STABLE HOMOTOPY OF SOME ELEMENTARY COMPLEXES

By
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Introduction

Let $S^n \cup \mathbb{R}^{n+1}$ be a complex obtained from an $n$-sphere $S^n$ by attaching an $(n+1)$-dimensional cell $e^{n+1}$, using a map $S^n \to S^n$ of degree 2. We denote by $\pi_k$ the $k$-th group of the stable homotopy of the complex, i.e.,

$$\pi_k = \text{Dir Lim}_{n \to \infty} [S^{n+k} \cup \mathbb{R}^{n+k+1}, S^n \cup \mathbb{R}^{n+1}].$$

Here the direct limit is taken with respect to suspensions and $\pi_k$ is endowed with the usual structure of an abelian group.

The present note attempts to determine $\pi_k$ by using the results of [1], [5], [6], [9] and [10]. The attempt was directly motivated by [1] and [11].

Let

$$S^n \xrightarrow{2} S^n \xrightarrow{i} S^n \cup \mathbb{R}^{n+1} \xrightarrow{p} S^{n+1} \xrightarrow{2} S^{n+1}$$

be the Puppe’s sequence [8], where the letter 2 denotes a map of degree 2, $i$ an inclusion, and $p$ a map shrinking $S^n$ to the base point of $S^{n+1}$. We shall directly determine the group extension induced by the above sequence (e.g., the typical group extension is given by $0 \leftarrow \mathbb{Z} \leftarrow \mathbb{Z} \leftarrow 0$), using the properties of the Toda brackets.

§ 1.

First we shall fix some notations:

- $CX$ : the cone of a space $X$
- $SX$ : the suspension of $X$
- $Y \cup_f CX$ : the mapping cone of $f: X \to Y$, which is obtained from $Y$ by attaching $CX$, using $f$ as attaching map
- $[X, Y]$ : the set of all homotopy classes of maps $X \to Y$
stable homotopy group of a sphere

\[ \pi_k(2) = \text{Dir Lim}_{n \to \infty} [S^{n+k-1} \cup \mathbb{Z} \mathbb{C}^{n+k}, S^n] \]
\[ \pi_k^*(2) = \text{Dir Lim}_{n \to \infty} [S^{n+k}, S^n \cup \mathbb{Z} \mathbb{C}^{n+1}] \]

For a map \( f: X \to Y \), we often use the same letter \( f \) to denote the homotopy class represented by \( f \), i.e., \( f \in [X, Y] \) when there is no danger of confusion.

**Proposition 1.1.** For any integer \( k \geq 0 \), the following sequence is exact.

\[ 0 \longrightarrow \text{Tor} (G_{k-1}, \mathbb{Z}_2) \longrightarrow \pi_k(2) \longrightarrow G_k \otimes \mathbb{Z}_2 \longrightarrow 0. \]

**Proof.** Let \( n \) be sufficiently large. We consider the following Puppe's sequence.

\[ S^{n+k-1} \longrightarrow S^{n+k-1} \cup \mathbb{Z} \mathbb{C}^{n+k} \longrightarrow S^{n+k} \longrightarrow S^{n+k}. \]

Then we obtain the exact sequence

\[ \pi_{n+k-1}(S^n) \longrightarrow \pi_{n+k}(S^n) \longrightarrow \pi_{n+k}(S^{n+k-1} \cup \mathbb{Z} \mathbb{C}^{n+k}, S^n) \longrightarrow \pi_{n+k}(S^n) \longrightarrow \pi_{n+k}(S^n), \]

leading to the proposition.

If \( k=0 \), we have \( \pi_0(2) \approx \mathbb{Z}_2 \). Its generator is the shrinking map \( p \).

**Proposition 1.2.** For any integer \( k \geq 0 \), the following sequence is exact.

\[ 0 \longrightarrow G_k \otimes \mathbb{Z}_2 \longrightarrow \pi_k^*(2) \longrightarrow \text{Tor} (G_{k-1}, \mathbb{Z}_2) \longrightarrow 0. \]

**Proof.** Let \( X \) be the mapping cylinder of a map \( S^n \to S^n \) of degree 2. The mapping cone \( S^n \cup \mathbb{Z} \mathbb{C}^{n+1} \) is obtained from \( X \) by collapsing \( S^n = S^n \times 0 \subset X \) to the base point. By using Theorem II of [3] we have the isomorphism \( \pi_{n+k}(X, S^n) \approx \pi_{n+k}(S^n \cup \mathbb{Z} \mathbb{C}^{n+1}) \) if \( n \) is sufficiently large. Thus the usual homotopy exact sequence of the pair \((X, S^n)\) implies the following dual Puppe's exact sequence.

\[ \pi_{n+k}(S^n) \longrightarrow \pi_{n+k}(S^n) \longrightarrow \pi_{n+k}(S^n \cup \mathbb{Z} \mathbb{C}^{n+1}) \longrightarrow \pi_{n+k}(S^n) \longrightarrow \pi_{n+k}(S^n) \]

This completes the proof.

If \( k=0 \), then \( \pi_k^*(2) \approx \mathbb{Z}_2 \) and \( i \) is its generator.

**Proposition 1.3.** For any integer \( k \geq -1 \), we have the following exact sequences.

1. \( 0 \longrightarrow \pi_{k+1}(2) \otimes \mathbb{Z}_2 \longrightarrow \pi_k \longrightarrow \text{Tor} (\pi_k, \mathbb{Z}_2) \longrightarrow 0, \)
2. \( 0 \longrightarrow \text{Tor} (\pi_k^*(2), \mathbb{Z}_2) \longrightarrow \pi_k \longrightarrow \pi_{k+1}^*(2) \otimes \mathbb{Z}_2 \longrightarrow 0. \)
PROOF. We obtain (1) by using the following dual Puppe’s exact sequence.

\[
\begin{align*}
[S^{n+k} \cup e^{n+k+1}, S^n] & \xrightarrow{\nu} [S^{n+k} \cup e^{n+k+1}, S^n] \\
& \xrightarrow{i} [S^{n+k} \cup e^{n+k+1}, S^n \cup e^{n+1}] \\
& \xrightarrow{\nu} [S^{n+k} \cup e^{n+k+1}, S^{n+1}] \\
& \xrightarrow{\alpha} [S^{n+k} \cup e^{n+k+1}, S^{n+1}],
\end{align*}
\]

where \( n \) is sufficiently large.

Similarly we obtain (2) by using the Puppe’s exact sequence. If \( k=-1 \), then we have \( \pi_{-1}=\mathbb{Z} \), and its generator is the composite \( ip \).

§ 2.

Now we shall give an account of the Toda bracket. Suppose given a sequence of spaces and maps

\[
W \xrightarrow{h} X \xrightarrow{g} Y \xrightarrow{f} Z
\]
such that \( gh: W \to Y \) and \( fg: X \to Z \) are null-homotopic.

We consider two maps \( \tilde{f}: Y \cup_{g} CX \to Z \) and \( \tilde{h}: SW \to Y \cup_{g} CX \) which are characterized by the following homotopy commutative diagrams.

Here \( Sh: SW \to SX \) denotes the suspension of \( h \), \( i \) an inclusion and \( p \) a map shrinking \( Y \) to the base point of \( SX \). \( \tilde{f} \) and \( \tilde{h} \) are called the extension of \( f \), the coextension of \( h \) respectively.

Put \( \alpha=[f] \), \( \beta=[g] \) and \( \gamma=[h] \), where \( [f] \) denotes the homotopy class represented by \( f \).

The Toda bracket \( \{\alpha, \beta, \gamma\} \) is represented the composite \( \tilde{f}\tilde{h}: SW \to Z \). It is easy to see that \( \{\alpha, \beta, \gamma\} \) is a double coset of two subgroups \( \alpha[SW, Y] \) and \( [SX, Z]_{\gamma} \) in \( [SW, Z] \).

We shall use notations \( \text{Ext} \alpha \) and \( \text{Coext} \gamma \) to denote homotopy classes (or sets of homotopy classes) represented by \( \tilde{f} \) and \( \tilde{h} \) respectively.
We state two properties about extensions and coextensions.

**Proposition 2.1.** Suppose given

\[ \alpha \in [Y, Z], \quad \beta \in [X, Y], \quad \gamma \in [W, X], \quad \alpha' \in [Z, Z'], \quad \gamma' \in [W, W] \]

such that

\[ \alpha \beta = \beta \gamma = 0. \]

Then we have

1. \( \alpha' \text{ Ext } \alpha \subset \text{Ext} (\alpha' \alpha) \),
2. \( (\text{Coext } \gamma) S' \subset \text{Coext} (\gamma' \gamma) \).

The proof is immediate.

**Proposition 2.2.** Suppose given

\[ \alpha_1, \alpha_2 \in [Y, Z], \quad \beta \in [X, Y], \quad \gamma_1, \gamma_2 \in [W, X] \]

such that

\[ \alpha_1 \beta = \alpha_2 \beta = \beta \gamma_1 = \beta \gamma_2 = 0. \]

Then we have

1. \( \text{Ext} (\alpha_1 + \alpha_2) = \text{Ext} \alpha_1 + \text{Ext} \alpha_2 \) if \( X = SX', \ Y = SY' \) and \( \beta = S \beta' \) for some \( \beta' \in [X', Y'] \).
2. \( \text{Coext} (\gamma_1 + \gamma_2) = \text{Coext} \gamma_1 + \text{Coext} \gamma_2 \) if \( W = SW' \).

The equalities hold as cosets of subgroups \([SX, Z](Sp)\) and \(i[SW, Y]\) in (1) and (2) respectively.

The proof is left to the reader.

One of the useful properties of the Toda brackets is given by the following proposition.

**Proposition 2.3.** (see [9], p. 11, Proposition 1.4) Suppose given

\[ \alpha \in [Y, Z], \quad \beta \in [X, Y], \quad \gamma \in [W, X], \quad \delta \in [V, W] \]

such that

\[ \alpha \beta = \beta \gamma = \gamma \delta = 0. \]

Then we have

\[ [\alpha, \beta, \gamma] S \delta = -\alpha [\beta, \gamma, \delta]. \]

The proof is done diagrammatically.

We now quote the basic theorem for computing \( \pi_k \) from [10].
THEOREM A. ([10], p. 307, Corollary) The relation

\[ 2.1 = i_\alpha \beta \]

holds in \( \pi_n \), where \( 1 \) denotes the homotopy class of the identity map of \( S^n \cup_2 e^{n+1} \), and \( \eta \) the generator of \( G_1 \approx \mathbb{Z}_2 \).

Remark that this theorem implies the important one in [9] which is asserted as follows:

\[ [2, \alpha, 2] \equiv \alpha \eta \text{ if } 2\alpha = 0 \text{ for } \alpha \in G_k. \]

From now on we only deal with extensions and coextensions of elements of order 2 with respect to the map \( S^n \to S^n \) of degree 2.

PROPOSITION 2.4. If \( 2\alpha = 0 \), then we have

1. \( \text{Ext } \alpha = [\alpha, 2, \beta] \text{ for } \alpha \in G_k \text{ or } \pi_k^s(2), \)
2. \( \text{Coext } \alpha = [i, 2, \alpha] \text{ for } \alpha \in G_k \text{ or } \pi_k(2). \)

This proposition is a special case of H. Toda's (see [9], p. 14, Proposition 1.8 and Proposition 1.9).

The Toda brackets \([\alpha, 2, \beta]\) and \([i, 2, \alpha]\) are defined since \( i \) and \( \beta \) are of order 2. We can take identity maps as an extension of \( i \) as well as a co-coextension of \( \beta \) respectively. Hence we obtain the proposition.

PROPOSITION 2.5. If \( 2\alpha = 0 \) for \( \alpha \in G_k \), then we have

1. \( 2 \text{Ext } \alpha = i_\alpha \beta \)
2. \( 2 \text{Coext } \alpha = i_\alpha \beta \)

This is a direct consequence of Theorem A.

PROPOSITION 2.6. \( 2 \text{Coext } (\alpha \beta) = 2 \text{Ext } (i\alpha) = i_\alpha \beta \) for any \( \alpha \in G_k \).

The proof is obvious.

PROPOSITION 2.7. Suppose given \( \alpha \in G_k \) such that \( 2\alpha = 0 \) and \( \eta \alpha \) is divisible by 2. Then we have

\[ 2 \text{Coext } (\text{Ext } \alpha) = i_\alpha \beta \text{ Ext } \alpha. \]

In particular assume that \( \eta \alpha = 0 \), then we have

\[ 2 \text{Coext } (\text{Ext } \alpha) = i[\eta, \alpha, 2] \]

By proposition 2.5. (1) and by the assumption, \( 2 \text{Ext } \alpha = \eta \alpha \beta = 0 \). So we can define \( \text{Coext } (\text{Ext } \alpha) \).

By Theorem A, we have
2 Coext (Ext \(\alpha\)) = i \bar{\eta} \text{ Ext } \alpha.

Further if \(\bar{\eta} \alpha = 0\), then by Proposition 2.4. (1) and Proposition 2.3 we have

\[ 2 \text{Coext} (\text{Ext } \alpha) = i \bar{\eta} [\alpha, 2, p] = i [\bar{\eta}, \alpha, 2, p]. \]

Remark. (1) By using Proposition 1.5 of [9] we obtain the following relation for \(\alpha \in G_\bar{k}\) such that \(2\alpha = 0\) and \(\bar{\eta} \alpha\) is divisible by 2:

\[ \text{Ext} (\text{Coext } \alpha) \equiv -\text{Coext} (\text{Ext } \alpha) \mod \{i, [2, \alpha, 2], p\}. \]

(2) Using the fact that the equality \(i \text{ Ext } \alpha = (\text{Coext } \alpha)p\) holds if \(\alpha \in G_\bar{k}\) is divisible by 2, we have

\[ 2 \text{Coext} (\text{Ext } \alpha) = 2 \text{ Ext} (\text{Coext } \alpha). \]

§ 3.

In this section we state our results of computations. They are obtained from the information about the 2-components of \(G_\bar{k}\) and using our propositions prepared in § 1 and § 2.

First we quote the following table from [5], [6] and [9], where \((G_\bar{k}; 2)\) denotes the 2-component of \(G_\bar{k}\).

**Theorem B.**

\[
\begin{array}{cccc}
\text{Generators} & k=0 & k=1 & k=2 & k=3 \\
(G_\bar{k}; 2) & Z & Z_2 & Z_3 & Z_8 \\
\eta & \eta & \bar{\eta} & \nu & \eta \\
k=4 & k=5 & k=6 & k=7 & k=8 \\
0 & 0 & Z_2 & Z_{16} & Z_2 + Z_2 \\
\nu & \sigma & \eta & \eta \sigma, \eta & \eta \\
k=9 & k=10 & k=11 & k=12 & k=13 \\
Z_2 + Z_2 + Z_2 & Z_2 & Z_8 & 0 & 0 \\
\eta \sigma, \bar{\eta} \epsilon, \mu & \eta \mu & \zeta & & \\
k=14 & k=15 & k=16 & k=17 & k=18 \\
Z_2 + Z_2 & Z_2 + Z_2 & Z_2 + Z_2 + Z_2 & Z_2 + Z_2 & Z_8 + Z_2 \\
\sigma^2, \epsilon & \rho, \eta & \eta \sigma, \bar{\eta}^* & \eta \rho, \eta \bar{\eta}^*, \nu \epsilon, \bar{\mu} & \nu ^*, \eta \bar{\mu} \\
k=19 & k=20 & k=21 & k=22 & \\
Z_8 + Z_2 & Z_8 & Z_2 + Z_2 & Z_2 + Z_2 & \\
\zeta, \sigma & \bar{\kappa} & \eta \kappa, \sigma^3 & \epsilon \kappa, \nu \sigma & \\
\end{array}
\]
The relations in $(G; 2) = \sum (G_k; 2)$ which are necessary for our computations are as follows:

Theorem C.

$$4\nu = \eta^3, \quad 4\zeta = \eta^2\mu, \quad 4\nu^* = \eta^2\eta^*, \quad 4\zeta^* = \eta^2\eta^*,$$

$$4\xi = \eta^2\xi, \quad 4\kappa = \nu^2\kappa, \quad \epsilon = \eta^2\kappa,$$

$$\eta_\nu = 0, \quad \eta_\sigma^2 = 0, \quad \eta_\xi^2 = 0,$$

$$\eta_\kappa = 0, \quad \eta_\nu^* = 0, \quad \eta_\sigma^* = 0,$$

$$\eta_\xi^* = 0.$$

It remains to examine the Toda brackets $\{\eta, \alpha, 2\}$ for $\alpha \in G_k$ such that $2\alpha = 0$ and $\eta_\alpha = 0$. The following theorem enables us to complete the computations of $\pi_k$ for $k \leq 21$.

Theorem D.

$$[\eta, \eta^2, 2] \ni \epsilon,$$

$$[\eta, \eta^2\sigma, 2] \ni \zeta,$$

$$[\eta, \sigma^2, 2] \ni \eta^*, \quad [\eta, \eta_\kappa, 2] \ni \nu_\kappa,$$

$$[\eta, 16\rho, 2] \ni \eta^2\xi,$$

$$[\eta, \sigma^2, 2] \ni \eta_\sigma^*.$$

The results of Theorem C and Theorem D are almost found in [6], [7] and [9]. The proof of the ones which we can not find there will be given in the next section.

We now compute $\pi_*(2)$ for $k \leq 22$, using Prop. 1.1, Prop. 2.1.1, Prop. 2.5.1, Th. B and Th. C. The results are as follows:

Theorem 3.1.

$$\begin{array}{cccc}
\pi_*(2) & k=0 & k=1 & k=2 & k=3 \\
\approx & Z_2 & Z_2 & Z_4 & Z_2 + Z_2 \\
Generators & p & \eta p & \text{Ext} \eta & \eta \text{Ext} \eta, \nu^2 p \\
k=4 & k=5 & k=6 & k=7 \\
Z_2 & 0 & Z_2 & Z_2 + Z_2 \\
\eta^2 \text{Ext} \eta & \nu^2 p & \text{Ext} \nu^2, \sigma p \\
k=8 & k=9 & k=10 & k=11 \\
Z_2 + Z_2 + Z_2 & Z_4 + Z_4 + Z_2 & Z_4 + Z_2 + Z_2 & Z_2 + Z_2 \\
\text{Ext} 8\sigma, \eta \sigma p, ep & \text{Ext} \epsilon, \sigma \text{Ext} \eta, \nu p & \text{Ext} \mu, \eta \sigma \text{Ext} \eta, \eta \text{Ext} \epsilon & \eta \text{Ext} \mu, \eta \xi^2 p \\
\end{array}$$
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\[
\begin{array}{ccc}
  k=12 & k=13 & k=14 \\
  Z_2 & 0 & Z_4+Z_2 \\
  \gamma^2 \text{ Ext } \mu & \gamma^2 \text{ Ext } \mu & \sigma^2 \text{p, } \kappa \text{p} \\
  k=15 & k=16 & k=17 \\
  Z_4+Z_2+Z_2 & Z_4+Z_2+Z_2+Z_2 & Z_4+Z_2+Z_2+Z_2 \\
\text{Ext } \kappa, \text{ Ext } \sigma^2, \rho \text{p} & \text{Ext } 16 \iota, \eta \text{ Ext } \kappa, \eta \mu \text{p, } \eta^* \text{p} & \text{Ext } \eta^*, \rho \text{ Ext } \eta, \overline{\eta} \rho, \nu \kappa \text{p} \\
  k=18 & k=19 & k=20 \\
  Z_4+Z_4+Z_2+Z_2 & Z_4+Z_2+Z_2+Z_2 & Z_2+Z_2+Z_2 \\
\text{Ext } \mu, \eta \rho \text{ Ext } \eta, \gamma \text{ Ext } \gamma^*, \gamma \text{ Ext } \overline{\mu}, \gamma^2 \text{ Ext } \gamma^*, \xi \text{p, } \rho \sigma \text{p} & \gamma^2 \text{ Ext } \overline{\mu}, \text{ Ext } \sigma, \kappa \text{p} & \\
  \nu \text{ Ext } \kappa, \nu^* \text{p} & & \\
  k=21 & k=22 & \\
  Z_2+Z_2+Z_2 & Z_4+Z_2+Z_2 & \\
  \nu^2 \text{ Ext } \kappa, \eta \kappa \text{p, } \sigma^2 \text{p} & \kappa \text{ Ext } \eta, \sigma \text{ Ext } \sigma^2, \nu \sigma \text{p} & \\
\end{array}
\]

Relations:

\[
\begin{align*}
2 \text{ Ext } \eta &= \eta^2 \text{p, } \\
2 \text{ Ext } \epsilon &= \epsilon \kappa \text{p, } \\
2 \text{ Ext } \kappa &= \kappa \kappa \text{p, } \\
2 \rho \text{ Ext } \eta &= \rho \eta^2 \text{p, } \\
2 \overline{\kappa} \text{ Ext } \eta &= \gamma^2 \kappa \text{p}. \\
\end{align*}
\]

We give an example of our computations:

**Example.** Let us compute \( \pi_{10}(2) \). Using Proposition 1.1 and Theorem B, we obtain the following short exact sequence:

\[
\begin{array}{c}
0 \longrightarrow [\eta^2 \sigma, \eta \epsilon, \mu] \longrightarrow \pi_{10}(2) \longrightarrow [\eta \mu] \longrightarrow 0. \\
\end{array}
\]

Choose the elements \( \eta \sigma \text{ Ext } \eta, \gamma \text{ Ext } \epsilon \) and \( \text{ Ext } \mu \) in \( \pi_{10}(2) \), then \( i^* \) maps them into the elements \( \gamma^2 \sigma, \gamma \epsilon \) and \( \mu \) in \( G_6 \) respectively. Since \( 2 \rho \sigma \text{ Ext } \eta = 2 \gamma \text{ Ext } \epsilon = 0 \) and by Proposition 2.5.1 the relation \( 2 \text{ Ext } \mu = \gamma \mu \text{p} \) holds, we can determine the group extension, i.e., we have

\[
\pi_{10}(2) = \{ \text{Ext } \mu, \eta \sigma \text{ Ext } \eta, \gamma \text{ Ext } \epsilon \} \approx Z_4+Z_2+Z_2.
\]

Exactly dual results are obtained for \( \pi_7(2) \).
THEOREM 3.2.

<table>
<thead>
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<th>k = 0</th>
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<th>k = 2</th>
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<td>$\pi^2(2)$</td>
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<td>$Z_4$</td>
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<tr>
<td>Generators</td>
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<td>$i\eta$</td>
</tr>
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<td>$k = 5$</td>
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<td>$0$</td>
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<td>(Coext $\eta$)$\gamma$, $i\nu$</td>
<td>(Coext $\eta$)$\gamma^2$</td>
<td>$i\nu^2$</td>
</tr>
<tr>
<td>$k = 7$</td>
<td>$k = 8$</td>
<td>$k = 9$</td>
</tr>
<tr>
<td>$Z_2 + Z_2$</td>
<td>$Z_2 + Z_2 + Z_2$</td>
<td>$Z_4 + Z_4 + Z_2$</td>
</tr>
<tr>
<td>Coext $\nu^2$, $i\sigma$</td>
<td>Coext $(8\sigma)$, $i\eta\sigma$, $i\epsilon$</td>
<td>Coext $\epsilon$, (Coext $\eta$)$\sigma$, Coext $\mu$, (Coext $\epsilon$)$\gamma$</td>
</tr>
</tbody>
</table>

| $k = 11$ | $k = 12$ | $k = 13$ | $k = 14$ |
| $Z_2 + Z_2$ | $Z_2$ | $0$ | $Z_2 + Z_2$ |
| (Coext $\mu$)$\gamma$, $i\xi$ | (Coext $\mu$)$\gamma^2$ | $i\xi^2$, $i\epsilon$ |

| $k = 15$ | $k = 16$ |
| $Z_4 + Z_2 + Z_2$ | $Z_2 + Z_2 + Z_2 + Z_2$ |
| Coext $\kappa$, Coext $\sigma^2$, $i\rho$ | Coext $(16\sigma)$, (Coext $\kappa$)$\gamma$, $i\eta\mu$, $i\gamma^*$ |

| $k = 17$ | $k = 18$ |
| $Z_4 + Z_4 + Z_2 + Z_2$ | $Z_4 + Z_2 + Z_2 + Z_2 + Z_2$ |
| (Coext $\eta$)$p$, Coext $\eta^*$, $i\kappa$, $i\mu$ | Coext $\mu$, (Coext $\eta$)$\eta^p$, (Coext $\eta^*$)$\eta$ | (Coext $\kappa$)$\nu$, $i\nu^*$ |

| $k = 19$ | $k = 20$ |
| $Z_2 + Z_2 + Z_2 + Z_2$ | $Z_2 + Z_2 + Z_2$ |
| (Coext $\mu$)$\gamma$, (Coext $\eta$)$\gamma^2$, $i\xi$, $i\sigma$ | (Coext $\mu$)$\gamma^2$, Coext $\sigma$, $i\xi$ |

| $k = 21$ | $k = 22$ |
| $Z_2 + Z_2 + Z_2$ | $Z_4 + Z_2 + Z_2$ |
| (Coext $\kappa$)$\nu^2$, $i\mu\kappa$, $i\sigma^3$ | (Coext $\eta$)$\kappa$, (Coext $\sigma^2$)$\sigma$, $i\nu\kappa$ |

Relations:

- $2\text{ Coext }\eta = i\eta^2$,
- $2\text{ Coext }\epsilon = i\eta\epsilon$,
- $2\text{ Coext }\kappa = i\eta\kappa$,
- $2\text{ Coext }\eta = i\eta\gamma^*$,
- $2\text{ Coext }\eta = i\eta\gamma^*$,
- $2\text{ Coext }\eta = i\gamma^2\kappa$.

- $2(\text{Coext }\eta)\sigma = i\gamma^2\sigma$,
- $2\text{ Coext }\mu = i\eta\mu$,
- $2(\text{Coext }\eta)p = i\gamma^2p$,
- $2\text{ Coext }\mu = i\eta\mu$.
For the proof of this theorem we make use of Prop. 1.2, Prop. 2.1.2, Prop. 2.5.2, Th. B and Th. C.

Now we state our main theorem. It partly generalizes Theorem 4.1 of [2].

**Theorem 3.3.**

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<th>$\pi_k \cong$</th>
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<th>$k=0$</th>
<th>$k=1$</th>
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<td>Generators</td>
<td>$Z_2$</td>
<td>$Z_4$</td>
<td>$Z_2+Z_2$</td>
</tr>
<tr>
<td>$k=2$</td>
<td>$i\gamma p$</td>
<td>1</td>
<td>$i\operatorname{Ext} \gamma$, $(\operatorname{Coext} \gamma)p$</td>
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</tr>
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<td>$Z_4+Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td></td>
</tr>
<tr>
<td>$i\gamma \operatorname{Ext} \gamma$, $(\operatorname{Coext} \gamma)\gamma p$, $i\gamma p$</td>
<td>$(\operatorname{Coext} \gamma)(\operatorname{Ext} \gamma)$, $(\operatorname{Coext} \gamma)\gamma(\operatorname{Ext} \gamma)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k=5$</td>
<td>$k=6$</td>
<td>$k=7$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Z_2$</td>
<td>$Z_2+Z_2+Z_2$</td>
<td>$Z_4+Z_4+Z_2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$i\gamma p$</td>
<td>$i\operatorname{Ext} \gamma$, $(\operatorname{Coext} \gamma)\gamma p$, $i\operatorname{Ext} \gamma$, $(\operatorname{Coext} \gamma)\gamma p$</td>
<td>Coext $(\gamma p)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k=8$</td>
<td>$k=9$</td>
<td>$k=10$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Z_4+Z_4+Z_4+Z_4$</td>
<td>$Z_4+Z_4+Z_2+Z_2$</td>
<td>Coext $(\operatorname{Ext} \gamma)$, $i\gamma \operatorname{Ext} \gamma$, $(\operatorname{Coext} \gamma)\gamma p$, $(\operatorname{Coext} \gamma)(\operatorname{Ext} \gamma)$, $(\gamma \gamma)(\operatorname{Ext} \gamma)$, $(\operatorname{Coext} \gamma)\gamma(\operatorname{Ext} \gamma)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k=11$</td>
<td>$k=12$</td>
<td>$k=13$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Z_4+Z_2$</td>
<td>$Z_2$</td>
<td>Coext $(\gamma p)$, Coext $(\zeta p)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(\operatorname{Coext} \gamma)(\operatorname{Ext} \gamma)$, Coext $(\gamma p)$</td>
<td>$(\operatorname{Coext} \gamma)\gamma(\operatorname{Ext} \gamma)$</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>$k=14$</td>
<td>$k=15$</td>
<td>$k=16$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Z_2+Z_2$</td>
<td>$Z_4+Z_4+Z_2+Z_2$</td>
<td>$i\gamma p$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$i\gamma p$, $i\gamma p$</td>
<td>$i\operatorname{Ext} \gamma$, $(\operatorname{Coext} \gamma)\gamma p$, $i\operatorname{Ext} \gamma$, $(\operatorname{Coext} \gamma)\gamma p$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k=15$</td>
<td>$k=16$</td>
<td>$k=17$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Z_4+Z_4+Z_4+Z_2+Z_2$</td>
<td>Coext $(\gamma p)$, Coext $(\gamma p)$, $i\gamma \operatorname{Ext} \gamma$, $(\operatorname{Coext} \gamma)\gamma p$, $i\operatorname{Ext} (16p)$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\[ k = 16 \]
\[ Z_k + Z_k + Z_k + Z_k + Z_k + Z_k \]
\[
(\text{Coext } \gamma)(\text{Ext } \kappa), \ (\text{Coext } \text{Ext } 16\rho), \ i\nu \text{ Ext } \gamma, \ (\text{Coext } \gamma)\rho p, \ i \text{ Ext } \gamma^*, \ (\text{Coext } \gamma^*)p
\]

\[ k = 17 \]
\[ Z_k + Z_k + Z_k + Z_k + Z_k + Z_k + Z_k + Z_k \]
\[
\ i\eta (\text{Ext } \gamma), \ (\text{Coext } \eta)\eta p, \ i \text{ Ext } \mu, \ (\text{Coext } \mu)p, \ i\nu \text{ Ext } \gamma^*, \ (\text{Coext } \gamma^*)\gamma p, \ i\nu \text{ Ext } \kappa, \ (\text{Coext } \kappa)\nu p, \ i\nu^* p
\]

\[ k = 18 \]
\[ Z_k + Z_k + Z_k + Z_k + Z_k + Z_k + Z_k + Z_k \]
\[
(\text{Coext } \gamma)(\text{Ext } \gamma^*), \ (\text{Coext } \gamma)\rho(\text{Ext } \eta), \ i\eta \text{ Ext } \mu, \ (\text{Coext } \mu)\eta p, \ 
(\text{Coext } \nu \text{ Ext } \kappa), \ (\text{Coext } \nu^* p), \ i\sigma p
\]

\[ k = 19 \]
\[ Z_k + Z_k + Z_k + Z_k + Z_k + Z_k + Z_k + Z_k \]
\[
(\text{Coext } \gamma)(\text{Ext } \mu), \ (\text{Coext } \gamma)(\text{Ext } \mu), \ i\nu \text{ Ext } \kappa, \ (\text{Coext } \sigma)\gamma p, \ i\kappa p
\]

\[ k = 20 \]
\[ Z_k + Z_k + Z_k + Z_k + Z_k + Z_k + Z_k + Z_k \]
\[
(\text{Coext } \kappa p), \ (\text{Coext } \gamma)(\text{Ext } \mu), \ i\nu^2 \text{ Ext } \kappa, \ i\nu^2 p, \ (\text{Coext } \text{Ext } \sigma + a\kappa p), \ 
\text{where } a = 0 \text{ or } 1
\]

\[ k = 21 \]
\[ Z_k + Z_k + Z_k + Z_k + Z_k + Z_k + Z_k + Z_k \]
\[
(\text{Coext } \nu^2)(\text{Ext } \kappa), \ i\kappa \text{ Ext } \gamma, \ (\text{Coext } \gamma)\kappa p, \ i\sigma \text{ Ext } \sigma^2, \ (\text{Coext } \sigma^2)\sigma p, \ i\sigma p
\]

**Relations:**

\[ 2 \text{ (Coext } \gamma)(\text{Ext } \eta) = i\eta\rho \text{ Ext } \eta , \]
\[ 2 \text{ Coext } (\sigma p) = i\gamma\sigma p , \]
\[ 2 \text{ Coext } (\sigma p) = i\gamma\sigma p , \]
\[ 2 \text{ (Coext } \gamma)(\text{Ext } \eta) = i\rho \text{ Ext } \eta , \]
\[ 2 \text{ Coext } \text{Ext } 8 \sigma = i\mu p , \]
\[ 2 \text{ Coext } \text{Ext } 8 \sigma = i\mu p , \]
\[ 2 \text{ Coext } (\text{Ext } \sigma^2) = i\eta\rho \text{ Ext } \sigma^2 , \]
\[ 2 \text{ Coext } (\sigma^2) = i\rho p , \]
\[ 2 \text{ Coext } \text{Ext } \kappa = i\nu \kappa p , \]
\[ 2 \text{ Coext } \text{Ext } \kappa = i\nu \kappa p , \]
\[ 2 \text{ (Coext } \gamma)(\text{Ext } \eta^*) = i\nu \rho \text{ Ext } \eta^* , \]
\[ 2 \text{ Coext } \gamma)(\text{Ext } \eta^*) = i\nu \rho \text{ Ext } \eta^* , \]
\[ 2 \text{ (Coext } \gamma)(\text{Ext } \mu) = i\kappa p , \]
\[ 2 \text{ Coext } (\text{Ext } \mu) = i\kappa p , \]

We prove this theorem by using Prop. 1.3.1, Prop. 2.1.2, Prop. 2.2.2, Prop. 2.6, Prop. 2.7, Th. 3.1, Th. C and Th. D. We shall show the computations of say \( \pi_7, \pi_{14} \) and \( \pi_{20} \). All of the remainders can be computed by trying the similar argument.
The computation of \( \pi_7 \).

By Th. 3.1, \( \pi_7(2)=|\text{Ext} \, \nu^2, \sigma p|=Z_2+Z_2 \) and \( \pi_6(2)=|\text{Ext} \, 8\sigma, \eta \sigma p, \epsilon p|=Z_2+Z_2+Z_2 \). It follows from Prop. 1.3.1 that we obtain the short exact sequence

\[
0 \longrightarrow \pi_6(2) \xrightarrow{i^*} \pi_7 \xrightarrow{p^*} \pi_7(2) \longrightarrow 0.
\]

Choose the elements \( \text{Coext} (\text{Ext} \, \nu^2) \) and \( \text{Coext} (\sigma p) \) in \( \pi_7 \) which project through \( p_* \) the elements \( \text{Ext} \, \nu^2 \) and \( \sigma p \) in \( \pi_7(2) \) respectively. By use of Prop. 2.6, \( 2 \text{Coext} (\sigma p)=i\eta \sigma p \), and so \( \text{Coext} (\sigma p) \) is of order 4. By Theorem C, \( \eta \nu^2=\eta \nu \nu=0 \). It follows by using Prop. 2.7 and Th. D that \( 2 \text{Coext} (\text{Ext} \, \nu^2)=i[\eta, \nu^2, 2]p=i\epsilon p \mod i\eta \sigma p \).

Hence we have

\[
\pi_7 = \langle \text{Coext} (\text{Ext} \, \nu^2), \text{Coext} (\sigma p), i \text{Ext} \, 8\sigma \rangle \approx Z_4+Z_4+Z_2.
\]

The computation of \( \pi_{14} \).

By Th. 3.1, \( \pi_{14}(2)=|\sigma^2 p, \epsilon p|=Z_2+Z_2 \) and \( \pi_{13}(2)=|\text{Ext} \, \kappa, \text{Ext} \, \sigma^2, \rho p|=Z_4+Z_2+Z_2 \). It follows from Prop. 1.3.1 that we have the short exact sequence

\[
0 \longrightarrow [i \text{Ext} \, \kappa, i \text{Ext} \, \sigma^2, i\epsilon p] \longrightarrow \pi_{14} \xrightarrow{p_{14}} \pi_{14}(2) \longrightarrow 0.
\]

By use of Prop. 2.1.2, \( \text{Coext} (\sigma^2 p)=(\text{Coext} \, \sigma^2)p \) and \( \text{Coext} (\epsilon p)=(\text{Coext} \, \kappa)p \). Since \( 2(\text{Coext} \, \sigma^2)p=2(\text{Coext} \, \kappa)p=0 \), the above exact sequence splits, i.e.,

\[
\pi_{14} = \langle i \text{Ext} \, \sigma^2, (\text{Coext} \, \sigma^2)p, i \text{Ext} \, \kappa, (\text{Coext} \, \kappa)p, i\epsilon p \rangle \approx Z_4+Z_4+Z_2+Z_2.
\]

The computation of \( \pi_{20} \).

Consider the short exact sequence

\[
0 \longrightarrow \pi_{20}(2) \xrightarrow{i^*} \pi_{20} \xrightarrow{p^*} \pi_{20}(2) \longrightarrow 0,
\]

where \( \pi_{20}(2)=|\nu^2 \text{Ext} \, \bar{\nu}, \text{Ext} \, \bar{\sigma}, \bar{\kappa} p|=Z_2+Z_2+Z_2 \) and \( \pi_{21}(2)=|\nu^2 \text{Ext} \, \kappa, \eta \kappa p, \sigma^2 p|=Z_2+Z_2+Z_2 \) by Th. 3.1. Choose the elements \( \langle \text{Coext} \, \eta \gamma (\text{Ext} \, \bar{\nu}), \text{Coext} \, (\bar{\kappa} p) \rangle \) and \( \text{Coext} (\text{Ext} \, \bar{\sigma}) \) in \( \pi_{20} \) which project through \( p_* \) the elements \( \gamma \bar{\nu}, \eta \bar{\kappa} p \) and \( \text{Ext} \, \bar{\sigma} \) respectively. \( 2(\text{Coext} \, \eta \gamma (\text{Ext} \, \bar{\nu}))=0 \), and \( 2(\text{Coext} \, (\bar{\kappa} p))=i\eta \kappa \bar{p} \) by Proposition 2.6. On the other hand by Prop. 2.7 and Theorem D, \( 2 \text{Coext} (\text{Ext} \, \bar{\sigma})=i[\eta, \bar{\sigma}, 2]p \geq 0 \). Since \( [\eta, \bar{\sigma}, 2] \) is a coset of \( \gamma(G_{20}; 2)+2G_{19}=|\gamma \bar{\kappa}|+2G_{21} \), we have \( 2 \text{Coext} (\text{Ext} \, \bar{\sigma})=a i\eta \kappa \bar{p} \) for some \( a \geq 0 \). By use of Proposi-
2.2.2. \[ \text{Coext}((\text{Ext } \bar{\sigma} + a\bar{\kappa}p)) = \text{Coext}((\text{Ext } \bar{\sigma}) + a \text{Coext}(\bar{\kappa}p)) \quad \text{and thus} \]
\[ 2 \text{Coext}((\text{Ext } \bar{\sigma} + a\bar{\kappa}p)) = 0. \]
Hence we obtain
\[ \pi_{20} = \{ \text{Coext}(\bar{\kappa}p), \{ \text{Coext} \eta \} \gamma(\text{Ext } \bar{\mu}), i\bar{\nu}^2 \text{Ext } \kappa, i\sigma^2p, \text{Coext}((\text{Ext } \bar{\sigma} + a\bar{\kappa}p)) \} \]
\[ \approx Z_1 + Z_2 + Z_2 + Z_2 + Z_2. \]

§ 4.

In this section we shall complete the proof of Theorem C and Theorem D in the previous section.

First we begin with Theorem C. We can find all the relations in Theorem 14.1 of [9] except for the following ones: \[ 4\bar{\kappa} = \nu^2 \kappa, \quad \kappa \varepsilon = \gamma \bar{\nu} \kappa, \quad \gamma \bar{\sigma} = 0 \quad \text{and} \quad \gamma \bar{\zeta} = 0. \]
The relation \[ 4\bar{\kappa} = \nu^2 \kappa \] is obtained from Lemma 15.4 of [6]. Therefore it is sufficient for us to prove the following lemma.

**Lemma 4.1.** (1) \[ \gamma \bar{\sigma} = \gamma \bar{\zeta} = 0, \]
(2) \[ \kappa \varepsilon = \gamma \bar{\nu} \kappa. \]

**Proof of (1).** By the definition, \[ \bar{\sigma} \in [\nu, \nu + \varepsilon, \sigma]. \]
By using (3.6) of [9], we have \[ \gamma \bar{\sigma} \in [\eta, \eta + \sigma, \sigma] = [\eta, \nu, \nu + \varepsilon + \sigma] = 0, \]
since \[ [\eta, \nu, \nu + \varepsilon] \] is a subset of \((G_{13}; 2) = 0. \]
Similarly by the definition \[ \bar{\zeta} \in [\zeta, 8, 2\sigma]. \]
Therefore \[ \gamma \bar{\zeta} \in \gamma [\zeta, 8, 2\sigma] = [\eta, \zeta, 8]2\sigma = 0. \]

**Proof of (2).** By Lemma 6.1 of [4], \[ \gamma \bar{\kappa} \in [\nu^2, 2, \kappa]. \]
By use of (3.6) and (6.1) of [9], we have \[ \gamma \bar{\kappa} \in [\nu^2, 2, \kappa] = [\eta, \nu^2, 2\kappa] \approx \kappa \varepsilon. \]
\[ [\eta, \nu^2, 2\kappa] \] is a coset of \( \gamma(G_{17}; 2) = [\sigma] \approx Z_1, \)
\[ G_6 \approx Z_2 + Z_2 + Z_2 + Z_2 \quad \text{and} \quad \sigma \kappa = 0 \]
from Proposition 7.2 of [5]. Hence we have \[ \gamma \bar{\kappa} = \kappa \varepsilon. \]

Next we proceed to Theorem D. We know the following results:

\[ [\eta, \nu^2, 2] \gamma \varepsilon, \quad \text{by (6.1) of [9].} \]
\[ [\eta, 8\sigma, 2] \gamma \mu, \quad \text{see page 235 of [7].} \]
\[ [\eta, \eta, 2] \gamma \zeta, \quad \text{by Lemma 9.1 of [9].} \]
\[ [\eta, \eta, 2] \gamma \nu, \quad \text{by Lemma 15.1 of [6].} \]
\[ [\eta, 16\rho, 2] \gamma \mu, \quad \text{see page 237 of [7].} \]

Therefore to complete the proof of Theorem D we must prove the following lemma.

**Lemma 4.2.** (1) \[ [\eta, \eta \sigma, 2] \gamma \zeta, \]
(2) \[ [\eta, \sigma^2, 2] \gamma \sigma, \]
(3) \[ [\eta, \eta \rho, 2] \gamma \zeta, \]
(4) \[ [\eta, \sigma, 2] \gamma 0. \]
Proof of (1). By (iii) of (3.5) of [9], \([\eta, \eta^s, \sigma, 2] \supseteq \{\sigma^s, \eta, 2\}\) and \([\eta^s, \eta, 2] \subset \{\sigma, \eta^s, 2\}\). \((\eta, \eta^s, \sigma, 2)\) is a coset of \(G_{10} ; 2 + 2G_{11} = [\eta^s, \mu] + 2G_{11} = 2G_{11}\), since \((G_{10} ; 2) = [\eta^s, \mu] \approx Z_2\) and \(\eta^s, \mu \equiv 4\zeta\). On the other hand \([\eta^s, \eta, 2] = \{\eta^s, \sigma, 2\}\) is a coset of \([\eta^s, \eta^s, 2] + 2G_{11} = 2G_{11}\), since \(\eta^s, 2 = 0\). Similarly since \(G_s = 0\), \([\sigma, \eta^s, 2] = \{\sigma, \eta^s, 2\}\). It follows from Lemma 9.1 of [9] that \(\{\eta, \eta^s, 2\} = [\eta^s, \sigma, 2] = \{\sigma, \eta^s, 2\} \equiv \zeta\).

Proof of (2). By (ii) of (3.9) of [9],

\[\{\eta, \sigma^s, 2\} + \{\sigma^s, 2, \eta\} + \{2, \eta, \sigma^s\} \equiv 0.\]

\([2, \eta, \sigma^s]\) is a coset of \(2G_{10} + [\eta^s, \sigma^s] = 0\) since \(G_{10} = [\eta^s, \eta^s] \approx Z_2 + Z_2\) and \(\eta^s, 2 = 0\). Thus \([2, \eta, \sigma^s]\) consists of a single element. Assume that \([2, \eta, \sigma^s] \equiv 0\), i.e., \([2, \eta, \sigma^s] = \eta^s, \eta^s + \eta^s, \eta^s\). Since \(\eta^s, \eta^s\) and \(\eta^s, \eta^s\) are linearly independent in \(G_{11}\), we have \([2, \eta, \sigma^s] = 0\). On the other hand by (3.6) of [9] and by Lemma 5.2 of [9], \(\{2, \eta, \sigma^s\} = \{\eta, 2, \eta \sigma^2 \equiv 2, \eta^s, 2 = 0\). This is a contradiction. So \([2, \eta, \sigma^s] = 0\) and hence

\[[\eta, \sigma^s, 2] + \{\sigma^s, 2, \eta\} \equiv 0.\]

By (iii) of (3.5) of [9], \([\sigma^s, 2, \eta] \subset \{\sigma, 2, \sigma, \eta\}\). The latter contains \(\eta^s\) by the definition. \([\sigma^s, 2, \eta]\) is a coset of \([\sigma^s, \eta^s] + (G_{11} ; 2) = [\eta^s, \sigma^s] = [\eta^s, \eta^s] = 0\) and \((G_{11} ; 2) = [\eta^s, \eta^s] \approx Z_2 + Z_2\). On the other hand \([\sigma, 2, \eta]\) is a coset of \([\sigma, \eta^s, 2] + (G_{11} ; 2) = [\sigma, \eta^s, \sigma^s, \mu] + [\eta^s, \sigma^s, \eta^s, \mu] = [\eta^s, \eta^s]\) since \(\sigma^s, 2 = 0\), and \(\sigma^s, \mu = \eta^s\) by Theorem 14.1 of [9]. Therefore \(\eta^s \in [\sigma, 2, \sigma, \eta] = [\sigma^s, 2, \eta]\). Hence we have

\[\eta^s \in [\eta, \sigma^s, 2].\]

Proof of (3). By i) of (3.9) of [9], \([\eta, \eta^p, 2] = [2, \eta^p, \eta]\). By Theorem 14.1 of [9] and by ii) of (3.5) of [9], \([2, \eta^p, \eta] = [2, \eta^p, \eta] \supset \{\sigma, \eta, \eta\}\) respectively. It is easy to see that these are cosets of \(2G_{10}\) respectively. So we have \([2, \eta^p, \eta] = [2, \eta^p, \eta]\).

By using (3.7) of [9], we have

\[\{[2, \eta^p, \eta] + [2, \eta^p, \eta], \eta] - [2, \eta^p, \eta] \equiv 0.\]

Since \([2, \eta^p, \eta] = 0\) and \([2, \eta^p, \eta] \supset \{\sigma, \eta, \eta\}\), we get \([2, \eta^p, \eta] = 0\). Therefore \([2, \eta^p, \eta] = [0, \eta, \eta] \equiv 0.\) It is a coset of \([\eta^p, (G_{10} ; 2) = [\eta^p, \eta^p] = [\eta^p, \eta^p] = 0\) and thus it consists of the single element 0. It follows from the fact \(\mu \in [2, \eta^p, \eta] = [2, \eta^p, \eta] = [2, \eta^p, \eta] \equiv 0.\) By the definition of \(\zeta\) and by ii) of (3.5) and (i) of (3.9) of [9],
we have \( \bar{Z} \in [\zeta, 8, 2, \sigma] \subset [\zeta, 8, \sigma, 2] = [2, 8, \sigma, \zeta] \). Hence we have \( \{\eta, \nu \eta^* \} = \{2, \mu, \nu \eta \} \approx \bar{Z} \).

**Proof of (4).** By (ii) of (3.9) of [9], we have

\[
\{\eta, \bar{a}, 2\} - \{\bar{a}, 2, \eta\} + [2, \eta, \bar{a}] \equiv 0.
\]

We shall show \( 0 \in [2, \eta, \bar{a}] \). By the relation \( \eta \sigma = \nu + \varepsilon \) and by (ii) of (3.5) of [9], \( \bar{a} \in [\nu, \nu + \varepsilon, \sigma] = [\nu, \eta \sigma, \sigma] \). \( \{\nu, \eta \sigma, \sigma\} \) is a coset of \( \nu(G_1; 2) + \sigma(G_1; 2) = \nu \eta \sigma = 0 \) by Theorem 14.1 of [9]. Therefore we have \( \bar{a} = [\nu, \nu + \varepsilon, \sigma] = [\nu, \eta \sigma, \sigma] \). By using (3.7) of [9], \( [2, \eta, \nu, \sigma] + [2, \eta, \nu, \sigma] = [2, \eta, \nu, \sigma] - [2, \eta, \nu, \sigma] \equiv 0 \). Since \( G_3 = G_1 = (G_1; 2) = 0 \), \( [2, \eta, \nu, \sigma] = [0, \sigma, \eta \sigma] = \eta \sigma(G_1; 2) = 0 \) and \( [2, \eta, \nu, \sigma] = [2, 0, \sigma] = 2G_2 \). Therefore \( [2, \eta, \bar{a}] \equiv 0 \).

Thus we have \( \{\eta, \bar{a}, 2\} - \{\bar{a}, 2, \eta\} \equiv 0 \). To obtain the assertion (4) it is sufficient to show \( \{\bar{a}, 2, \eta\} \equiv 0 \).

Consider the exact sequence

\[
\pi_{12}^{13} \xrightarrow{d} \pi_{23}^{12} \xrightarrow{E} \pi_{26}^{12},
\]

where \( \pi_{12}^{13} = [\sigma_{13}, \kappa_{13}] \approx Z_{16} + Z_2 \) (See (4.4) of [9]). By (12.17) of [9], \( 2\bar{a}_7 = 0 \). Therefore by the above exact sequence we have \( 2\bar{a}_6 = d(x\sigma_{12} + y\kappa_{13}) \), where \( x \) and \( y \) are integers. By use of Proposition 2.6 of [9], we obtain

\[
H(\bar{a}_7, 2\bar{a}_6, \eta_{20}) = -J^{-1}(2\bar{a}_6)\eta_{27} \equiv x\sigma_{12} \eta_{27} + y\kappa_{13} \eta_{27}.
\]

By (7.1) and (10.18) of [9], \( \sigma_{12} \eta_{27} = \sigma_{13}(\bar{a}_{20} + \varepsilon_{20}) = 0 \). By (10.23) of [9], \( \kappa_{12} \eta_{27} = \eta_{13} \kappa_{14} \). Thus \( H(\bar{a}_7, 2\bar{a}_6, \eta_{20}) \equiv y\eta_{13} \kappa_{14} \). On the other hand, by Proposition 2.2 and Lemma 5.14 of [9], we have \( H(\sigma^* \kappa_{14}) = H(\sigma^* \kappa_{14}) = \eta_{13} \kappa_{14} \).

It follows from the exact sequence (4.4) of [9] that there exists \( \alpha \in [\bar{a}_7, 2\bar{a}_6, \eta_{27}] \) such that \( \alpha = y \sigma^* \kappa_{14} \equiv E \pi_{27}^{12} \). We have \( E(\sigma^* \kappa_{14}) = 2\sigma \kappa = 0 \) and \( E \pi_{27}^{12} = [\eta, \kappa_{13}] \approx Z_2 \) by Theorem A of [5], so we obtain \( E^* \alpha \subset [\eta, \kappa_{13}] \). This conclude the assertion \( 0 \in [\bar{a}, 2, \eta] \) since \( [\bar{a}, 2, \eta] \) is a coset of \( [\eta, \kappa_{13}] \).

§ 5.

In this section we apply our methods to the results of [1] and then we obtain some general results about \( \pi_4(2), \pi_5(2) \) and \( \pi_6 \) respectively.

We denote by \( J_0 \) the generator of the 2-component of the image of \( J \)-homomorphism \( J: \pi_3(\text{SO}) \to G_k \). The results of [1] which are necessary for us are summarized as follows:
THEOREM E. \( G_8 \) contains a direct summand (notation: \( \oplus \)) which is isomorphic to the corresponding group in the following table \((k>1)\):

<table>
<thead>
<tr>
<th>( k= )</th>
<th>( 8s )</th>
<th>( 8s+1 )</th>
<th>( 8s+2 )</th>
<th>( 8s+3 )</th>
<th>( 8s+4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_k \oplus )</td>
<td>( Z_2 )</td>
<td>( Z_2 \oplus Z_2 )</td>
<td>( Z_2 )</td>
<td>( Z_2 )</td>
<td>( Z_2 )</td>
</tr>
<tr>
<td>Generators</td>
<td>( j_{8s} )</td>
<td>( j_{8s+1} ), ( \mu_{8s+1} )</td>
<td>( \mu_{8s+2} )</td>
<td>( j_{8s+1} )</td>
<td></td>
</tr>
</tbody>
</table>

Remark that we can take \( j_{8s+1} \sim j_{8s+1}^2 \) and \( \mu_{8s+2} = \mu_{8s+1}^2 \).

Our results are summarized as follows:

THEOREM 5.1. \( \pi_k(2) \), \( \pi_n^i(2) \) and \( \pi_k \) contain direct summands which are isomorphic to the corresponding groups in the following table \((k>2)\):

<table>
<thead>
<tr>
<th>( k )</th>
<th>( 8s )</th>
<th>( 8s+1 )</th>
<th>( 8s+2 )</th>
<th>( 8s+3 )</th>
<th>( 8s+4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi_k(2) \oplus )</td>
<td>( Z_2 )</td>
<td>( Z_2 \oplus Z_2 )</td>
<td>( Z_2 )</td>
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</tr>
<tr>
<td>Generators</td>
<td>( j_{8s} p )</td>
<td>( \text{Ext} j_{8s}, \mu_{8s+1} p )</td>
<td>( \text{Ext} \mu_{8s+1}, \eta \text{Ext} j_{8s} )</td>
<td>( \eta \text{Ext} \mu_{8s+1}, j_{8s+1} )</td>
<td>( \eta^2 \text{Ext} \mu_{8s+1} )</td>
</tr>
<tr>
<td>( \pi_n^i(2) \oplus )</td>
<td>( Z_2 )</td>
<td>( Z_2 \oplus Z_2 )</td>
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</tr>
<tr>
<td>Generators</td>
<td>( i j_{8s} )</td>
<td>( \text{Coext} j_{8s}, i j_{8s+1} )</td>
<td>( \text{Coext} \mu_{8s+1}, ) ( \text{(Coext } j_{8s}) \eta )</td>
<td>( \text{(Coext } \mu_{8s+1}) \eta )</td>
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<tr>
<td>( \pi_k \oplus )</td>
<td>( Z_2 \oplus Z_2 )</td>
<td>( Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 )</td>
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</tr>
</tbody>
</table>

Relations:

\[
2 \text{ Ext } j_{8s} = j_{8s+1} \quad 2 \text{ Ext } j_{8s} = j_{8s+2} \quad 2 \text{ Coext } j_{8s} = j_{8s+1} \quad 2 \text{ Coext } j_{8s} = j_{8s+2} \\
2 (\text{Coext } \eta \text{Ext } \mu_{8s+1}) = i j_{8s+1}^2 \text{ Ext } \mu_{8s+1}.
\]

The proof is left to the reader.

REMARK. If the assertion that \( G_k \) contains the \( J \)-image as a direct summand for \( k=8s+7 \) is true, then we can obtain better results. For examples \( \pi_{8s} \) contains a direct summand \( Z_4 + Z_2 + Z_2 \).
References


