A TREATMENT OF GEOMETRIC QUANTITIES IN THE MANIFOLD OF SURFACE-ELEMENTS

By

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1. Introduction. About ten years ago J. DOUGLAS ([1])⁽¹⁾ discussed the theory of K -spreads in an N -dimensional manifold which are defined by a system of partial differential equations of the second order. The manifold of all K -dimensional surface-elements of order 1 was based in this case and also in a theory of generalized metrics of A. KAWAGUCHI ([5]). Later, A. KAWAGUCHI and the present authors ([7], [8]) have also studied the theory of K -spreads of higher order. A K -dimensional surface-element will be determined by several components $p_{a(r)}^i$ with indices of two kinds and designated to be *analytic* in the same sense as in the projective geometry an analytic point means a set of homogeneous coordinates. In the present note we aim to give the geometries in the manifold of *geometric* surface-elements an analytic foundation by adopting the components $p_{a(r)}^i$ of analytic elements and associating to each analytic element a group $G_{(v)}$ of those transformations which transform the analytic element into another belonging to the same geometric element. This is a generalization of the study of the manifold of all curve-elements ([3], [4]) and the study of J. DOUGLAS cited above. We derive the infinitesimal operators of the group $G_{(v)}$ and its extended group $H_{(v)}$. These will rather be useful in other kinds of geometries such as the projective geometry of K -spreads.

⁽¹⁾ Numbers in square brackets refer to the bibliography at the end of the paper.

2. Transformation of parameters. Let the function f of K parameters u^a ($a = 1, 2, \dots, K$) be analytic in its arguments. By a transformation of parameters $u^a \rightarrow u^\alpha$

$$(1) \quad u^\alpha = u^\alpha(u^a) \quad (a = 1', 2', \dots, K')$$

the partial derivatives

$$f_{a(r)} \equiv f_{a_1 a_2 \dots a_r} = \frac{\partial^r f}{\partial u^{a_1} \dots \partial u^{a_r}} \quad (r = 1, 2, \dots)$$

are transformed in such a way:

$$f_a = u_a^\beta f_\beta, \quad f_{a_1 a_2} = u_{a_1}^{\beta_1} u_{a_2}^{\beta_2} f_{\beta_1 \beta_2} + u_{a_1 a_2}^\beta f_\beta, \dots,$$

where

$$u_{a(r)}^\beta \equiv u_{a_1 a_2 \dots a_r}^\beta = \frac{\partial^r u^\beta}{\partial u^{a_1} \dots \partial u^{a_r}},$$

and in general

$$(2) \quad f_{a(r)} = \sum_{s=1}^r A_{a(r)}^{\beta(s)} f_{\beta(s)},$$

in which the summation convention with respect to β 's is adopted and the coefficients $A_{a(r)}^{\beta(s)}$ are polynomials of the derivatives u_b^α , $u_{b_1 b_2}^\alpha$ and etc.⁽²⁾ By differentiation we have

$$f_{a(r+1)} = \frac{\partial f_{a(r)}}{\partial u^{a_{r+1}}} = \sum_{s=2}^{r+1} A_{a(r)}^{\beta(s-1)} u_{a_{r+1}}^{\beta_s} f_{\beta(s)} + \sum_{s=1}^r \frac{\mathfrak{d}}{\mathfrak{d} u^{a_{r+1}}} A_{a(r)}^{\beta(s)} f_{\beta(s)};$$

$\frac{\mathfrak{d}}{\mathfrak{d} u^b}$ represents the total derivation with respect to u^b , i.e.,

$$\frac{\mathfrak{d}}{\mathfrak{d} u^b} = \sum_a u_{ab}^\alpha \frac{\partial}{\partial u_a^\alpha} + \sum_{(a_1, a_2)} u_{a(a_2)b}^\alpha \frac{\partial}{\partial u_{a_2}^\alpha} + \sum_{(a_1, a_2, a_3)} u_{a(a_3)b}^\alpha \frac{\partial}{\partial u_{a_3}^\alpha} + \dots$$

Hence, the polynomials $A_{a(r)}^{\beta(s)}$ will be determined by the recursion formulae

$$(3) \quad A_{a(r)}^{\beta(s)} = \begin{cases} u_{a_1 a_2 \dots a_r}^{\beta_1} & \text{for } s = 1, \\ A_{a(r-1)}^{\beta(s-1)} u_{a_r}^{\beta_s} + \frac{\mathfrak{d}}{\mathfrak{d} u^{a_r}} A_{a(r-1)}^{\beta(s)} & \text{for } 2 \leq s \leq r-1, \\ u_{a_1}^{(\beta_1)} u_{a_2}^{\beta_2} \dots u_{a_r}^{\beta_r} & \text{for } s = r. \end{cases}$$

⁽²⁾ We adopt various notations by the method in [7], Chapter 1.

Noting that $u_{a(r)}^\alpha$ is symmetric with respect to the subscripts $a_1, a_2, \dots, a_r$, we modify each of usual partial derivatives $\partial f / \partial u_{a(r)}^\alpha$ by a numerical factor in such a manner

$$\frac{\bar{\partial}}{\bar{\partial} u_{a(r)}^\alpha} = \frac{\nu_1! \nu_2! \dots \nu_s!}{r!} \frac{\partial}{\partial u_{a(r)}^\alpha},$$

when the indices $a_1, a_2, \dots, a_r$ consist of $\nu_1, \nu_2, \dots$ and ν_s same indices ($\nu_1 + \nu_2 + \dots + \nu_s = r$). Then we can write various formulae in the largely simplified form as in the tensor calculus. Specially we note the relations

$$\frac{\bar{\partial}}{\bar{\partial} u_{a(r)}^\alpha} u_{b(r)}^\beta = \delta_\alpha^\beta \delta_{(b_1}^{a_1} \delta_{b_2}^{a_2} \dots \delta_{b_r)}^{a_r} = \delta_\alpha^\beta \delta_{(b(r))}^{a(r)},$$

$$df = \sum_{r \geq 1} \frac{\bar{\partial} f}{\bar{\partial} u_{a(r)}^\alpha} du_{a(r)}^\alpha.$$

By the second relation the total derivatives of a function f of $u_{a(r)}^\alpha$'s are given in the form

$$\frac{df}{du^b} = \sum_{r \geq 1} u_{a(r)b}^\alpha \frac{\bar{\partial} f}{\bar{\partial} u_{a(r)}^\alpha}.$$

Concerning the derivations with respect to some variables which are symmetric in some indices, we adopt the same method, but, to make typography easy, use the notations of usual partial derivatives, omitting the bar in $\bar{\partial}$.

Now we prove the following formula which is a generalization of the formula of A. KAWAGUCHI

$$(4) \quad \frac{\partial}{\partial u_{b(t)}^\beta} A_{a(r)}^{\alpha(s)} = \binom{r}{t} \delta_\beta^{(\alpha_1} A_{(a_1 \dots a_{r-t}}^{\alpha_2 \dots \alpha_s)} \delta_{a_{r-t+1} \dots a_r)}^{b(t)} \\ = \binom{r}{t} \delta_\beta^{(\alpha_1} A_{(a(r-t))}^{\alpha(s-1)} \delta_{a(t)}^{b(t)},$$

where $A_{a(r)}^0 = 0$ for $r \geq 1$, $A_0^0 = 1$, $A_{a(r)}^{\alpha(s)} = 0$ for $s > r$, and $A_{a(r)}^{\alpha(s)} = 0$ for $r < 0$.

Proof. First we notice the relation

$$(5a) \quad \frac{\partial}{\partial u_{b(t)}^\beta} \left(\frac{\mathfrak{d}f}{\mathfrak{d}u^a} \right) = \frac{\mathfrak{d}}{\mathfrak{d}u^a} \left(\frac{\partial f}{\partial u_{b(t)}^\beta} \right) + \frac{\partial f}{\partial u_{b(t-1)}^\beta} \delta_{a(t)}^{b(t)},$$

$$(5b) \quad \frac{\partial}{\partial u_b^\beta} \left(\frac{\mathfrak{d}f}{\mathfrak{d}u^a} \right) = \frac{\mathfrak{d}}{\mathfrak{d}u^a} \left(\frac{\partial f}{\partial u_b^\beta} \right).$$

We establish (4) by induction with respect to r and s . In the case $t \geq 2$ we have from (3) and (5a)

$$\begin{aligned} \frac{\partial}{\partial u_{b(t)}^\beta} A_{a(r)}^{\alpha(s)} &= \frac{\partial}{\partial u_{b(t)}^\beta} \left\{ A_{a(r-1)}^{\alpha(s-1)} u_{a_r}^{\alpha(s)} + \frac{\mathfrak{d}}{\mathfrak{d}u_{a_r}} A_{a(r-1)}^{\alpha(s)} \right\} \\ &= \binom{r-1}{t} \delta_\beta^{\alpha_1} \left\{ A_{a(r-t-1)}^{\alpha(s-2)} u_{a_r}^{\alpha(s)} + \frac{\mathfrak{d}}{\mathfrak{d}u_{a_r}} A_{a(r-t-1)}^{\alpha(s-1)} \right\} \delta_{a(t)}^{b(t)} \\ &\quad + \binom{r-1}{t-1} \delta_\beta^{\alpha_1} A_{a(r-t)}^{\alpha(s-1)} \delta_{a(t-1)}^{b(t-1)} \delta_{a_r}^{b(t)} \\ &= \binom{r-1}{t} \delta_\beta^{\alpha_1} A_{a(r-t-1)}^{\alpha(s-1)} \delta_{a_r}^{b(t)} \delta_{a(t)}^{b(t)} + \binom{r-1}{t-1} \delta_\beta^{\alpha_1} A_{a(r-t)}^{\alpha(s-1)} \delta_{a(t-1)}^{b(t-1)} \delta_{a_r}^{b(t)} \\ &= \binom{r}{t} \delta_\beta^{\alpha_1} A_{a(r-t)}^{\alpha(s-1)} \delta_{a(t)}^{b(t)}. \end{aligned}$$

When $t = 1$, we see likewise from (3) and (5b) that the relation (4) holds good. On the other hand, (4) is true for $s = 1$ and $s = r$ by virtue of (3). Hence the formula is established in general.

By virtue of (4) we see that

$$\frac{\mathfrak{d}}{\mathfrak{d}u_{a_{r+1}}} A_{a(r)}^{\beta(s)} = \sum_{1 \leq t \leq r-s+1} \binom{r}{t} A_{a(r-t)}^{(\beta(s-1))} u_{a(t)}^{\beta(s)} a_{r+1},$$

hence we get from (3) the algebraic recursion formulae of $A_{a(r)}^{\beta(s)}$:

$$(6) \quad \boxed{A_{a(r+1)}^{\beta(s)} = \sum_{0 \leq t \leq r-s+1} \binom{r}{t} A_{a(r-t)}^{(\beta(s-1))} u_{a(t)}^{\beta(s)} a_{r+1};}$$

because of the symmetric property of $A_{a(t+1)}^{\beta(s)}$ and $u_{a(t+1)}^\beta$ in $a_{(t+1)}$, these formulae may be written in the form

$$(6') \quad A_{a(r+1)}^{\beta(s)} = \sum_{0 \leq t \leq r-s+1} \binom{r}{t} A_{a(r-t)}^{(\beta(s-1))} u_{a(t+1)}^{\beta(s)}.$$

3. K-dimensional surface-elements. In an n -dimensional manifold, let us consider a K -dimensional surface with parametric representation

$$x^i = x^i(u^a) \quad (a = 1, 2, \dots, K).$$

Then the system of quantities

$$(7) \quad x^i, p_{a_1}^i \equiv \frac{\partial x^i}{\partial u^{a_1}}, p_{a_{(2)}}^i \equiv \frac{\partial^2 x^i}{\partial u^{a_1} \partial u^{a_2}}, \dots, p_{a_{(\nu)}}^i \equiv \frac{\partial^\nu x^i}{\partial u^{a_1} \dots \partial u^{a_\nu}}.$$

is called the surface-element of the ν -th order of this surface. It determines the infinitesimal neighborhood of the ν -th order of the surface. Evidently this system depends on the parameters u^a , and, to define the surface-element which is independent of auxiliary parameters, we merely ought to take K coordinates of the n -dimensional manifold as the parameters of the surface. The surface-element of both kinds will be called to be *analytic* and *geometric*, respectively.

The geometric surface-element of the ν -th order can be uniquely calculated from an analytic surface-element of the same order, but its converse is not true, i.e., the latter is not uniquely determined from the former. The necessary and sufficient condition that two analytic surface-elements $(x^i, p_{a_{(r)}}^i)$ and $(x^i, {}'p_{a_{(r)}}^i)$ correspond to the same geometric surface-element is, as can be seen from (2), that there exists a system of scalars which are symmetric in the subscripts: u_a^b with the non-vanishing determinant $U = \det |u_a^b|$, $u_{a_{(2)}}^b, \dots, u_{a_{(\nu)}}^b$, such that the relation

$$(8) \quad {}'p_{a_{(r)}}^i = \sum_{s=1}^r A_{a_{(r)}}^{b_{(s)}} {}'p_{b_{(s)}}^i \quad (r = 1, 2, \dots, \nu)$$

holds good, where $A_{a_{(r)}}^{b_{(s)}}$ are polynomials of the $u_{a_{(r)}}^c$'s and given by the following recursion formulae

$$(6'') \quad A_a^b = u_a^b, \quad A_{a_{(r+1)}}^{b_{(s)}} = \sum_{0 \leq t \leq r-s+1} \binom{r}{t} A_{a_{(r-t)}}^{(b_{(s-1)}} u_{a_{(t+1)}}^{b_{(s)}}.$$

The relation can also be written in the form

$$(8') \quad {}'p_{a_{(r)}}^i = \sum_{s=1}^r B_{a_{(r)}}^{b_{(s)}} p_{b_{(s)}}^i \quad (r = 1, 2, \dots, \nu)$$

where $B_{a_{(r)}}^{b_{(s)}}$ are rational functions of the $u_{a_{(r)}}^c$'s with the denominator $U^{r(r+1)/2}$. If the relations of the form (8) are considered to be transformations of those analytic surface-elements which correspond to the same geometric surface-element, the totality of these transformations form a finite continuous group, whose parameters are

$u_a^b, u_{a_1 a_2}^b, \dots$. We denote this group by $G_{(\nu)}$. The parameters of the identical transformation are given by

$$(9) \quad u_a^b = \delta_a^b, \quad u_{a_{(r)}}^b = 0 \quad \text{for } r > 1.$$

The analytic and geometric surface-elements defined above are dependent on a K -dimensional *surface* and their laws of transformations under a transformation of coordinate systems will be calculated easily. If a system of numbers which is independent of any surface has the same law of transformation as the analytic or geometric surface-elements of a K -dimensional surface under transformations of coordinate systems, we shall call them also *analytic* or *geometric surface-elements in the n -dimensional manifold*. The manifold of all surface-elements of the ν -th order will be denoted by $F_n^{(\nu)}$ or $\mathfrak{F}_n^{(\nu)}$ according as the elements are analytic or geometric. In the study of a differential geometry in the manifold $\mathfrak{F}_n^{(\nu)}$, it will be rather convenient for us to adopt the homogeneous components $(x^i, p_{a_{(\nu)}}^i)$ of the analytic element in $F_n^{(\nu)}$ instead of the non-homogeneous components of the geometric element, demanding the invariance of the theory under the group $G_{(\nu)}$.

Any quantity in the manifold $F_n^{(\nu)}$ is, if each component is invariant under the group $G_{(\nu)}$, of course a function in the manifold $\mathfrak{F}_n^{(\nu)}$. However, if an object in $F_n^{(\nu)}$ is closed not only under the group of transformations of coordinate systems but also under the group $G_{(\nu)}$, we can consider it a geometric object in $\mathfrak{F}_n^{(\nu)}$ in the generalized sense, even though each component is not invariant under $G_{(\nu)}$.

4. Δ -Operators. Let us denote by $[g]$ the value of a function g , for the values of its arguments $u_{a_{(r)}}^b$'s given by (9). Since by (6'')

$$[A_{a_{(r)}}^{b_{(s)}}] = \delta_{(a_{(r)})}^{b_{(s)}} \quad \text{for } s = r, \quad = 0 \quad \text{for } s \neq r,$$

the relation (4) yields to

$$(4') \quad \left[\frac{\partial}{\partial u_{a_{(s)}}^b} A_{c_{(r)}}^{b_{(t)}} \right] = \binom{r}{s} \delta_b^{(b_1)} \delta_{(c_{(r-s)})}^{b_{(r-s)}} \delta_{c_{(s)}}^{a_{(s)}} \quad \text{for } t = r - s + 1, \\ = 0 \quad \text{otherwise.}$$

From (8) it follows by differentiation

$$0 = \sum_{t=1}^r A_{c(r)}^{b(t)} \frac{\partial' p_{b(t)}^i}{\partial u_{a(s)}^b} + \sum_{t=1}^r \frac{\partial A_{c(r)}^{b(t)}}{\partial u_{a(s)}^b} p_{b(t)}^i$$

and hence, on account of (4'),

$$\left[\frac{\partial' p_{c(r)}^i}{\partial u_{a(s)}^b} \right] = - \left[\frac{\partial A_{c(r)}^{b(r-s+1)}}{\partial u_{a(s)}^b} \right] p_{b(r-s+1)}^i = - \binom{r}{s} \delta_{(c(s))}^{a(s)} p_{c(r-s)}^i.$$

Therefore, the infinitesimal operators of the group $G_{(\nu)}$

$$\Delta_b^{a(s)} f \equiv - \left[\frac{\partial}{\partial u_{a(s)}^b} f(x, p_{c_1}, p_{c_2}, \dots, p_{c(\nu)}) \right] = - \sum_{r \leq s} \frac{\partial f}{\partial p_{c(r)}^i} \left[\frac{\partial' p_{c(r)}^i}{\partial u_{a(s)}^b} \right]$$

are given by

$$(10) \quad \Delta_b^{a(s)} f = \sum_{r \leq s} \binom{r}{s} \frac{\partial f}{\partial p_{a(s)c(r-s)}^i} p_{c(r-s)}^i = \sum_{r \leq s} \binom{r}{s} f^{(a(s)c(r-s))}_i p_{c(r-s)}^i$$

($s = 1, 2, \dots, \nu$). Thus, in order that a function f may be that of the geometric surface element, it is necessary and sufficient that the following relations are fulfilled:

$$(11) \quad \Delta_b^{a(s)} f = 0 \quad (s = 1, 2, \dots, \nu).$$

From (10) we have

$$\begin{aligned} \Delta_b^{a(s)} \Delta_d^{c(t)} f &= \sum_{r \leq s} \sum_{q \geq t} \binom{r}{s} \binom{q}{t} p_{a(s)c(r-s)}^i \frac{\partial}{\partial p_{a(s)c(r-s)}^i} \left\{ p_{\gamma(q-t)d}^j \frac{\partial f}{\partial p_{c(t)\gamma(q-t)}^j} \right\} \\ &= \sum_{q \geq s+t-1} \binom{q-t+1}{s} \binom{q}{t} p_{a(s)c(q-s-t+1)}^j \delta_{\gamma(q-t)d}^{a(s)c(q-s-t+1)} \frac{\partial f}{\partial p_{c(t)\gamma(q-t)}^j} + \dots \end{aligned}$$

where the unwritten terms are partial derivatives of the second order of the function f . Since

$$\begin{aligned} (q-t+1) \delta_{\gamma(q-t)d}^{a(s)c(q-s-t+1)} &= (q-s-t+1) \delta_{(\gamma(q-t))}^{a(s)(c(q-s-t))} \delta_d^{c(q-s-t+1)} \\ &\quad + s \delta_{(\gamma(q-t))}^{c(q-s-t+1)(a(s-1))} \delta_d^{a(s)}, \end{aligned}$$

we can write the last expression in the form

$$\begin{aligned} \Delta_b^{a(s)} \Delta_d^{c(t)} f &= \frac{(s+t-1)!}{s! t!} \sum_{q \geq s+t-1} \binom{q}{s+t-1} \left\{ (q-s-t+1) \right. \\ &\times p_{\alpha(q-s-t)bd}^j \frac{\partial f}{\partial p_{a(s)c(t)\alpha(q-s-t)}^j} + s p_{\alpha(q-s-t+1)b}^j \frac{\partial f}{\partial p_{a(s-1)c(t)\alpha(q-s-t+1)}^j} \left. \right\} + \dots \end{aligned}$$

Hence, the commutator of the operators $\Delta_b^{a(s)} f$ and $\Delta_d^{c(t)} f$ is

$$(12) \quad \boxed{[\Delta_b^{a(s)} \Delta_d^{c(t)}] f = \frac{(s+t-1)!}{s! t!} \left\{ s \delta_d^{(a_1)} \Delta_b^{a(s-1)c(t)} f - t \delta_b^{(c_1)} \Delta_d^{c(t-1)a(s)} f \right\}.}$$

If we put, according to A. KAWAGUCHI ([5]),

$$(13) \quad \Delta^{a(s-1)} f \equiv \sum_{b=1}^K \Delta_b^{ba(s-1)} f,$$

the commutator of two operators $\Delta^{a(s)} f$ and $\Delta^{c(t)} f$ may be derived immediately from (12):

$$(14) \quad [\Delta^{a(s)} \Delta^{c(t)}] f = \frac{(s-t) \cdot (s+t+1)!}{(s+1)! (t+1)!} \Delta^{a(s)c(t)} f.$$

From (14) we have, in particular,

$$\Delta^{a(s+1)} f = \frac{2}{(s-1)(s+2)} [\Delta^{a(s)} \Delta^{a(s+1)}] f.$$

From (12) for $t = 2$, we get by contraction of the two indices d and c_2

$$\Delta_b^{a(s+1)} f = \frac{2}{(s-1)(s+1)} [\Delta_b^{a(s)} \Delta^{a(s+1)}] f + \frac{1}{s-1} \delta_b^{a_{s+1}} \Delta^{a(s)} f.$$

The last two relations show us that from the operators

$$(15) \quad \boxed{\Delta_b^{a_1} f, \Delta_b^{a_1 a_2} f, \Delta^{a_1 a_2} f = \sum_{b=1}^K \Delta_b^{ba_1 a_2} f \quad (a_1, a_2, b=1, 2, \dots, K)}$$

all the other operators can be derived by repeated constructions of commutators. The operators (15) are fundamental in this sense.

As a matter of fact, the fundamental operators reduce to the following four⁽³⁾:

(A) two operators $\Delta_b^a f$ ($a, b=1, 2, \dots, K$), for example

$$(\Delta_2^1 f + \Delta_1^2 f) + (\Delta_3^1 f + \Delta_1^3 f) + \dots + (\Delta_K^{K-1} f + \Delta_{K-1}^K f)$$

and

$$\Delta_1^1 f \text{ or } \Delta_K^K f;$$

(B) one operator of $\Delta_b^{a_1 a_2} f$ ($a_1, a_2, b=1, 2, \dots, K$), for example

$$\Delta_a^{aa} f \text{ or } \sum_{b=1}^K \Delta_a^{bb} f \quad (a: \text{arbitrary});$$

(C) one operator of $\Delta^{a_1 a_2} f$ ($a_1, a_2=1, 2, \dots, K$), for example

$$\Delta^{a_1 a_2} f \text{ or } \sum_{b=1}^K \Delta^{bb} f.$$

6. $\tilde{A}$ -Operators. If we apply a transformation of the group $G_{(v)}$ to each analytic surface-element in the manifold $F_n^{(v)}$, the differentials of the original and the transformed surface-element are related by

$$(16) \quad dp_{a(r)}^i = \sum_s A_{a(r)}^{b(s)} d'p_{b(s)}^i + \sum_{s,t} \frac{\partial}{\partial u_{e(t)}^c} A_{a(r)}^{b(s)} \cdot 'p_{b(s)}^i du_{e(t)}^c.$$

Hence, we get the transformation of analytic surface-elements and their differentials $(x^i, p_{a(r)}^i, dp_{a(r)}^i) \rightarrow (x^i, 'p_{a(r)}^i, 'dp_{a(r)}^i)$ in the form

$$(17) \quad \begin{cases} p_{a(r)}^i = \sum_{s=1}^r A_{a(r)}^{b(s)}(u) \cdot 'p_{b(s)}^i, \\ dp_{a(r)}^i = \sum_{s=1}^r A_{a(r)}^{b(s)}(u) \cdot 'dp_{b(s)}^i + \sum_{s=1}^r *A_{a(r)}^{b(s)}(u, v) \cdot 'p_{b(s)}^i \quad (4). \end{cases}$$

The second set of equations (17) is obtained from (16) by replacing $d'p_{b(s)}^i$ and $du_{e(t)}^c$ with $'dp_{b(s)}^i$ and $v_{e(t)}^c$ respectively; hence, the polynomial $*A_{a(r)}^{b(s)}$ is given by

$$(18) \quad *A_{a(r)}^{b(s)}(u, v) = \sum_t \frac{\partial}{\partial u_{e(t)}^c} A_{a(r)}^{b(s)} \cdot v_{e(t)}^c = \sum_{1 \leq t \leq r-s+1} \binom{r}{t} A_{a(r-t)}^{(b(s-1))} v_{a(t)}^{b(s)}.$$

⁽³⁾ As to the detail of this discussion, see the Note I.

⁽⁴⁾ The variables $dp_{a(r)}^i$ and $p_{a(r)}^i$ are regarded to be independent of each other.

In the transformation (17), the $u_{b(r)}^a$'s and the $v_{b(r)}^a$'s are arbitrary but for the only condition $|u_b^a| \neq 0$. As in the case $K=1$, we can see that the transformations of the form (17) generate a finite continuous group $H_{(v)}$ whose parameters are $u_{b(r)}^a$'s and $v_{b(r)}^a$'s⁽⁵⁾. The values of parameters corresponding to the identical transformation are

$$u_b^a = \delta_b^a, u_{b(r)}^a = 0 \text{ for } r > 1, v_{b(r)}^a = 0 \text{ for } r \geq 1.$$

It is obvious that the necessary and sufficient condition that a PFAFFian form of the analytic element may be that of the geometric element is the invariance under the group $H_{(v)}$.

If we represent by $[g]$ the value of $g(u, v)$ for the parameters of the identical transformation of $H_{(v)}$:

$$[g] = [g(u, v)] u_b^a = \delta_b^a, u_{b(r)}^a = 0 \text{ (} r > 1 \text{)}, v_{b(r)}^a = 0 \text{ (} r \geq 1 \text{)},$$

then the infinitesimal operators of the group $H_{(v)}$ are defined by

$$\Delta_b^{a(s)} P(d) = - \left[\frac{\partial' P(d)}{\partial u_{a(s)}^b} \right], \bar{\Delta}_b^{a(s)} P(d) = - \left[\frac{\partial' P(d)}{\partial v_{a(s)}^b} \right],$$

where $P(d)$ is a PFAFFian form in the manifold $F_n^{(v)}$:

$$(19) \quad P(d) = \sum_{i=0}^v T_i^{a(r)} dp_{a(r)}^i \quad (p_{a(0)}^i = x^i).$$

The first of these operators is calculated in the same manner as in (10):

$$(20a) \quad \begin{aligned} \Delta_b^{a(s)} P(d) &= \sum_{r \geq s} \binom{r}{s} \left\{ \frac{\partial P(d)}{\partial p_{a(s)c(r-s)}^i} p_{c(r-s)b}^i + \frac{\partial P(d)}{\partial p_{a(s)c(r-s)}^i} dp_{c(r-s)b}^i \right\} \\ &= \sum_{i \geq 0} \Delta_b^{a(s)} T_i^{c(r)} dp_{c(r)}^i + \sum_{i \geq s} \binom{r}{s} T_i^{a(s)c(r-s)} dp_{c(r-s)b}^i. \end{aligned}$$

Further, to calculate the operator of the second kind, let us differentiate the second equation of (17) with respect to $v_{c(s)}^b$ and evaluate the result for the parameters of the identical transformation of $H_{(v)}$, then we have

$$\left[\frac{\partial}{\partial v_{a(s)}^b} dp_{c(r)}^i \right] + \sum_{t=1}^r \left[\frac{\partial}{\partial v_{a(s)}^b} * A_{c(r)}^{b(t)} \right] p_{b(t)}^i = 0.$$

(5) See: H. HOMBU [3], Chapter 2, §9.

On the other hand, it follows from (18) that

$$\left[\frac{\partial}{\partial v_{a(s)}^b} * A_{c(r)}^{b_{(t)}} \right] = \left[\frac{\partial}{\partial u_{a(s)}^b} A_{c(r)}^{b_{(t)}} \right] = \binom{r}{s} \delta_{b(c(r-s))}^{(b_{(r-s+1)})} \delta_{c(s)}^{a_{(s)}} \text{ for } t=r-s+1, \\ = 0 \text{ otherwise.}$$

Hence we have

$$\left[\frac{\partial}{\partial v_{a(s)}^b} ' dp_{c(r)}^i \right] = - \binom{r}{s} p_{b(c(r-s))}^i \delta_{c(s)}^{a_{(s)}}.$$

By virtue of this we get the operators sought in the form

$$(20b) \quad \tilde{J}_b^{a_{(s)}} P(d) = \sum_{r \geq s} \binom{r}{s} \frac{\partial P(d)}{\partial dp_{a(s)c(r-s)}^i} p_{c(r-s)b}^i = \sum_{r \geq s} \binom{r}{s} T_i^{a_{(s)}c(r-s)} p_{c(r-s)b}^i.$$

The commutator of two operators $\tilde{J}_b^{a_{(r)}} P(d)$ and $\tilde{J}_d^{c_{(s)}} P(d)$ is identical with (12), since the laws of composition of the parameters $u_{b_{(r)}}^a$'s of the two groups $G_{(v)}$ and $H_{(v)}$ are the same. From (20a) and (20b) we have

$$\tilde{J}_b^{a_{(s)}} \tilde{J}_d^{c_{(t)}} P(d) = \sum_{r \geq t} \binom{r}{t} \frac{\partial P(d)}{\partial dp_{c(t)\gamma(r-t)}^i} \tilde{J}_b^{a_{(s)}} (p_{\gamma(r-t)d}^i) + \dots,$$

$$\tilde{J}_d^{c_{(t)}} \tilde{J}_b^{a_{(s)}} P(d) = \sum_{r \geq s} \binom{r}{s} \frac{\partial P(d)}{\partial dp_{a(s)\alpha(r-s)}^i} \tilde{J}_d^{c_{(t)}} (dp_{\alpha(r-s)b}^i) + \dots,$$

where the unwritten terms are of the partial derivatives of the second order of the form $P(d)$. Hence, in the same way as in (12) we have the commutator $[\tilde{J}_b^{a_{(s)}} \tilde{J}_d^{c_{(t)}}] P(d)$, as follows:

$$(21) \quad \left[\tilde{J}_b^{a_{(s)}} \tilde{J}_d^{c_{(t)}} \right] P(d) = \frac{(s+t-1)!}{s! t!} \left\{ s \delta_b^{a_1} \tilde{J}_d^{a_{(s-1)}c_{(t)}} P(d) \right. \\ \left. - t \delta_b^{c_1} \tilde{J}_d^{c_{(t-1)}a_{(s)}} P(d) \right\}.$$

From (21) for $t=1$, we have, by contraction for c_1 and d , the relation

$$[\tilde{J}_b^{a_{(s)}}, \sum_{c=1}^K \tilde{J}_c^c] P(d) = (s-1) \tilde{J}_b^{a_{(s)}} P(d),$$

which shows that all the $\tilde{J}$ -operators can be derived by constructing commutators of the operators $\tilde{J}_b^a P(d)$ and $\tilde{J}_b^{a_{(r)}} P(d)$. It is seen further that the one operator of the $\tilde{J}_b^a P(d)$, for example

$$(D) \quad \tilde{J}_a^a P(d) \quad (a: \text{arbitrary})$$

is fundamental (see the Note II).

6. Functions and PFAFFian forms in $\mathfrak{F}_n^{(\nu)}$. Since the necessary and sufficient condition that a quantity f or a PFAFFian form $P(d)$ may be that of the geometric surface-element is the invariance under the group $G_{(\nu)}$ or $H_{(\nu)}$ respectively, we can write the conditions by means of the infinitesimal operators of the group, as follows:

$$(11) \quad \Delta_b^{a(s)} f = 0 \quad \text{for all } b, a_1, a_2, \dots, a_s \quad (s \geq 1)$$

or

$$(22) \quad \Delta_b^{a(s)} P(d) = \tilde{J}_b^{a(s)} P(d) = 0 \quad \text{for all } b, a_1, \dots, a_s \quad (s \geq 1).$$

The conditions may be reduced respectively to four or five relations corresponding to the fundamental operators (A), (B), (C) and (D), because the other relations are identically satisfied by virtue of these. However, it is more convenient in practical uses to write the conditions in the form

$$(11') \quad \Delta_b^{a_1} f = \Delta_b^{a_1 a_2} f = \Delta_b^{a_1 a_2} f = 0,$$

$$(22') \quad \Delta_b^{a_1} P(d) = \Delta_b^{a_1 a_2} P(d) = \Delta_b^{a_1 a_2} P(d) = 0, \quad \tilde{J}_b^{a_1} P(d) = 0$$

for all b, a_1 and a_2 .

We remark that the conditions (11) and (22) can also be deduced without any aid of the theory of group by deriving some simple formulae. For the sake of convenience, we consider the transformation of analytic surface-elements under the change of parameters

$$p_{a(r)}^i = \sum_{s=1}^r A_{a(r)}^{\beta(s)} p_{\beta(s)}^i$$

instead of (8). By differentiation with respect to $u_{c(s)}^\beta$ we have, on account of the relation (4),

$$\frac{\partial p_{a(r)}^i}{\partial u_{c(s)}^\beta} = \binom{r}{s} \sum_{t=1}^{r-s+1} A_{(a(r-s))}^{\gamma(t-1)} \delta_{a(s)}^{\beta c(s)} p_{\gamma(t-1)\beta}^i$$

where we regard that $p_{\beta(s)}^i$ is independent of $u_{c(s)}^\beta$. Therefore, it follows

$$\frac{\partial f}{\partial u_{c(s)}^\beta} = \sum_{r \geq s} \binom{r}{s} \frac{\partial f}{\partial p_{c(s)a(r-s)}^i} \sum_{t=1}^{r-s+1} A_{a(r-s)}^{\gamma(t-1)} p_{\gamma(t-1)\beta}^i$$

for a function f in the manifold $F_n^{(\nu)}$. Now, we calculate, noticing $p_\beta^i = u_\beta^b p_b^i$ and the LEIBNIZ formula,

$$(23) \quad \sum_{t=1}^{r+1} A_{a(r)}^{\gamma(t-1)} p_{\gamma(t-1)\beta}^i = \frac{\delta^r p_\beta^i}{\delta u_{a(r)}} = \sum_{t=0}^r \binom{r}{t} \frac{\delta^t u_\beta^b}{\delta u_{a(t)}} p_{a(r-t)b}^i$$

where $\delta^r / \delta u_{a(r)} = \delta^r / \delta u_{a_1} \delta u_{a_2} \dots \delta u_{a_r}$. From the last two relations there follows

$$\frac{\partial f}{\partial u_{c(s)}^\beta} = \sum_{t \geq 0} \binom{s+t}{t} \frac{\delta^t u_\beta^b}{\delta u_{a(t)}} \sum_{r \geq s+t} \binom{r}{s+t} \frac{\partial f}{\partial p_{c(s)a(r-s)}^i} p_{a(r-s-t)b}^i$$

or

$$(24) \quad \boxed{\frac{\partial f}{\partial u_{c(s)}^\beta} = \sum_{t \geq 0} \binom{s+t}{t} \frac{\delta^t u_\beta^b}{\delta u_{a(t)}} \cdot \mathcal{A}_b^{c(s)a(t)} f.}$$

For PFAFFian form in the manifold $F_n^{(\nu)}$ we can derive similar relations. Since we regard that $du_{c(s)}^\beta$ is independent of $u_{c(s)}^\beta$, the two operators d and $\partial / \partial u_{c(s)}^\beta$ are commutative:

$$\frac{\partial}{\partial u_{c(s)}^\beta} dp_{a(r)}^i = d \frac{\partial p_{a(r)}^i}{\partial u_{c(s)}^\beta}.$$

Noting this and using (23) the relation follows

$$(25a) \quad \boxed{\frac{\partial P(d)}{\partial u_{c(s)}^\beta} = \sum_{t \geq 0} \binom{s+t}{t} \left\{ \frac{\delta^t u_\beta^b}{\delta u_{a(t)}} \cdot \mathcal{A}_b^{c(s)a(t)} P(d) + d \left(\frac{\delta^t u_\beta^b}{\delta u_{a(t)}} \right) \cdot \tilde{\mathcal{A}}_b^{c(s)a(t)} P(d) \right\}.}$$

And, by virtue of the relation

$$\frac{\partial dp_{a(r)}^i}{\partial du_{c(s)}^\beta} = \frac{\partial p_{a(r)}^i}{\partial u_{c(s)}^\beta}$$

we have at once

$$(25b) \quad \boxed{\frac{\partial P(d)}{\partial du_{c(s)}^\beta} = \sum_{t \geq 0} \binom{s+t}{t} \frac{\delta^t u_\beta^b}{\delta u_{a(t)}} \cdot \tilde{\mathcal{A}}_b^{c(s)a(t)} P(d).}$$

One can easily see that these formulae (24), (25a) and (25b) may be applied also for the transformation (8). However, in this case the $u_\beta^{b'}$'s must be expressed in terms of the u_a^a 's and the term $du_{e(r)}^c$ be replaced by $v_{e(r)}^c$. $\delta^i u_\beta^b / \delta u^{a(i)}$ and $d(\delta^i u_\beta^b / \delta u^{a(i)})$ are then rational functions of the $u_{e(r)}^c$'s and $v_{e(r)}^c$'s. In virtue of the formulae we see the necessary and sufficient conditions (11) and (22) are now evident, since these are completely equivalent to

$$\frac{\partial f}{\partial u_{c(s)}^\beta} = 0 \quad \text{and} \quad \frac{\partial P(d)}{\partial u_{c(s)}^\beta} = 0, \quad \frac{\partial P(d)}{\partial v_{c(s)}^\beta} = 0$$

for all $\beta, c_1, c_2, \dots, c_s \ (1 \leq s \leq \nu)$.

The formulae (24) and (25) will be also of use in other problems.

7. Linear connection. When the parallel displacement of vectors in the manifold $F_n^{(\nu)}$ is defined by the vanishing of a covariant differential of the form

$$(26) \quad Dv^i = dv^i + \omega_j^i(d) v^j, \quad \omega_j^i(d) = \sum_{r=0}^{\nu} \Gamma_{jk}^{ic(r)} dp_{c(r)}^k,$$

the manifold $F_n^{(\nu)}$ is said to be affinely connected in a general sense. In order to study the differential geometry in such a manifold by the usual tensor analysis, it is most convenient for us if we are able to replace the differentials $dp_{a(r)}^i$ with PFAFFIAN forms with vector character. We call such PFAFFIAN forms the covariant differentials of the element⁽⁶⁾. The most simple form is

$$(27) \quad \delta p_{a(r)}^i = dp_{a(r)}^i + \sum_{s=0}^{r-1} Q_{a(r)j}^{ib(s)} dp_{b(s)}^j \quad (r = 1, 2, \dots, \nu)$$

where $Q_{a(r)j}^{ib(s)}$ is a function of the analytic surface-element, and $\delta p_{a(r)}^i$ is supposed to be contravariant with respect to the super-script i .

Using these covariant differentials of the element of the manifold, we can define the covariant derivatives $\Gamma_k^{a(r)} v^i$ of a vector field v^i in the manifold by

$$(28) \quad Dv^i = \sum_{r=0}^{\nu} \Gamma_k^{a(r)} v^i \cdot \delta p_{a(r)}^k.$$

(6) See T. SUGURI ([8]), Chapter 2, §7.

The covariant derivatives are obtained in the form

$$(29) \quad \begin{cases} \Gamma_k^{a(\nu)} v^i = v^i{}_{;k}^{(a(\nu))} + \Gamma_{jk}^{ia(\nu)} v^j, \\ \Gamma_k^{a(r)} v^i = v^i{}_{;k}^{(a(r))} + \Gamma_{jk}^{ia(r)} v^j - \sum_{s=r+1}^{\nu} Q_{b(s)k}^{ja(r)} \Gamma_j^{b(s)} v^i, \end{cases}$$

$$(r = 0, 1, \dots, \nu - 1).$$

$\Gamma_k^{a(r)} v^i$ is an affinor with respect to i and k . If we define $R_{a(q)j}^{ib(r)}$ by the recursion formulae

$$(30) \quad \begin{cases} R_{a(q)j}^{ib(q-1)} = Q_{a(q)j}^{ib(q-1)}, \\ R_{a(q)j}^{ib(r)} = Q_{a(q)j}^{ib(r)} - \sum_{s=r+1}^{q-1} R_{a(q)k}^{ic(s)} Q_{c(s)j}^{kb(r)} \quad (r = 0, 1, \dots, q-2), \end{cases}$$

the covariant derivatives can be written in the form

$$(29') \quad \Gamma_k^{a(r)} v^i = v^i{}_{;k}^{(a(r))} - \sum_{s=r+1}^{\nu} R_{b(s)k}^{la(r)} v^i{}_{;l}^{(b(s))} + \Gamma_{jk}^{*ia(r)} v^j$$

($r = 0, 1, \dots, \nu$), where we put

$$(31) \quad \Gamma_{jk}^{*ia(r)} = \Gamma_{jk}^{ia(r)} - \sum_{s=r+1}^{\nu} R_{b(s)k}^{la(r)} \Gamma_{jl}^{ib(s)} \quad (r = 0, 1, \dots, \nu).$$

Now, let us consider the necessary and sufficient condition that the linear connection defined by (26) may be that of the manifold $\mathfrak{F}_n^{(\nu)}$. If v^i is a vector in the manifold $\mathfrak{F}_n^{(\nu)}$, we have

$$\Delta_c^{b(t)} v^i = 0 \text{ for all } b_1, b_2, \dots, b_t \text{ and } c \quad (t = 1, 2, \dots, \nu);$$

hence the conditions of invariance

$$\Delta_c^{b(t)} (Dv^i) = 0, \quad \bar{\Delta}_c^{b(t)} (Dv^i) = 0$$

reduce simply to

$$(32) \quad \text{(a) } \Delta_c^{b(t)} \omega_j^i(d) = 0, \quad \text{(b) } \bar{\Delta}_c^{b(t)} \omega_j^i(d) = 0,$$

which means that $\omega_j^i(d)$ should be PFAFFIAN form of the geometric surface-element.

For the covariant differentials of the surface-element of the form (27) we have

$$\Delta_c^b (\delta p_{a(r)}^i) = r \delta_{(a_1}^b dp_{a(r-1)c}^i + \dots, \quad \bar{\Delta}_c^{b(r)} (\delta p_{a(r)}^i) = \delta_{(a(r))}^{b(r)} p_c^i;$$

therefore we can not demand the conditions

$$\Delta_c^{b(s)}(\delta p_{a(r)}^i) = 0, \quad \bar{\Delta}_c^{b(s)}(\delta p_{a(r)}^i) = 0,$$

i.e., $\delta p_{a(r)}^i$ is never PFAFFian form of the geometric surface-element. However, for the purpose of defining the covariant derivatives $\nabla_k^{a(r)} v^i$ of a vector field v^i which constitutes a homogeneous function-system⁽⁷⁾ with respect to $a_{(r)}$, it is sufficient to suppose that the differentials $\delta p_{a(r)}^i$ are not necessarily contravariant but *contravariant* (*mod* p_e^i) with respect to i and satisfy the conditions

$$(33) \quad \begin{aligned} (a) \quad \Delta_c^{b(s)}(\delta p_{a(r)}^i) &\equiv r \delta_{(a_1}^b \delta p_{a_{(r-1))c}}^i \pmod{p_e^i} \text{ for } s = 1, \\ &\equiv 0 \pmod{p_e^i} \text{ for } s > 1; \\ (b) \quad \bar{\Delta}_c^{b(s)}(\delta p_{a(r)}^i) &\equiv 0 \pmod{p_e^i} \text{ for } s \geq 1, \end{aligned}$$

where $P^i \equiv 0 \pmod{p_e^i}$ means $P^i = \sum_{e=1}^K \varphi^e p_e^i$ with suitable functions or PFAFFian forms φ^e . A special case is important in which the differentials $\delta p_{a(r)}^i$ are of the form

$$(34) \quad \delta p_{a(r)}^i = dp_{a(r)}^i + \sum_{s=0}^{r-1} Q_{a(r)j}^{ib(s)} dp_{b(s)}^j + p_c^i W_{a(r)}^c(d)$$

with PFAFFian forms $W_{a(r)}^c(d)$, contravariant with respect to i and constitute a homogeneous function-system with respect to $a_{(r)}$, i.e.,

$$(35) \quad \begin{aligned} (a) \quad \Delta_c^{b(s)}(\delta p_{a(r)}^i) &= r \delta_{(a_1}^b \delta p_{a_{(r-1))c}}^i \text{ for } s = 1, \\ &= 0 \text{ for } s > 1; \\ (b) \quad \bar{\Delta}_c^{b(s)}(\delta p_{a(r)}^i) &= 0 \text{ for } s \geq 1. \end{aligned}$$

The principal part of (34)

$$dp_{a(r)}^i + \sum_{s=0}^{r-1} Q_{a(r)j}^{ib(s)} dp_{b(s)}^j$$

satisfies then the conditions (33). It is not difficult to interpret such covariant differentials geometrically⁽⁸⁾.

Let us now prove the above statement. From $\Delta_c^{b(s)}(Dv^i) = 0$, it is shown that the coefficients $\nabla_k^{a(r)} v^i$ of $dp_{a(r)}^i$ in Dv^i constitute a homogeneous function-system with respect to $a_{(r)}$, such that

(7) See the Note III.

(8) A. KAWAGUCHI and H. HOMBU ([7]), Chapter 2.

$$(36) \quad \begin{aligned} \Delta_c^{b(s)} (\Gamma_k^{a(v)} v^i) &= - \nu \delta_c^{(a_1} \Gamma_k^{a(v-1))b} v^i & \text{for } s = 1, \\ &= 0 & \text{for } s > 1. \end{aligned}$$

From $\tilde{\Delta}_c^{a(v)} (Dv^i) = 0$ we have

$$\Gamma_k^{a(v)} v^i \cdot p_c^k = 0.$$

From these relations and (33) it follows immediately the Δ - and $\tilde{\Delta}$ -invariance of the expression $\Gamma_k^{a(v)} v^i \cdot \delta p_{a(v)}^k$ which is contravariant with respect to i . Hence, the expression

$$Dv^i - \Gamma_k^{a(v)} v^i \cdot \delta p_{a(v)}^k$$

which lacks the terms of $dp_{a(v)}^k$ is also Δ - and $\tilde{\Delta}$ -invariant, and the coefficients $\Gamma_k^{a(v-1)} v^i$ of $dp_{a(v-1)}^k$ in this expression therefore constitute a homogeneous function-system with respect to $a_{(v-1)}$ and satisfy the condition

$$\Gamma_k^{a(v-1)} v^i \cdot p_c^k = 0.$$

Repeating this process step by step we have at last all the covariant derivatives $\Gamma_k^{a(r)} v^i$ ($r \geq 0$) analogous to those given by (28), (29), (30) and (31) which constitute a homogeneous function-system with respect to $a_{(r)}$ and satisfy the condition

$$\Gamma_k^{a(r)} v^i \cdot p_c^k = 0 \quad (r \geq 1).$$

Further, let a vector $v_{b(s)}^{a(r), i}$ constitute a homogeneous function-system with respect to $r + s$ indices $a_{(r)}$ and $b_{(s)}$ and define again its covariant differential by (26), then it is readily proved that the differentials $Dv_{b(s)}^{a(r), i}$ constitute a homogeneous system and consequently the covariant derivatives also.

Thus, we can conclude that it is adequate to consider not only those quantities which are invariant under the group $G_{(v)}$, but also those which constitute homogeneous function-systems.

Note I.

1. From (12) we have

$$[\Delta_b^a \Delta_d^c] f = \delta_d^a \Delta_b^c f - \delta_b^c \Delta_d^a f;$$

by means of this formula we see that all the operators $\Delta_b^a f$ may be obtained from the operators

$$(A) \quad \mathcal{A}_1^* f = (\mathcal{A}_2^1 + \mathcal{A}_1^2) f + (\mathcal{A}_3^2 + \mathcal{A}_2^3) f + \dots + (\mathcal{A}_K^{K-1} + \mathcal{A}_{K-1}^K) f$$

and $\mathcal{A}_1^1 f$

by the process of constructing commutators. Indeed, at first the operators $\mathcal{A}_2^1 f$ and $\mathcal{A}_1^2 f$ are obtained from the relations

$$[\mathcal{A}_1^1 \mathcal{A}_1^*] f = -\mathcal{A}_2^1 f + \mathcal{A}_1^2 f, \quad [\mathcal{A}_1^1, -\mathcal{A}_2^1 + \mathcal{A}_1^2] f = \mathcal{A}_2^1 f + \mathcal{A}_1^2 f,$$

and therefore we know

$$\mathcal{A}_2^* f = (\mathcal{A}_3^2 + \mathcal{A}_2^3) f + \dots + (\mathcal{A}_K^{K-1} + \mathcal{A}_{K-1}^K) f.$$

On the other hand, the operator $\mathcal{A}_2^2 f$ may be obtained from

$$[\mathcal{A}_2^1 \mathcal{A}_1^2] f = \mathcal{A}_2^2 f - \mathcal{A}_1^1 f.$$

Hence, repeating this process we can derive the operators

$$\mathcal{A}_1^1 f, \mathcal{A}_2^2 f, \dots, \mathcal{A}_K^K f;$$

$$\mathcal{A}_2^1 f, \mathcal{A}_3^2 f, \dots, \mathcal{A}_K^{K-1} f \text{ and } \mathcal{A}_1^2 f, \mathcal{A}_2^3 f, \dots, \mathcal{A}_{K-1}^K f.$$

The remaining operators are given, as follows:

$$\begin{aligned} \mathcal{A}_b^a f &= [\mathcal{A}_b^{b-1} [\dots [\mathcal{A}_{a+3}^{a+2} [\mathcal{A}_{a+2}^{a+1} \mathcal{A}_{a+1}^a] \dots]] f & \text{for } b > a, \\ &= [[\dots [\mathcal{A}_b^{b+1} \mathcal{A}_{b+1}^{b+2} \mathcal{A}_{b+2}^{b+3} \dots] \mathcal{A}_{a-1}^a] f & \text{for } b < a. \end{aligned}$$

2. For the operators $\mathcal{A}^{a_1 a_2} f$ we have from (12)

$$[\mathcal{A}_b^a \mathcal{A}^{c_1 c_2}] f = -\delta_b^{(c_1} \mathcal{A}^{c_2) a} f,$$

in particular

$$[\mathcal{A}^{cc} \mathcal{A}_c^a] f = \mathcal{A}^{ca} f, \quad [\mathcal{A}^{bc} \mathcal{A}_b^a] f = \frac{1}{2} \mathcal{A}^{ca} f \quad (b \neq c).$$

Therefore, when the operator

$$(C_1) \quad \mathcal{A}^{12} f$$

is known, all the operators $\mathcal{A}^{a_1 a_2} f$ may be led from the operators $\mathcal{A}^{12} f$ and $\mathcal{A}_b^a f$ in the following way:

$$[\mathcal{A}^{12} \mathcal{A}_1^a] f = \frac{1}{2} \mathcal{A}^{2a} f, \quad [\mathcal{A}^{2c} \mathcal{A}_2^a] f = \frac{1}{2} \mathcal{A}^{ca} f.$$

The operator

$$(C_2) \quad \mathcal{A}^{11} f$$

can also be taken as fundamental, since the case is reducible to the case (C₁) on account of the relation

$$[\Delta^{11} \Delta_1^2] f = \Delta^{12} f.$$

Further, if we take

$$(C_3) \quad \Delta^* f = \sum_{a=1}^K \Delta^{aa} f$$

to be fundamental, $\Delta^{a_1 a_2} f$ may be obtained more simply by virtue of

$$[\Delta^* \Delta_c^a] f = \Delta^{ca} f.$$

3. At last, for the operators $\Delta^{a_1 a_2} f$ we have from (12)

$$(37) \quad [\Delta^{a_1 a_2} \Delta_d^c] f = \partial_d^{a_1} \Delta_b^{a_2 c} f + \partial_d^{a_2} \Delta_b^{a_1 c} f - \partial_b^c \Delta_d^{a_1 a_2} f.$$

Hence, if we take

$$(B_1) \quad \Delta_a^{aa} f \quad \text{or} \quad \sum_{b=1}^K \Delta_a^{bb} f$$

as the fundamental operator, all the other operators may be obtained by the following process. In virtue of the relation

$$[\sum_b \Delta_a^{bb}, \Delta_a^a] f = 2\Delta_a^{aa} f - \sum_b \Delta_a^{bb} f$$

it is sufficient to consider only the operator $\Delta_a^{aa} f$.

In case $K \geq 3$, various operators may be derived successively:

$$\begin{aligned} \Delta_b^{aa} f & \text{ from } [\Delta_a^{aa} \Delta_b^a] f = -\Delta_b^{aa} f, \\ \Delta_b^{ab} f & \text{ ,, } [\Delta_b^{aa} \Delta_a^b] f + \Delta_a^{aa} f = 2\Delta_b^{ab} f, \\ \Delta_c^{ab} f & \text{ ,, } [\Delta_b^{ab} \Delta_c^c] f = -\Delta_c^{ab} f \quad (a, b, c: \text{all distinct}), \\ \Delta_b^{bc} f & \text{ ,, } [\Delta_b^{ab} \Delta_a^c] f = \Delta_b^{bc} f \quad (b \neq a, c), \\ \Delta_c^{bb} f & \text{ ,, } [\Delta_c^{ab} \Delta_a^b] f = \Delta_c^{bb} f, \\ \Delta_b^{bb} f & \text{ ,, } [\Delta_c^{bb} \Delta_b^c] f - 2\Delta_c^{bc} f = -\Delta_b^{bb} f \quad (a, b, c: \text{all distinct}). \end{aligned}$$

It is readily seen that continuing this process we can obtain all the operators $\Delta^{a_1 a_2} f$. In similar manner, it is shown that any one of the operators

$$(B_2) \quad \Delta_b^{aa} f, \Delta_b^{ab} f, \Delta_c^{ab} f \quad (a, b, c: \text{all distinct})$$

is fundamental in case $K \geq 3$.

In case $K = 2$, we see from (37) that the processes of constructing commutators give rise to five derivations

$$\begin{aligned} \Delta_a^{aa} f & \rightarrow \Delta_a^{ab} f, \quad \Delta_a^{aa} f \rightarrow \Delta_b^{aa} f, \quad \Delta_b^{ab} f \rightarrow \Delta_b^{aa} f, \\ \Delta_b^{aa} f & \rightarrow 2\Delta_b^{ab} f - \Delta_a^{aa} f, \quad \Delta_a^{ab} f \rightarrow \Delta_a^{aa} f - \Delta_b^{ab} f. \end{aligned}$$

Hence any one of the operators

$$\mathcal{A}_1^{11}f, \mathcal{A}_2^{22}f, \mathcal{A}_1^{12}f, \mathcal{A}_2^{21}f$$

can be taken as fundamental operator, however non of the operators $\mathcal{A}_2^{11}f, \mathcal{A}_1^{22}f$ is fundamental, since

$$\mathcal{A}_2^{11}f, \mathcal{A}_1^{22}f, 2\mathcal{A}_2^{12}f - \mathcal{A}_1^{11}f, 2\mathcal{A}_1^{21}f - \mathcal{A}_2^{22}f$$

and together with operations (A) and (C) can be regarded to be operators of a subgroup of $G_{(\vee)}$.

Note II.

4. For $\tilde{\mathcal{A}}_a^b P(d)$ we have from (21)

$$[\mathcal{A}_b^a \tilde{\mathcal{A}}_a^b]f = \partial_a^a \tilde{\mathcal{A}}_b^a P(d) - \partial_b^b \tilde{\mathcal{A}}_a^a P(d);$$

hence, by the operators $\mathcal{A}_b^a P(d)$ and the only $\tilde{\mathcal{A}}$ -operator

$$(D) \quad \tilde{\mathcal{A}}_a^a P(d) \quad (a: \text{arbitrary})$$

all the other $\tilde{\mathcal{A}}$ -operators are expressible. Indeed, we have successively

$$\begin{aligned} \tilde{\mathcal{A}}_b^a P(d) & \text{ from } [\mathcal{A}_b^a \tilde{\mathcal{A}}_a^a] P(d) = \tilde{\mathcal{A}}_b^a P(d), \\ \tilde{\mathcal{A}}_b^b P(d) & \text{ ,, } [\mathcal{A}_a^b \tilde{\mathcal{A}}_b^a] P(d) = \tilde{\mathcal{A}}_a^a P(d) - \tilde{\mathcal{A}}_b^b P(d), \\ \tilde{\mathcal{A}}_c^b P(d) & \text{ ,, } [\mathcal{A}_c^b \tilde{\mathcal{A}}_b^b] P(d) = \tilde{\mathcal{A}}_c^b P(d). \end{aligned}$$

On the contrary, the operator $\tilde{\mathcal{A}}_b^a P(d)$ ($a \neq b$) or $\sum_{b=1}^K \tilde{\mathcal{A}}_b^b P(d)$ is not fundamental.

Note III. Homogeneity in a generalized sense. Projective parameters.

5. **Affine and projective transformations of parameters.** As we have already remarked, the operators (15) are fundamental in the sense that all the other $\mathcal{A}$ -operators can be derived from these by the process of commutators. On the other hand, some of the fundamental operators are closely related with affine and projective transformations of parameters.

Since

$$\begin{aligned} [\Delta^a \Delta^b]f &= 0, \quad [\Delta_b^a \Delta^c]f = -\delta_b^c \Delta^a f, \\ [\Delta_b^a \Delta_d^c]f &= \delta_d^a \Delta_b^c f - \delta_b^c \Delta_d^a f, \end{aligned}$$

we see that the operators

$$(38) \quad \Delta_b^a f, \quad \Delta^a f = \sum_{b=1}^K \Delta_b^{ba} f \quad (a, b = 1, 2, \dots, K)$$

generate a subgroup of $G_{(v)}$, which is related with the group of projective transformations of parameters

$$\bar{u}^a = \frac{L_a^\alpha u^a + L^\alpha}{L_a u^a + L}$$

where $L_a^\alpha, L^\alpha, L_a, L$ are all arbitrary constants but for the condition

$$\det \begin{vmatrix} L_a^\alpha & L^\alpha \\ L_a & L \end{vmatrix} \neq 0.$$

Indeed, for a function of u^a 's and the surface-element the infinitesimal operators of the projective group are expressible in terms of the operator (38) and ordinary infinitesimal operators of the same group. By an infinitesimal transformation

$$\bar{u}^a = u^a + (l_a^\alpha u^a + l^\alpha) - u^a (l_a u^a + l)$$

in which the constants $l_a^\alpha, l^\alpha, l_a, l$ are supposed to be infinitesimal, we have

$$\bar{u}_a^\alpha = \delta_a^\alpha + l_a^\alpha - u^a l_a - \delta_a^\alpha (l_c u^c + l), \quad \bar{u}_{a_1 a_2}^\alpha = -(\delta_{a_1}^\alpha l_{a_2} + \delta_{a_2}^\alpha l_{a_1}),$$

by means of which the operators in question may be calculated readily in the form

$$\left[\frac{\partial f}{\partial l^\alpha} \right]_{l_a^\alpha = l^\alpha = l_a = l = 0} = \frac{\partial f}{\partial u^\alpha},$$

$$\left[\frac{\partial f}{\partial l_a^\alpha} \right] = \frac{\partial f}{\partial u^\alpha} u^a - \Delta_a^\alpha f,$$

$$\left[\frac{\partial f}{\partial l} \right] = - \sum_{\beta} \frac{\partial f}{\partial u^\beta} u^\beta + \Delta f,$$

$$\left[\frac{\partial f}{\partial l_a} \right] = - \sum_{\beta} \frac{\partial f}{\partial u^\beta} u^\beta u^a + \sum_{\beta} \Delta_\beta^a f \cdot u^\beta + \Delta f \cdot u^a + 2 \Delta^a f.$$

It is to be noted that the infinitesimal operators of the first two kinds are those of the affine group of parameters.

In this way we find the necessary and sufficient condition that a function of u^a 's and the surface-element may be invariant in functional form respectively under the affine and projective group of parameters is given by

$$(39) \quad \frac{\partial f}{\partial u^a} = 0, \quad \Delta_b^a f = 0$$

and

$$(40) \quad \frac{\partial f}{\partial u^a} = 0, \quad \Delta_b^a f = 0, \quad \Delta^a f = 0.$$

6. Homogeneous function-system. The infinitesimal operators

$$(41) \quad \Delta_b^{a_1 a_2} f, \quad \Delta_b^{a_1 a_2 a_3} f, \quad \dots, \quad \Delta_b^{a_{(\nu)}} f$$

generate, as one can see by virtue of (12), a subgroup of those transformations of $G_{(\nu)}$ which are obtained by putting the parameters u_a^b equal to δ_a^b . Of the function-systems which are invariant under this group there are an important class of systems designated as *homogeneous*. The functions $T_{b_1 b_2 \dots b_\sigma}^{a_1 a_2 \dots a_\rho}$ of such a system are transformed, under any transformation of $G_{(\nu)}$, in *functional form* as they are components of an affinor in the manifold of parameters, i.e.,

$$(42) \quad u_{b_1}^{f_1} u_{b_2}^{f_2} \dots u_{b_\sigma}^{f_\sigma} T_{f_1 f_2 \dots f_\sigma}^{a_1 a_2 \dots a_\rho} = u_{e_1}^{a_1} u_{e_2}^{a_2} \dots u_{e_\rho}^{a_\rho} T_{b_1 b_2 \dots b_\sigma}^{e_1 e_2 \dots e_\rho},$$

in which, it is specially to be noticed, $T_{b_{(\sigma)}}^{a_{(\rho)}}$ means the same function $T_{b_{(\sigma)}}^{a_{(\rho)}}$ whose arguments $p_{b_{(r)}}^i$ however are replaced by $p_{b_{(r)}}^i$. Differentiate the relation with respect to u_c^d and evaluate for the values of parameters of the identical transformation. Then, noticing the definition of the operator $\Delta_d^c f$ we have

$$(43) \quad \begin{aligned} \Delta_d^c T_{b_1 \dots b_\sigma}^{a_1 \dots a_\rho} &= \sum_{r=1}^{\nu} r \frac{\partial}{\partial p_{c e_{(r-1)}}^i} T_{b_1 \dots b_\sigma}^{a_1 \dots a_\rho} \cdot p_{d e_{(r-1)}}^i \\ &= \delta_{b_1}^c T_{d b_2 \dots b_\sigma}^{a_{(\rho)}} + \delta_{b_2}^c T_{b_1 d \dots b_\sigma}^{a_{(\rho)}} + \dots + \delta_{b_\sigma}^c T_{b_1 \dots b_{\sigma-1} d}^{a_{(\rho)}} \\ &\quad - \delta_d^{a_1} T_{b_{(\sigma)}}^{c a_2 \dots a_\rho} - \delta_d^{a_2} T_{b_{(\sigma)}}^{a_1 c \dots a_\rho} - \dots - \delta_d^{a_\rho} T_{b_{(\sigma)}}^{a_1 \dots a_{\rho-1} c}. \end{aligned}$$

This condition alone means that the function-system be transformed under affine group of parameters according to (42), and together

with those of invariance under the operators (41) gives the necessary and sufficient condition for homogeneity in a generalized sense⁽⁹⁾.

7. Projective parameters of K-spreads. Let a system of partial differential equations of the m -th order

$$p_{a_1 a_2 \dots a_m}^i + H_{a_1 a_2 \dots a_m}^i(u, x, p_{b_1}, \dots, p_{b_{(m-1)}}) = 0$$

be given. Since we can not expect existence of any solution of this system, it is adequate to regard the system as determining a correspondence of u 's and analytic surface-element of the $(m-1)$ -th order to an analytic surface-element of the m -th order. All the corresponding surface-elements are seen to be non-holonomic in generalized sense and called K -spreads of the m -th order. Under this circumstance it has no meaning at first sight to say the parameters of K -spreads to be affine or projective, however, by this we shall mean that the system be preserved in the form of equations under affine or projective group of parameters. Thus we can readily see it is necessary and sufficient for affine or projective parameters that the functions, say $T_{a_{(m)}}^i$, on the left of differential equations be transformed under any transformation of the group $u^a \rightarrow \bar{u}^a$ according to

$$T_{a_{(m)}}^i = \bar{u}_{a_1}^{\alpha_1} \bar{u}_{a_2}^{\alpha_2} \dots \bar{u}_{a_m}^{\alpha_m} \bar{T}_{\alpha_{(m)}}^i.$$

By the same arguments as in the case of homogeneous function-systems we may replace the condition by the following relations, the first for affine parameters and both for projective parameters⁽¹⁰⁾:

$$(44) \quad \sum_{r=1}^{m-1} r H_{a_{(m)}; j}^i (cb_{(r-1)}) p_{b_{(r-1)}d}^j = m \delta_{(a_1}^c H_{a_{(m-1)}d}^i,$$

$$(45) \quad \sum_{r=2}^{m-1} \binom{r}{2} H_{a_{(m)}; j}^i (cb_{(r-1)}) p_{b_{(r-1)}}^j = - \binom{m}{2} p_{(a_{(m-1)}}^i \delta_{a_m)}^c.$$

⁽⁹⁾ Compare J. DOUGLAS [1].

⁽¹⁰⁾ H. HOMBU [2], (1.28 b) and (1.37).

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