mu^n}(y) - U^\mu(y))\mu^n(dy) \leq \varepsilon \quad (3.7)$$

Next, we shall treat of the second term of the right side in (3.1).

Noticing the uniform boundedness of the energies $||\mu^n||$ and weak convergence of μ^n to μ , we can get the equality

$$\lim_{n \rightarrow \infty} \int U^{\mu^n}(y)\nu'(dy) = \int U^\mu(y)\nu'(dy) \quad (3.8)$$

for any W -measure ν' . (Refer to [2]).

Now, for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\begin{aligned} & \int_{R^N} U(x, y)(U^\mu(y) - U^{\mu^n}(y))\mu(dy) \\ &= \int_{R^N} U_\delta(x, y)(U^\mu(y) - U^{\mu^n}(y))\mu(dy) + \xi'_n(\varepsilon), \end{aligned} \quad (3.9)$$

where $|\xi'_n(\varepsilon)| < \varepsilon$. Introduce the W -measure $\nu'(dy) = U_\delta(x, y)\mu(dy)$, then, from (3.8) and (3.9) it follows

$$\overline{\lim}_{n \rightarrow \infty} \int_{R^N} U(x, y)(U^{\mu^n}(y) - U^\mu(y))\mu(dy) \leq \varepsilon.$$

Consequently, we get

$$\lim_{n \rightarrow \infty} E_x[\varphi_\infty^0(\mu^n) - \varphi_\infty^0(\mu)]^2 = 0. \quad (\text{Q.E.D.})$$

(Proof of Proposition 3)

From the facts (1) and (2) the potentials $U^{\mu^n}(y)$ are equicontinuous on F and converge to $U^\mu(y)$. Furthermore the convergence is uniform on F . Using

(3.1) it is seen easily

$$\lim_{n \rightarrow \infty} E_x [\varphi_\infty^0(\mu^n) - \varphi_\infty^0(\mu)]^2 = 0. \quad (\text{Q.E.D.})$$

We now come to the case of S -measures. We write μ_K for the restriction of a measure onto K .

THEOREM; Let μ^n ($n=1, 2, \dots$) and μ be S -measures. If the restricted measures μ_K^n converge weakly to μ_K , for any compact set K , then there exists a subsequence $\{\mu^{n_k}\} \subset \{\mu^n\}$ and a set P of Newtonian capacity 0, such that, for any $\varepsilon > 0$ and each

$$x \notin P; \lim_{n_k \rightarrow \infty} P_x \left\{ \sup_{0 \leq u \leq t} |\varphi_u^0(\mu^{n_k}) - \varphi_u^0(\mu)| > \varepsilon \right\} = 0.$$

To prove the theorem, we quote the following Ninomiya's theorem.

Lemma 2 [N. Ninomiya [4]]; Let μ^n be a sequence of measures on $(R^N, \mathfrak{B}(R^N))$, supported by a compact set, satisfying the following conditions:

- (i) μ^n converges weakly to μ .
- (ii) $\mu^n(R^N)$, $n=1, 2, \dots$ are bounded in n .
- (iii) $\lim_{n \rightarrow \infty} \mu^n(E) = \mu(E)$, for any open set E .

Then there exists a subsequence $\{\mu^{n_k}\} \subset \{\mu^n\}$ such that corresponding potentials $U^{\mu^{n_k}}(x)$ converge to $U^\mu(x)$ except on a set of outer capacity 0. Furthermore, the convergence is uniform except on some open set, which can be chosen so that its capacity is less than an arbitrary given positive number. (Proof of theorem)

For each n let $\{I_m^{(n)'}\}$, $m=1, 2, \dots$ be a sequence of compact sets corresponding to S -measure μ^n , and $\{I_m''\}$ the one corresponding to μ . Let us begin with the following simple lemmas. For each fixed m ,

- (a) $\mu_{I_m'}^{n_{I_m'}}$ converges weakly to $\mu_{I_m'}$ as $n \uparrow \infty$.
- (b) $\lim_{n \rightarrow \infty} \mu_{I_m'}^{n_{I_m'}}(E) = \mu_{I_m'}(E)$ for any open set E .
- (c) $\mu_{I_m'}^{n_{I_m'}}(R^N)$, $n=1, 2, \dots$ are bounded in n , if necessary in n greater than some n_0 .

Because, from the condition of the theorem (a) is evident. As for (b),

$$\mu(I_m'') = \mu(I_m' - E) + \mu_{I_m'}(E); \quad \mu^n(I_m'') = \mu^n(I_m' - E) + \mu_{I_m'}^{n_{I_m'}}(E) \quad (3.10)$$

on letting $n \uparrow \infty$ in (3.10). (c) is immediate from (a).

By Lemma 2 and in view of the fact that, Newtonian potential is continuous except on some open set which can be chosen so that its capacity is less than an arbitrary given positive number, for each fixed m , we can extract a subsequence $\{\mu^{n_k}\} \subset \{\mu^n\}$ such that, potentials $U_{I_m'}^{\mu^{n_k}}(x)$ converge uniformly

to $U^{\mu_{\Gamma'_m}}(x)$ as $k \uparrow \infty$, except on an open set 0_m , $\text{Cap}(0_m) \leq \frac{1}{2^m}$.

In addition, noticing that $\mu_{\Gamma'_m}$ is a W -measure, it follows easily that, $\sup_{x \in 0_m^c} U^{\mu_{\Gamma'_m}}(x)$ and $\sup_{x \in \mathbb{R}^N} U^{\mu_{\Gamma'_m}}(x)$ are bounded in k for fixed m (*). By a diagonal method, we can extract μ^{n_k} , common to Γ'_m , $m=1, 2, \dots$

Now, introduce the following compact sets $\{\Gamma_m\}: \Gamma_m = \Gamma'_m - \bigcup_{k=m}^{\infty} 0_k$. Let $P = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} 0_k$, then $\text{Cap}(P)=0$, and $\Gamma_m \uparrow \mathbb{R}^N - P$ as $m \uparrow \infty$.

Next, we shall prove that for each fixed m the sequence $\{\mu_{\Gamma'_m}^{n_k}\}$ satisfies the conditions of Proposition 3.

For the first, weak convergence of $\mu_{\Gamma'_m}^{n_k}$ to $\mu_{\Gamma'_m}$ as $k \uparrow \infty$ is evident. Next, we shall treat of the condition of the uniform boundedness of potentials. In view of the fact (*), and the following inequalities, $U^{\mu_{\Gamma'_m}^{n_k}}(x) \geq U^{\mu_{\Gamma'_m}}(x)$ and $U^{\mu_{\Gamma'_m}}(x) \geq U^{\mu_{\Gamma_m}}(x)$, it follows, $\sup_{x \in 0_m^c} U^{\mu_{\Gamma'_m}^{n_k}}(x)$ and $\sup_{x \in \mathbb{R}^N} U^{\mu_{\Gamma'_m}}(x)$ are bounded in k . Therefore, noticing $0_m \subset \Gamma_m^c$, the condition follows from the maximum principle of Newtonian potential. For the last, we shall treat of the condition (3·B) on Γ_m . It is seen easily that, potentials $U^{\mu_{\Gamma'_m}^{n_k}}(x)$ are equicontinuous on $\Gamma'_m \cap 0_m^c (\supset \Gamma_m)$. Therefore, by the fact (3), $\{\mu_{\Gamma'_m}^{n_k}\}$ satisfies the condition (3·B) on $\Gamma'_m \cap 0_m^c$. Then, immediately, the sequence $\{\mu_{\Gamma'_m}^{n_k}\}$ satisfies the condition (3·B) on Γ_m .

Consequently, from proposition 3, it follows that

$$\lim_{n_k \rightarrow \infty} E_x[\varphi_i^0(\mu_{\Gamma'_m}^{n_k}) - \varphi_i^0(\mu_{\Gamma'_m})]^2 = 0. \quad (3.11)$$

With the notations $\mu_m^{n_k} = \mu_{\Gamma'_m}^{n_k}$ and $\mu_m = \mu_{\Gamma'_m}$, we have

$$\varphi_i^s(\mu) = \varphi_i^s(\mu_m), \quad 0 \leq s \leq t < \tau_{\Gamma'_m}, \quad (3.12)$$

$$\varphi_i^s(\mu^{n_k}) = \varphi_i^s(\mu_m^{n_k}), \quad 0 \leq s \leq t < \tau_{\Gamma'_m}. \quad (3.13)$$

Introduce the notations $0'_m = \bigcup_{k=m}^{\infty} 0_k$, and $\sigma_{0'_m} = \inf\{t \geq 0, x_t \in 0'_m\}$, then $\tau_{\Gamma'_m} = \min(\tau_{\Gamma'_m}, \sigma_{0'_m})$. In view of the fact that both $\tau_{\Gamma'_m}$ and $\sigma_{\Gamma'_m}$ increase to ∞ a.s. (P_x) as $m \uparrow \infty$ for each $x \notin P$, it follows

$$\tau_{\Gamma'_m} \uparrow \infty \text{ a.s. } (P_x) \text{ as } m \uparrow \infty \text{ for each } x \notin P. \quad (3.14)$$

From (3.12) and (3.13), we have

$$\begin{aligned} P_x\{\sup_{0 \leq u \leq t} |\varphi_u^0(\mu^{n_k}) - \varphi_u^0(\mu)| > \varepsilon\} &\leq P_x\{\tau_{\Gamma'_m} \leq t\} \\ &+ P_x\{\sup_{0 \leq u \leq t} |\varphi_u^0(\mu_m^{n_k}) - \varphi_u^0(\mu_m)| > \varepsilon\}. \end{aligned} \quad (3.15)$$

From (3.11), the second term of the right side in (3.15) tends to zero as $k \uparrow \infty$.

$$\text{Consequently, } \lim_{n_k \rightarrow \infty} P_x\{\sup_{0 \leq u \leq t} |\varphi_u^0(\mu^{n_k}) - \varphi_u^0(\mu)| > \varepsilon\} \leq P_x\{\tau_{\Gamma'_m} \leq t\}. \quad (3.16)$$

But, from (3.14),

$$\lim_{n \rightarrow \infty} P_x(\tau_{\Gamma_m} \leq t) = 0, \text{ if } x \notin P.$$

Therefore, for any $\varepsilon > 0$ and each $x \notin P$

$$\lim_{n_k \rightarrow \infty} P_x \left\{ \sup_{0 \leq u \leq t} |\varphi_u^0(\mu^{n_k}) - \varphi_u^0(\mu)| > \varepsilon \right\} = 0. \quad (\text{Q.E.D.})$$

COROLLARY; Under the same conditions as in the Theorem, we have

$$\lim_{n \rightarrow \infty} P_\nu \left\{ \sup_{0 \leq u \leq t} |\varphi_u^0(\mu^n) - \varphi_u^0(\mu)| > \varepsilon \right\} = 0,$$

where ν is an initial distribution such that for any set A of capacity zero, $\nu(A) = 0$.

§ 4. Examples.

Example 1; (μ^n converges strictly to μ , but at origin 0, $\lim E_0[\varphi_t^0(\mu^n) - \varphi_t^0(\mu)]^2 \neq 0$). Consider 3-dimensional Euclidean space. Let μ^n be $f_n(x, y, z) dx dy dz$,

$$\text{where } f_n(x, y, z) = \begin{cases} n^3 \left(\frac{1}{n} - \sqrt{x^2 + y^2 + z^2} \right); & 0 \leq \sqrt{x^2 + y^2 + z^2} \leq \frac{1}{n} \\ 0 & ; \frac{1}{n} < \sqrt{x^2 + y^2 + z^2} \end{cases}$$

and μ be $\mu \equiv 0$. μ^n , ($n=1, 2, \dots$) are supported by a compact set, and $\mu^n(R^3) = \text{Const.} \times \frac{1}{n}$. Noticing $U^{\mu^n}(0) = \text{Const.}$ (independent of n), and $U^{\mu^n}(x)$, ($n=1, 2, \dots$) are uniformly bounded, it follows immediately, μ^n converges strictly to μ (of course the convergence is fine and weak). We have

$$\begin{aligned} E_0[\varphi_t^0(\mu^n) - \varphi_t^0(\mu)] &= E_0[\varphi_t^0(\mu)] \\ &= \frac{1}{2\pi^{\frac{3}{2}}} \int_{R^3} \frac{1}{|x|} \left\{ \int_{|x|^2/2t}^{\infty} e^{-s} s^{-\frac{1}{2}} ds \right\} \mu^n(dx) \\ &\geq \frac{M}{2\pi^{\frac{3}{2}}}, \end{aligned}$$

where $M = U^{\mu^n}(0) \cdot \int_{1/2t}^{\infty} e^{-s} s^{-\frac{1}{2}} ds$; (a positive constant independent of n).

Therefore, it follows easily $\lim E_0[\varphi_t^0(\mu^n) - \varphi_t^0(\mu)]^2 \neq 0$.

Example 2; Consider a cube C of total mass 1 in R^3 . Divide C into mutually congruent small cubes $A_1, \dots, A_{2^3}$. Translate μ^1 of example 1, from origin to each center of A_i and denote them $\nu^1, \dots, \nu^{2^3}$. Dividing each A_i into mutually congruent small 2^3 cubes, we get cubes B_i ($i=1, \dots, 2^6$). Translate μ^2 of example 1 from origin to each center of B_i , and denote them $\nu^{2^3+1}, \dots, \nu^{2^3+2^6}$. Continue these procedures, then we get a sequence

of W -measures $\{\nu^n\}$. It is seen, ν^n converges strictly to $\nu \equiv 0$. But for each $x \in C$ $\overline{\lim}_{n \rightarrow \infty} E_x[\varphi_i^0(\nu^n) - \varphi_i^0(\nu)]^2 \neq 0$.

References

- [1] E. B. DYNKIN, *Markov processes*, Moscow, 1963.
- [2] H. CARTAN, *Théorie du potentiel newtonien; énergie, capacité, suites de potentiels*, Bull. Soc. Math. France, 73 (1945), 74–106.
- [3] J. Deny, *Sur la convergence des suites de potentiels*. C. R. Acad. Sci. Paris, 218 (1944), 497–499.
- [4] N. NINOMIYA, *Sur une suite convergente de distributions de masses et leurs potentiels correspondants*, Math. J. Okayama Univ., 2 (1952), 1–7.