A MATRIX THEORY OF MULTI-LEVEL DIFFERENCE EQUATIONS ANALOGOUS TO LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

By

Wasao SIBAGAKI and Goichi MARUKI

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Introduction.

The initial value problem of a hyperbolic or parabolic partial differential equation is properly posed in the Hilbert space of L^2 functions, and an approximation theory can be based on L^2 -norm in accordance with Lax's theory, but in this paper we shall try to develop our theory of approximation from the view-point of sup-norm since the problem is also properly posed in the space of smooth functions. From practical standpoint this view will be more convenient.

In §1 we define multi-level difference problems analogous to initial value problems of linear partial differential equations with constant coefficients. In §2 we develop a general theory of the linear difference equations which will be utilized afterwards. In §3 we obtain the solutions of the analogous linear difference equations in case the coefficients are all special tridiagonal matrices. In §4 we introduce the stability concept on the difference scheme and discuss its significance in error estimation and also in departure estimation.

§ 1. Difference problems analogous to initial value problems of linear partial differential equations.

As a simple and typical example we consider the initial value problem of a vibrating string stretched in the interval $0 \leq x \leq 1$, which is formulated mathematically in

$$\left\{ \begin{array}{l} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad (0 \leq x \leq 1, 0 \leq t), \\ u(0, t) = u(1, t) = 0 \quad (0 \leq t), \\ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x) \quad (0 \leq x \leq 1). \end{array} \right. \quad \begin{array}{l} (1.1) \\ (1.2) \\ (1.3) \end{array}$$

In order to construct a difference approximation to the above problem we consider those values of u at the lattice points (jh, nk) ($j=1, \dots, M; n=1, 2, \dots$), where

$$h = \Delta x = \frac{1}{M+1}, \quad k = \Delta t, \quad (1.4)$$

and write

$$u(jh, nk) = u_j^n. \quad (1.5)$$

According to the boundary condition (1.2) we have always

$$u_0^n = u_{M+1}^n = 0, \quad (1.6)$$

and the remaining M values $(u_j^n, j=1, 2, \dots, M)$ can be taken as the components of an M -dimensional vector u_n which we shall call the n -th *level vector*. Now for smooth functions $u(x, t)$ it holds:

$$\frac{\partial^2 u(x, t)}{\partial t^2} = \frac{u(x, t+k) - 2u(x, t) + u(x, t-k)}{k^2} + O(k^2), \quad (1.7)$$

$$\frac{\partial^2 u(x, t)}{\partial x^2} = \frac{u(x+h, t) - 2u(x, t) + u(x-h, t)}{h^2} + O(h^2), \quad (1.8)$$

and also

$$\frac{\partial^2 u(x, t)}{\partial x^2} = \frac{u(x+h, t+k) - 2u(x, t+k) + u(x-h, t+k)}{h^2} + O\left(h^2 + k + \frac{k^3}{h^2}\right). \quad (1.9)$$

Therefore, if we write down at the lattice points (jh, nk) the relation (1.1) which holds for smooth solutions by using (1.7) for $\partial^2 u / \partial t^2$ and (1.8) for $\partial^2 u / \partial x^2$, we get the so-called 'explicit' difference equation with remainder:

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{k^2} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} + O(h^2 + k^2), \quad (1.10)$$

while if we use (1.7) for $\partial^2 u / \partial t^2$ and (1.9) for $\partial^2 u / \partial x^2$, we get the so-called 'implicit' difference equation with remainder:

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{k^2} = \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2} + O\left(h^2 + k + \frac{k^3}{h^2} + k^2\right). \quad (1.11)$$

In our theory of approximation we shall ultimately make $k \downarrow 0$ holding

$$r = \frac{k}{h} = \text{constant}. \quad (1.12)$$

Taking this into account the explicit formula (1.10) will be rewritten as

$$u_j^{n+1} = r^2 u_{j+1}^n + 2(1-r^2)u_j^n + r^2 u_{j-1}^n - u_j^{n-1} + O(k^4), \quad (1.13)$$

which may be represented in matrix form as

$$u_{n+1} = A^{(0)}u_n + A^{(1)}u_{n-1} + \delta_{n+1}, \quad (1.14)$$

where

$$A^{(0)} = \begin{bmatrix} 2(1-r^2) & r^2 & & & \\ r^2 & 2(1-r^2) & r^2 & & 0 \\ & 0 & r^2 & 2(1-r^2) & r^2 \\ & & & r^2 & 2(1-r^2) \\ & & & & r^2 \end{bmatrix}, \quad (1.15)$$

$$A^{(1)} = -I \quad (I = M \times M \text{ unit matrix}) \quad (1.16)$$

and δ_{n+1} is a term of order $O(k^4)$.

The implicit formula (1.11) which is rewritten as

$$r^2 u_{j+1}^{n+1} - (1+2r^2)u_j^{n+1} + r^2 u_{j-1}^{n+1} = -2u_j^n + u_j^{n-1} + O(k^3) \quad (1.17)$$

may be represented in matrix form as

$$Fu_{n+1} = -2Iu_n + Iu_{n-1} + O(k^3), \quad (1.18)$$

where

$$F = \begin{bmatrix} -(1+2r^2) & r^2 & & & \\ r^2 & -(1+2r^2) & r^2 & & 0 \\ & 0 & r^2 & -(1+2r^2) & r^2 \\ & & & r^2 & -(1+2r^2) \\ & & & & r^2 \end{bmatrix}, \quad (1.19)$$

but it may be multiplied by the inverse matrix F^{-1} so that we may have again an equation of the same form as (1.14) with different tridiagonal matrix $A^{(0)}$ and $A^{(1)}$ and with $\delta_{n+1} = O(k^3)$. (1.14) is a linear inhomogeneous difference equation of order 2 or of 3 levels with constant coefficients $A^{(0)}$ and $A^{(1)}$.

We shall have occasion to indicate that it may be effective in some respects to replace $\partial^2 u / \partial t^2$ in (1.1) by a linear combination to obtain the following relation which holds for smooth solutions of the problem (1.1):

$$\begin{aligned} (1+p) \left(\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{k^2} \right) - p \left(\frac{u_j^n - 2u_j^{n-1} + u_j^{n-2}}{k^2} \right) \\ = \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2} + O \left(h^2 + k + \frac{k^3}{h^2} + k^2 + h + \frac{h^3}{k^2} \right). \end{aligned} \quad (1.20)$$

It can be represented in the following matrix form

$$u_{n+2} = A^{(0)} u_{n+1} + A^{(1)} u_n + A^{(2)} u_{n-1} + \delta_{n+2}, \quad (1.21)$$

where $A^{(0)}$, $A^{(1)}$, $A^{(2)}$ are some tridiagonal matrices and $\delta_{n+2} = O(k^3)$. This is a linear difference equation of order 3 or of 4 levels.

Generally, a linear difference equation of order $(m+1)$ or of $(m+2)$ levels

$$u_{n+m} = A^{(0)} u_{n+m-1} + A^{(1)} u_{n+m-2} + \cdots + A^{(m)} u_{n-1}, \quad (1.22)$$

which comes from a second order linear hyperbolic partial differential equation with constant coefficients such as (1.1) is said to be *analogous* to the latter if it satisfies the following condition: the difference

$$\delta_{n+m} = u_{n+m} - A^{(0)} u_{n+m-1} - A^{(1)} u_{n+m-2} - \cdots - A^{(m)} u_{n-1}, \quad (1.23)$$

in which u is substituted by a smooth solution of the partial differential equation tends to zero in the order $o(k^2)$ when the mesh length k tends to zero keeping $r = k/h = \text{constant}$, and the *accuracy* of the approximate formula (1.22) is said to be *higher* if δ_{n+m} tends to zero faster. In this terminology the three formulae deduced above are all analogous to (1.1), and the explicit one is of higher accuracy compared with the other two.

Lastly, we must approximate the initial data (1.3). Setting

$$u_j^0 = \varphi(jh) \quad (j=1, \dots, M) \quad (1.24)$$

should be obvious. As for the derivative u_t if we take the simplest approximation

$$u_t(x, t) = \frac{u(x, t+k) - u(x, t)}{k} + O(k), \quad (1.25)$$

we get

$$u_j^1 = \varphi(jh) + k\psi(jh) + O(k^2) \quad (j=1, \dots, M). \quad (1.26)$$

The column vectors $u_0=(\varphi(jh))$, $u_1=(\varphi(jh)+k\psi(jh))$ can be taken as the initial data necessary to determine the solution of the explicit or implicit difference equation. But to perform the solution of the 4 level equation we must know one more column u_2 which may be defined conveniently as the one obtainable from the explicit or implicit equation.

We should make some comments on problems with more general boundary conditions, for instance

$$\begin{cases} u(0, t) + \alpha u_x(0, t) = \beta, \\ u(1, t) + \gamma u_x(1, t) = \delta, \end{cases} \quad (0 \leq t) \quad (1.27)$$

instead of (1.2) where $\alpha, \beta, \gamma, \delta$ are given constants. These are approximated respectively by

$$\begin{cases} u(0, t) + \alpha \frac{u(h, t) - u(0, t)}{h} = \beta + O(h), \\ u(1, t) + \gamma \frac{u(1, t) - u(1-h, t)}{h} = \delta + O(h), \end{cases} \quad (1.29)$$

or

$$\begin{cases} (-\alpha + h)u_0^n + \alpha u_1^n = \beta h + O(h^2), \\ (\gamma + h)u_{M+h}^n - \gamma u_M^n = \delta h + O(h^2). \end{cases} \quad (1.31)$$

Now the initial data functions (1.3) being given the values $u_1^0, u_1^1, u_M^0, u_M^1$ are known so that from (1.31) and (1.32) for $n=0$ and $n=1$ the values $u_0^0, u_0^1, u_{M+1}^0, u_{M+1}^1$ can be determined. Then the boundary conditions for $n=2$ may be included in the main system of difference equations so that we shall have, for instance, in the case of explicit formula $M+2$ equations for $M+2$ unknowns $u_0^{n+1}, u_1^{n+1}, \dots, u_M^{n+1}, u_{M+1}^{n+1}$:

$$\begin{cases} (-\alpha + h)u_0^{n+1} + \alpha u_1^{n+1} = \beta h + O(h^2), \\ u_j^{n+1} = r^2 u_{j+1}^n + 2(1-r^2)u_j^n + r^2 u_{j-1}^n - u_j^{n-1} + O(h^4) \quad (j=1, \dots, M), \\ (\gamma + h)u_{M+1}^{n+1} - \gamma u_M^{n+1} = \delta h + O(h^2). \end{cases} \quad (1.33)$$

In matrix form this is

$$Gu_{n+1} = E^{(0)}u_n + E^{(1)}u_{n-1} + \delta_{n+1}, \quad (1.34)$$

where u_n is the $(M+2)$ -dimensional column

$$[u_0^n, u_1^n, \dots, u_M^n, u_{M+1}^n]', \quad (1.35)$$

and

$$G = \begin{bmatrix} -\alpha + h & \alpha & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -\gamma & \gamma + h \end{bmatrix}, \quad (1.36)$$

$$E^0 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ r^2 & 2(1-r^2) & r^2 & \cdots & 0 \\ 0 & \cdots & r^2 & 2(1-r^2) & r^2 \end{bmatrix}, \quad (1.37)$$

$$E^{(1)} = \begin{bmatrix} 0 & \cdots & 0 \\ -1 & \cdots & -1 \\ 0 & \cdots & 0 \end{bmatrix}, \quad (1.38)$$

$$\delta_{n+1} = \begin{bmatrix} \beta h + O(k^2) \\ O(k^4) \\ \vdots \\ O(k^4) \\ \delta h + O(k^2) \end{bmatrix}. \quad (1.39)$$

Thus multiplying (1.34) by the inverse matrix G^{-1} we shall have an analogous linear difference equation of 3 levels

$$u_{n+1} = \bar{A}^{(0)} u_n + \bar{A}^{(1)} u_{n-1} + \bar{\delta}_{n+1}, \quad (1.35)$$

where $\bar{A}^{(0)}$ and $\bar{A}^{(1)}$ are almost tridiagonal so that the following theory may be reasonably applied.

§ 2. General theory of linear difference equations of $(m+2)$ levels.

Let us consider in this paragraph generally the linear difference equation of $(m+2)$ levels of the following form:

$$u_{n+m} = A^{(0)} u_{n+m-1} + A^{(1)} u_{n+m-2} + \cdots + A^{(m)} u_{n-1} + \delta_{n+m}, \quad (2.1)$$

where $A^{(0)}, A^{(1)}, \dots, A^{(m)}$ are arbitrary constant (that is, not dependent on the level variable n) $M \times M$ matrices and δ_{n+m} is a given M -dimensional column function of n , and consequently u_{n+m} is the unknown M -dimensional column

function of n . For general discussions the variable n may take arbitrary integral values, but corresponding to the initial value problem considered in §1 we shall usually consider u_n for non-negative integer n only, the $m+1$ consecutive values $u_0, u_1, \dots, u_m$ being taken as given *initial values*.

First we consider for simplicity the case $m=1$, i.e. the equation

$$u_{n+1} = A^{(0)}u_n + A^{(1)}u_{n-1} + \delta_{n+1} \quad (2.2)$$

and its associate homogeneous equation

$$u_{n+1} = A^{(0)}u_n + A^{(1)}u_{n-1}. \quad (2.3)$$

By obvious successive substitutions we shall have

$$\begin{aligned} u_2 &= A^{(0)}u_1 + A^{(1)}u_0 + \delta_2, \\ u_3 &= \{A^{(0)^2} + A^{(1)}\}u_1 + A^{(0)}A^{(1)}u_0 + A^{(0)}\delta_2 + \delta_3, \\ u_4 &= \{A^{(0)^3} + A^{(0)}A^{(1)} + A^{(1)}A^{(0)}\}u_1 + \{A^{(0)^2}A^{(1)} + (A^{(1)})^2\}u_0 \\ &\quad + \{A^{(0)^2} + A^{(1)}\}\delta_2 + A^{(1)}\delta_3 + \delta_4, \\ &\dots\dots\dots \end{aligned}$$

and in general we may write

$$u_{n+1} = P_n^{(0)}u_1 + P_n^{(1)}u_0 + q_{n+1}, \quad (2.4)$$

where the coefficients $P_n^{(0)}, P_n^{(1)}$ are $M \times M$ matrices that are generated to be some polynomials in $A^{(0)}$ and $A^{(1)}$, and q_{n+1} is a linear form in $\delta_2, \delta_3, \dots, \delta_{n+1}$ with similar polynomial coefficients in $A^{(0)}$ and $A^{(1)}$. We wish to examine the properties of these coefficients and also of q_{n+1} .

Writing down

$$u_{n+2} = A^{(0)}u_{n+1} + A^{(1)}u_n + \delta_{n+2}$$

and inserting in it the expression (2.4) we get

$$u_{n+2} = A^{(0)}\{P_n^{(0)}u_1 + P_n^{(1)}u_0 + q_{n+1}\} + A^{(1)}\{P_{n-1}^{(0)}u_1 + P_{n-1}^{(1)}u_0 + q_n\} + \delta_{n+2},$$

which should be

$$= P_{n+1}^{(0)}u_1 + P_{n+1}^{(1)}u_0 + q_{n+2},$$

so that comparing corresponding coefficients on both sides we get the relations

$$P_{n+1}^{(0)} = A^{(0)}P_n^{(0)} + A^{(1)}P_{n-1}^{(0)}, \quad (2.5)$$

$$P_{n+1}^{(1)} = A^{(0)}P_n^{(1)} + A^{(1)}P_{n-1}^{(1)}, \quad (2.6)$$

$$q_{n+1} = A^{(0)}q_n + A^{(1)}q_{n-1} + \delta_{n+1}. \quad (2.7)$$

The first two relations show that the matrices $P_n^{(0)}$ and $P_n^{(1)}$ satisfy respectively the homogeneous equation (2.3), and (2.7) shows that q_n satisfies the general equation (2.2). Now in the defining relation (2.4) of the matrices $P_n^{(0)}$, $P_n^{(1)}$ and q_{n+1} we may put $n=0$ and $n=1$ getting

$$u_1 = P_0^{(0)}u_1 + P_0^{(1)}u_0 + q_1,$$

$$u_2 = P_1^{(0)}u_1 + P_1^{(1)}u_0 + q_2 \text{ which is } = A^{(0)}u_1 + A^{(1)}u_0 + \delta_2.$$

These must hold for arbitrary u_0 , u_1 so that we must have the following initial conditions:

$$P_0^{(0)} = I, \quad P_1^{(0)} = A^{(0)}, \quad (2.8)$$

$$P_0^{(1)} = 0, \quad P_1^{(1)} = A^{(1)}, \quad (2.9)$$

and $q_1=0$, $q_2=\delta_2$. On the other hand putting $n=1$ in (2.7) we have

$$q_2 = A^{(0)}q_1 + A^{(1)}q_0 + \delta_2,$$

so that we must have $q_0=0$. Thus we have the initial condition:

$$q_0=0, \quad q_1=0. \quad (2.10)$$

Besides the relation (2.6) it holds the more useful relation

$$P_n^{(1)} = P_{n-1}^{(0)}A^{(1)}. \quad (2.11)$$

In fact, setting $Q_n = P_{n-1}^{(0)}A^{(1)}$ we may verify that it satisfies the same conditions as $P_n^{(1)}$ does. Now,

$$Q_1 = P_0^{(0)}A^{(1)} = A^{(1)} = P_1^{(1)}, \quad Q_2 = P_1^{(0)}A^{(1)} = A^{(0)}A^{(1)} = P_2^{(1)},$$

and

$$\begin{aligned} A^{(0)}Q_n + A^{(1)}Q_{n-1} &= A^{(0)}P_{n-1}^{(0)}A^{(1)} + A^{(1)}P_{n-2}^{(0)}A^{(1)} \\ &= (A^{(0)}P_{n-1}^{(0)} + A^{(1)}P_{n-2}^{(0)})A^{(1)} \\ &= P_n^{(0)}A^{(1)} = Q_{n+1}. \end{aligned}$$

Also we can show that the non-linear difference equation of order 1 for square matrices C_n :

$$C_n = A^{(0)} + A^{(1)}C_{n-1}^{-1} \quad (2.12)$$

with the initial condition $C_0 = A^{(0)}$ has the solution

$$C_n = P_{n+1}^{(0)}(P_n^{(0)})^{-1}. \quad (2.13)$$

In fact,

$$\begin{aligned} C_1 &= A^{(0)} + A^{(1)}C_0^{-1} = A^{(0)} + A^{(1)}(A^{(0)})^{-1} \\ &= \{(A^{(0)})^2 + A^{(1)}\}(A^{(0)})^{-1} = P_2^{(0)}(P_1^{(0)})^{-1}, \end{aligned}$$

and assuming (2.13) for $n=k$ we have

$$\begin{aligned} C_{k+1} &= A^{(0)} + A^{(1)}C_k^{-1} \\ &= A^{(0)} + A^{(1)}\{P_{k+1}^{(0)}(P_k^{(0)})^{-1}\}^{-1} \\ &= \{A^{(0)}P_{k+1}^{(0)} + A^{(1)}P_k^{(0)}\}(P_{k+1}^{(0)})^{-1} \\ &= P_{k+2}^{(0)}(P_{k+1}^{(0)})^{-1}. \end{aligned}$$

Now, writing (2.13) for $n=0, 1, \dots, n-1$ and multiplying the resulting relations we get

$$C_{n-1}C_{n-2}\cdots C_1C_0 = P_n^{(0)}(I)^{-1} = P_n^{(0)}. \quad (2.14)$$

This relation may be considered as giving an algorithm for calculating the matrices $P_n^{(0)}$.

We can show that the solution of the non-homogeneous difference equation (2.2) for the initial values u_0, u_1 is given by

$$u_{n+1} = P_n^{(0)}u_1 + P_n^{(1)}u_0 + q_{n+1}, \quad (2.15)$$

where

$$q_{n+1} = \sum_{i=2}^{n+1} P_{n+1-i}^{(0)} \delta_i. \quad (2.16)$$

In fact, since $u_{n+1} = P_n^{(0)}u_1 + P_n^{(0)}u_0$ is the solution of the homogeneous equation (2.3) for the initial values u_0, u_1 , it remains to show that $u_{n+1} = q_{n+1}$ is the solution of the non-homogeneous equation (2.2) for the initial values $u_0 = u_1 = 0$. Now, by (2.10) the initial condition is satisfied, and substituting the expression (2.16) on the right hand side of (2.7) we have

$$\begin{aligned} &A^{(0)}q_n + A^{(1)}q_{n-1} + \delta_{n+1} \\ &= A^{(0)} \sum_{i=2}^n P_{n-i}^{(0)} \delta_i + A^{(1)} \sum_{i=2}^{n-1} P_{n-i-1}^{(0)} \delta_i + \delta_{n+1} \\ &= A^{(0)}\{P_{n-2}^{(0)}\delta_2 + P_{n-3}^{(0)}\delta_3 + \cdots + P_0^{(0)}\delta_n\} \\ &\quad + A^{(1)}\{P_{n-3}^{(0)}\delta_2 + P_{n-4}^{(0)}\delta_3 + \cdots + P_0^{(0)}\delta_{n-1}\} + \delta_{n+1} \\ &= P_{n-1}^{(0)}\delta_2 + P_{n-2}^{(0)}\delta_3 + \cdots + P_2^{(0)}\delta_{n-1} + P_1^{(0)}\delta_n + P_0^{(0)}\delta_{n+1} \\ &= q_{n+1}. \end{aligned}$$

The above facts can be shown to hold for general multi-level equation (2.1) and its associate homogeneous one

$$+A^{(m)}(P_{n-m}^{(0)}u_m+P_{n-m}^{(1)}u_{m-1}+\cdots+P_{n-m}^{(m)}u_0),$$

which should be written as

$$= P_{n+1}^{(0)} u_m + P_{n+1}^{(1)} u_{m-1} + \cdots + P_{n+1}^{(m)} u_0.$$

$u_0, u_1, \dots, u_m$ being independent we get by comparing coefficients the relations

$$P_{n+1}^{(0)} = A^{(0)}P_n^{(0)} + A^{(1)}P_{n-1}^{(0)} + \dots + A^{(m)}P_{n-m}^{(0)}, \quad (2.23)$$

$$P_{n+1}^{(i)} = A^{(0)}P_n^{(i)} + A^{(1)}P_{n-1}^{(i)} + \cdots + A^{(m)}P_{n-m}^{(i)} \quad (i=1, \cdots, m), \quad (2.24)$$

which when $n+m$ is written for n show the first part of the theorem.

To obtain initial conditions we proceed recurrently. First, from

$$u_m = P_0^{(0)} u_m + P_0^{(1)} u_{m-1} + \cdots + P_0^{(m)} u_0,$$

which must hold for arbitrary $u_m, u_{m-1}, \dots, u_0$ we have

$$P_0^{(0)}=I, \quad P_0^{(i)}=0.$$

Next, from

$$u_{1+m} = P_1^{(0)} u_m + P_1^{(1)} u_{m-1} + \cdots + P_1^{(m)} u_0,$$

which must be

$$= A^{(0)}u_m + A^{(1)}u_{m-1} + \cdots + A^{(m)}u_0,$$

we get

$$P_1^{(0)}=A^{(0)}, \quad P_1^{(i)}=A^{(i)}.$$

Next, from

$$\begin{aligned} u_{2+m} &= P_2^{(0)}u_m + P_2^{(1)}u_{m-1} + \cdots + P_2^{(m)}u_0 \\ &= A^{(0)}u_{1+m} + A^{(1)}u_m + A^{(2)}u_{m-1} + \cdots + A^{(m)}u_1 \\ &= A^{(0)}(P_1^{(0)}u_m + P_1^{(1)}u_{m-1} + \cdots + P_1^{(m)}u_0) \\ &\quad + A^{(1)}u_m + A^{(2)}u_{m-1} + \cdots + A^{(m)}u_1, \end{aligned}$$

we have

$$P_2^{(0)} = A^{(0)} P_1^{(0)} + A^{(1)},$$

and

$$\begin{aligned} P_2^{(1)} &= A^{(0)} P_1^{(1)} + A^{(2)}, \\ &\dots\dots\dots, \\ P_2^{(m-1)} &= A^{(0)} P_1^{(m-1)} + A^{(m)}, \\ P_2^{(m)} &= A^{(0)} P_1^{(m)}. \end{aligned}$$

But we can rewrite

$$\begin{aligned} P_2^{(1)} &= A^{(0)}A^{(1)} + A^{(2)} = P_1^{(0)}A^{(1)} + P_0^{(0)}A^{(2)}, \\ &\dots\dots\dots, \\ P_2^{(m-1)} &= A^{(0)}A^{(m-1)} + A^{(m)} = P_1^{(0)}A^{(m-1)} + P_0^{(0)}A^{(m)}, \\ P_2^{(m)} &= A^{(0)}A^{(m)} = P_1^{(0)}A^{(m)} + P_0^{(0)}A^{(m+1)}. \end{aligned}$$

Thus we have shown the correctness of the third line in (2.20). After this we may proceed inductively. Assume that for $i=1, \dots, m$ the representation is true as far as n so that

$$P_n^{(i)} = P_{n-1}^{(0)}A^{(i)} + P_{n-2}^{(0)}A^{(i+1)} + \dots.$$

Then using the already proved relation (2.24)

$$\begin{aligned} P_{n+1}^{(i)} &= A^{(0)}P_n^{(i)} + A^{(1)}P_{n-1}^{(i)} + \dots + A^{(m)}P_{n-m}^{(i)} \\ &= A^{(0)}(P_{n-1}^{(0)}A^{(i)} + P_{n-2}^{(0)}A^{(i+1)} + \dots) \\ &\quad + A^{(1)}(P_{n-2}^{(0)}A^{(i)} + P_{n-3}^{(0)}A^{(i+1)} + \dots) \\ &\quad + \dots\dots\dots \\ &\quad + A^{(m-1)}(P_{n-m}^{(0)}A^{(i)} + P_{n-m-1}^{(0)}A^{(i+1)} + \dots) \\ &\quad + A^{(m)}(P_{n-m-1}^{(0)}A^{(i)} + P_{n-m-2}^{(0)}A^{(i+1)} + \dots) \\ &= (A^{(0)}P_{n-1}^{(0)} + A^{(1)}P_{n-2}^{(0)} + \dots + A^{(m)}P_{n-m-1}^{(0)})A^{(i)} \\ &\quad + (A^{(0)}P_{n-2}^{(0)} + A^{(1)}P_{n-3}^{(0)} + \dots + A^{(m)}P_{n-m-2}^{(0)})A^{(i+1)} \\ &\quad + \dots\dots\dots \\ &= P_n^{(0)}A^{(i)} + P_{n-1}^{(0)}A^{(i+1)} + \dots. \end{aligned}$$

THEOREM 2. *The non-linear difference equation of order m for square matrices C_n :*

$$\begin{aligned} C_n &= A^{(0)} + A^{(1)}C_{n-1}^{-1} + A^{(2)}(C_{n-1}C_{n-2})^{-1} + \dots \\ &\quad + A^{(m)}(C_{n-1}C_{n-2}\dots C_{n-m})^{-1} \end{aligned} \quad (2.25)$$

with the initial condition

$$C_n = P_{n+1}^{(0)}(P_n^{(0)})^{-1} \quad (n=0, 1, \dots, m-1), \quad (2.26)$$

has the solution

$$C_n = P_{n+1}^{(0)}(P_n^{(0)})^{-1} \quad (n=m, m+1, \dots). \quad (2.27)$$

PROOF. We verify it by substitution.

$$\begin{aligned}
& A^{(0)} + A^{(1)}\{P_n^{(0)}(P_{n-1}^{(0)})^{-1}\}^{-1} + A^{(2)}\{P_n^{(0)}(P_{n-1}^{(0)})^{-1}P_{n-1}^{(0)}(P_{n-2}^{(0)})^{-1}\}^{-1} \\
& \quad + \cdots + A^{(m)}\{P_n^{(0)}(P_{n-1}^{(0)})^{-1}P_{n-1}^{(0)}(P_{n-2}^{(0)})^{-1} \cdots P_{n-m+1}^{(0)}(P_{n-m}^{(0)})^{-1}\}^{-1} \\
& = A^{(0)} + A^{(1)}P_{n-1}^{(0)}(P_n^{(0)})^{-1} + A^{(2)}P_{n-2}^{(0)}(P_n^{(0)})^{-1} + \cdots + A^{(m)}P_{n-m}^{(0)}(P_n^{(0)})^{-1} \\
& = \{A^{(0)}P_n^{(0)} + A^{(1)}P_{n-1}^{(0)} + A^{(2)}P_{n-2}^{(0)} + \cdots + A^{(m)}P_{n-m}^{(0)}\}(P_n^{(0)})^{-1} \\
& = P_{n+1}^{(0)}(P_n^{(0)})^{-1}.
\end{aligned}$$

Now, writing (2.27) for $n=0, 1, \dots, n-1$ and multiplying the resulting relations we get

$$C_{n-1}C_{n-2} \cdots C_1C_0 = P_n^{(0)}. \quad (2.28)$$

THEOREM 3. *The solution of the non-homogeneous difference equation (2.1) for the initial values $u_0, u_1, \dots, u_m$ is given by*

$$u_{n+m} = P_n^{(0)}u_m + P_n^{(1)}u_{m-1} + \cdots + P_n^{(m)}u_0 + q_{n+m},$$

where

$$q_{n+m} = \sum_{i=m+1}^{n+m} P_{n+m-i}^{(0)} \delta_i.$$

The sequence (q_{n+m}) is the solution of the non-homogeneous difference equation for the homogeneous initial condition

$$u_0 = u_1 = \cdots = u_m = 0. \quad (2.29)$$

The proof goes similarly as in the case $m=1$.

§ 3. The solution of an analogous linear difference equation in which the coefficients are all special tridiagonal matrices.

Let

$$u_{n+m} = A^{(0)}u_{n+m-1} + A^{(1)}u_{n+m-2} + \cdots + A^{(m)}u_{n-1} + \delta_{n+m} \quad (3.1)$$

be a linear difference equation analogous to an initial value problem of a second order hyperbolic linear partial differential equation with constant coefficients such that

$$A^{(i)} = a_i I + \alpha_i J \quad (i=0, 1, \dots, m), \quad (3.2)$$

where I is the $M \times M$ unit matrix and J is the following tridiagonal matrix

$$J = \begin{bmatrix} 0 & 1 & & 0 \\ 1 & 0 & 1 & \\ & \ddots & \ddots & \ddots \\ 0 & & 1 & 0 \end{bmatrix} \quad (3.3)$$

In this case it is possible to transform the equation (3.1) into a similar equation

$$v_{n+m} = B^{(0)}v_{n+m-1} + B^{(1)}v_{n+m-2} + \cdots + B^{(m)}v_{n-1} + \bar{\delta}_{n+m}, \quad (3.4)$$

in which the coefficient matrices are all diagonal. In fact, by the linear transformation

$$v_n = Tu_n, \quad (3.5)$$

where

$$T = \left(\sin \frac{ij\pi}{M+1}, \quad i \downarrow 1, \dots, M \right)_{j \rightarrow 1, \dots, M} \quad (3.6)$$

and

$$T^{-1} = \frac{2}{M+1} T, \quad (3.7)$$

we have the similarities

$$TA^{(i)}T^{-1} = B^{(i)} \quad (i=0, 1, \dots, M) \quad (3.8)$$

such that $B^{(i)}$'s are all diagonal matrices of the form

$$B^{(i)} = \alpha_i I + 2\alpha_i G, \quad G = \text{Diag} \left(\cos \frac{\pi}{M+1}, \cos \frac{2\pi}{M+1}, \dots, \cos \frac{M\pi}{M+1} \right). \quad (3.9)$$

Setting

$$x_j^{(i)} = \alpha_i + 2\alpha_i \cos \frac{ij\pi}{M+1} \quad (i=0, 1, \dots, M; j=1, \dots, M) \quad (3.10)$$

we have

$$B^{(i)} = \text{Diag} (x_1^{(i)}, x_2^{(i)}, \dots, x_M^{(i)}). \quad (3.11)$$

First let us try to obtain the general solution of the homogeneous equation associated with (3.4)

$$v_{n+m} = B^{(0)}v_{n+m-1} + B^{(1)}v_{n+m-2} + \cdots + B^{(m)}v_{n-1}. \quad (3.12)$$

Assume a particular solution

$$v_n = \rho^n c, \quad (3.13)$$

where

$$c = \text{column matrix } (\gamma_i, i \downarrow 1, \dots, M). \quad (3.14)$$

Then it must hold the relation:

$$(\rho^{m+1}I - \rho^m B^{(0)} - \rho^{m-1} B^{(1)} - \cdots - B^{(m)})c = 0. \quad (3.15)$$

When we take into account of the diagonality (3.11) of the matrices $B^{(i)}$ the above equation takes the form:

$$\begin{bmatrix} \rho^{m+1} - x_1^{(0)}\rho^m - x_1^{(1)}\rho^{m-1} - \cdots - x_1^{(m)}, & 0, & \dots, & 0 \\ 0, & \rho^{m+1} - x_2^{(0)}\rho^m - x_2^{(1)}\rho^{m-1} - \cdots - x_2^{(m)}, & 0, & \dots, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & \dots, & 0, & \rho^{m+1} - x_M^{(0)}\rho^m - x_M^{(1)}\rho^{m-1} - \cdots - x_M^{(m)} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_M \end{bmatrix} = 0. \quad (3.16)$$

The determinant of the coefficient matrix which is equal to the polynomial

$$\begin{aligned} & (\rho^{m+1} - x_1^{(0)}\rho^m - x_1^{(1)}\rho^{m-1} - \cdots - x_1^{(m)}) \\ & \dots (\rho^{m+1} - x_M^{(0)}\rho^m - x_M^{(1)}\rho^{m-1} - \cdots - x_M^{(m)}) \end{aligned} \quad (3.17)$$

is the *characteristic polynomial* of the difference equation (3.12). Let

$$\rho_{j,s} \quad (s=1, \dots, m+1) \quad (3.18)$$

be the $(m+1)$ roots of the j -th equation

$$\rho^{m+1} - x_j^{(0)}\rho^m - x_j^{(1)}\rho^{m-1} - \cdots - x_j^{(m)} = 0 \quad (3.19)$$

on which we make the assumption that they are all distinct. Then we can take the characteristic vectors corresponding to them to be the same for which we may choose the column matrix

$$c_j = (\delta_{kj}, k \downarrow 1, \dots, M) \quad (3.20)$$

where δ_{kj} is the Kronecker symbol. Then under the assumption made the required general solution of (3.12) is given by

$$v_n = \sum_{j=1}^M \left\{ \sum_{s=1}^{m+1} C_{j,s} \rho_{j,s}^n c_j \right\}, \quad (3.21)$$

where $C_{j,s}$ ($j=1, \dots, M$; $s=1, \dots, m+1$) are $(m+1)M$ arbitrary constants.

Now, by making use of this solution we can easily determine the (now diagonal) matrices $\tilde{P}_n^{(0)} = TP_n^{(0)}T^{-1}$ which enter the defining equation

$$v_{n+m} = \tilde{P}_n^{(0)}v_m + \tilde{P}_n^{(1)}v_{m-1} + \cdots + \tilde{P}_n^{(m)}v_0. \quad (3.22)$$

According to the general theory in §2 $\tilde{P}_n^{(0)}$ satisfies the homogeneous linear equation (3.12) with the initial condition corresponding to (2.19) so that it must hold:

$$\begin{cases} \tilde{P}_0^{(0)} = I, \\ \tilde{P}_1^{(0)} = B^{(0)}\tilde{P}_0^{(0)}, \\ \tilde{P}_2^{(0)} = B^{(0)}\tilde{P}_1^{(0)} + B^{(1)}\tilde{P}_0^{(0)}, \\ \dots\dots\dots \\ \tilde{P}_m^{(0)} = B^{(0)}\tilde{P}_{m-1}^{(0)} + B^{(1)}\tilde{P}_{m-2}^{(0)} + \cdots + B^{(m-1)}\tilde{P}_0^{(0)}. \end{cases} \quad (3.23)$$

Obviously the j -th column vector of $\tilde{P}_n^{(0)}$ is the solution of the homogeneous equation (3.12) satisfying the initial condition of the same form as given in (3.23). By observing this fact each of the column vector of $\tilde{P}_n^{(0)}$ and consequently the matrix itself can be determined.

For instance, let us try to determine the first column, say v_n of $\tilde{P}_n^{(0)}$ for which we may assume the form (3.21), the arbitrary constants $C_{j,s}$ being determined so that the initial condition (3.23) may be satisfied. Accordingly we have the relations

$$\begin{cases} \sum C_{1,s} = 1, \quad \sum C_{1,s}\rho_{1,s} = x_1^{(0)}, \quad \sum C_{1,s}\rho_{1,s}^2 = (x_1^{(0)})^2 + x_1^{(1)}, \quad \dots, \quad \sum C_{1,s}\rho_{1,s}^m = (*), \\ \sum C_{2,s} = 0, \quad \sum C_{2,s}\rho_{2,s} = 0, \quad \sum C_{2,s}\rho_{2,s}^2 = 0, \quad \dots, \quad \sum C_{2,s}\rho_{2,s}^m = 0, \\ \dots\dots\dots \\ \sum C_{M,s} = 0, \quad \sum C_{M,s}\rho_{M,s} = 0, \quad \sum C_{M,s}\rho_{M,s}^2 = 0, \quad \dots, \quad \sum C_{M,s}\rho_{M,s}^m = 0, \end{cases} \quad (3.24)$$

where $\sum$ means $\sum_{s=1}^{m+1}$ and $(*)$ is a polynomial in $x_1^{(0)}, x_1^{(1)}, \dots, x_1^{(m-1)}$. Firstly, by the assumption that the $(m+1)$ roots $\rho_{1,s}$ ($s=1, \dots, m+1$) are all distinct the first line determines the $(m+1)$ quantities $C_{1,s}$ ($s=1, \dots, m+1$) uniquely, say $C_{1,s} = \gamma_{1,s}$. Next, by the assumption that the $(m+1)$ roots $\rho_{2,s}$ ($s=1, \dots, m+1$) are all distinct the second line determines each of the $(m+1)$ quantities $C_{2,s}$ to be equal to zero. In the same way by the corresponding assumptions we have $C_{3,s} = \dots = C_{M,s} = 0$ ($s=1, \dots, m+1$). Thus the first column of $P_n^{(0)}$ is determined to

$$\left[\sum_{s=1}^{m+1} \gamma_{1,s}\rho_{1,s}^2, 0, \dots, 0 \right]. \quad (3.25)$$

In the same way the other columns of $\tilde{P}_n^{(0)}$ can be determined so that we have eventually

$$\tilde{P}_n^{(0)} = \begin{pmatrix} \sum_{s=1}^{m+1} \gamma_{1,s} \rho_{1,s}^n, & 0, & \dots, & 0 \\ 0, & \sum_{s=1}^{m+1} \gamma_{2,s} \rho_{2,s}^n, & \dots, & 0 \\ \dots & \dots & \dots & \dots \\ 0, & 0, & \dots, & 0, \sum_{s=1}^{m+1} \gamma_{M,s} \rho_{M,s}^n \end{pmatrix}, \quad (3.26)$$

which shows clearly the dependency of the sequence of matrices $(\tilde{P}_n^{(0)}, n=0, 1, \dots)$ on the zeros of the characteristic polynomials of the difference equation (3.12) but only under the assumed condition that each of the sets of zeros $\rho_{1,s}, \rho_{2,s}, \dots, \rho_{M,s}$ consists of members of non-equal values. The case in which this condition is not satisfied can be dealt with precisely, but in this paper we shall not enter its discussions.

Now let us consider the other matrices $P_n^{(i)} (i=1, \dots, m)$. To them there exist remarkable representations (2.21) expressing each of them in terms of the matrices $P_n^{(0)}$. If we make on them similarity transformations by T we shall have corresponding representations of $\tilde{P}_n^{(i)} = TP_n^{(i)}T^{-1}$, i.e.

$$\tilde{P}_n^{(i)} = \tilde{P}_{n-1}^{(0)} B^{(i)} + \tilde{P}_{n-2}^{(0)} B^{(i+1)} + \dots \quad (i=1, \dots, m). \quad (3.27)$$

Here the matrices entering the relations are all diagonal. Now from the representations of $\tilde{P}_n^{(0)}$ in (3.26) we know that the matrices $\tilde{P}_n^{(i)}$ are of the form:

$$\tilde{P}_n^{(i)} = \text{Diag} (\text{Polynomial in } (\rho_{1,s}) \text{ of degree } (n-1), \text{ Polynomial in } (\rho_{2,s}) \text{ of degree } (n-1), \dots, \text{ Polynomial in } (\rho_{M,s}) \text{ of degree } (n-1)). \quad (3.28)$$

§ 4. Stability condition and its significance in error and departure estimation.

Let $u=u(x, t)$ be the smooth solution of our initial value problem of partial differential equation in the interval $0 \leq t \leq \tau$, corresponding to a given smooth initial data $u(x, 0) = \varphi(x)$, $u_t(x, 0) = \psi(x)$.

Let $U=U(x, t)$ be the exact solution of an analogous initial value problem of difference equation corresponding to the same initial data on the domain of lattice points $x=jh$, $t=nk$, where $j=1, 2, \dots, M$; $(M+1)h=1$ and $n=0, 1, \dots, N$; $Nk \leq \tau$.

Then we call

$$e = U - u \quad (4.1)$$

the *error* of the approximation U to u . We wish to estimate the error in sup-norm and examine the possibility $\|e\| \rightarrow 0$ when we let $k \downarrow 0$ holding $r = k/h = \text{constant}$. By the definition of “analogous difference equation” we have

$$u_{n+m} = A^{(0)}u_{n+m-1} + \cdots + A^{(m)}u_{n-1} + \delta_{n+m}, \quad (4.2)$$

$$\delta_{n+m} = O(k^{2+\alpha}), \quad \alpha > 0 \quad (4.3)$$

and

$$U_{n+m} = A^{(0)}U_{n+m-1} + \cdots + A^{(m)}U_{n-1}, \quad (4.4)$$

so that e satisfies the equation

$$e_{n+m} = A^{(0)}e_{n+m-1} + \cdots + A^{(m)}e_{n-1} + (-\delta_{n+m}). \quad (4.5)$$

According to Theorem 3 we have the representation

$$e_{n+m} = P_n^{(0)}e_m + P_n^{(1)}e_{m-1} + \cdots + P_n^{(m)}e_0 + (-q_{n+m}), \quad (4.6)$$

but here $e_m = e_{m-1} = \cdots = e_0 = 0$, so that

$$e_{n+m} = -q_{n+m} = -\sum_{i=m+1}^{n+m} P_{n+m-i}^{(0)} \delta_i. \quad (4.7)$$

Therefore

$$\|e_{n+m}\| \leq \sum_{i=m+1}^{n+m} \|P_{n+m-i}^{(0)}\| \|\delta_i\|. \quad (4.8)$$

Recall that the matrices $P_{n+m-i}^{(0)}$ depend on the positive parameter $r = k/h$ in some closed interval I_r and the level variable $n = 0, 1, \dots, N$ and also on k since they are $M \times M$ matrices where $(M+1)h = 1$ so that $M+1 = r/k$.

Now by (4.3) $\|\delta_i\| = O(k^{2+\alpha})$, $\alpha > 0$, so that if we have

$$\|P_n^{(0)}\| = O\left(\frac{1}{k^{1+\beta}}\right), \quad \beta < \alpha \quad (4.9)$$

uniformly in $r \in I_r$ and $n = 0, 1, \dots, N$ the inequality (4.8) will give us the required estimation

$$\|e_{n+m}\| = O\left(\frac{1}{k^{1+\beta}}\right) \cdot O(k^{2+\alpha}) \cdot \frac{\tau}{k} = O(k^{\alpha-\beta}) = o(1). \quad (4.10)$$

We shall call the stated condition (4.9) the *stability condition* on the analo-

gous difference scheme. If this condition is satisfied we shall have *convergence* $U \rightarrow u$ in sup-norm.

Now let us look the other side of the coin. Let $u^* = u^*(x, t)$ be an approximate solution of the same initial value problem of difference equation, i.e. a lattice function satisfying

$$u_{n+m}^* = A^{(0)}u_{n+m-1}^* + \cdots + A^{(m)}u_{n-1}^* + \delta_{n+m}^* \quad (4.11)$$

and the initial condition

$$u_0^* = u_0 + \delta_0^*, \dots, u_m^* = u_m + \delta_m^*, \quad (4.12)$$

where $\delta_0^*, \dots, \delta_m^*$ represents, respectively, the *numerical error* in computing $u_0, \dots, u_m$, and δ_{n+m}^* means the total numerical error arising in computing the expression $A^{(0)}u_{n+m-1}^* + \cdots + A^{(m)}u_{n-1}^*$ for given $u_{n+m-1}, \dots, u_{n-1}$. We call

$$d = u^* - U \quad (4.13)$$

the *departure* of the approximation u^* to U . Then the *numerical error* of the approximation u^* to the smooth solution u of the partial differential equation problem is given by

$$\begin{aligned} u^* - u &= (u^* - U) + (U - u) \\ &= \text{departure} + \text{error}. \end{aligned} \quad (4.14)$$

Now subtracting (4.4) from (4.11) we see that d satisfies the equation

$$d_{n+m} = A^{(0)}d_{n+m-1} + \cdots + A^{(m)}d_{n-1} + \delta_{n+m}^* \quad (4.15)$$

and the initial condition

$$d_0 = \delta_0^*, \dots, d_m = \delta_m^*. \quad (4.16)$$

According to Theorem 3 we have the representation

$$d_{n+m} = P_n^{(0)}d_m + P_n^{(1)}d_{m-1} + \cdots + P_n^{(m)}d_0 + q_{n+m}^*, \quad (4.17)$$

$$q_{n+m}^* = \sum_{i=m+1}^{n+m} P_{n+m-i}^{(0)} \delta_i^*. \quad (4.18)$$

Therefore

$$\|d_{n+m}\| \leq \|P_n^{(0)}\| \|d_m\| + \|P_n^{(1)}\| \|d_{m-1}\| + \cdots + \|P_n^{(m)}\| \|d_0\| + \|q_{n+m}^*\|, \quad (4.19)$$

where

$$\|q_{n+m}^*\| \leq \sum_{i=m+1}^{n+m} \|P_{n+m-i}^{(0)}\| \|\delta_i^*\|. \quad (4.20)$$

Let

$$\max \{ \|\delta_i^*\|, i=0, 1, \dots, m; m+1, \dots, N \} = \bar{\sigma} \quad (4.21)$$

be the maximum numerical error at each level which as we may reasonably suppose can be made as small as we please. Now if the stability condition holds, we shall have

$$\|q_{n+m}^*\| = O\left(\frac{1}{k^{1+\beta}}\right) \cdot \frac{\tau}{k} \bar{\sigma} = \bar{\sigma} \cdot O\left(\frac{1}{k^{2+\beta}}\right) \quad (4.22)$$

and from (4.19) making use of the relations (2.21)

$$\|d_{n+m}\| = \bar{\sigma} \cdot O\left(\frac{1}{k^{2+\beta}}\right). \quad (4.23)$$

When $\bar{\sigma}$ is fixed and k is made to tend to 0, the departure will tend to infinity, but this procedure has no sense in numerical computation. In numerical computation it is essential that the departure can be made as small as we please by choosing $\bar{\sigma}$ sufficiently small while k is kept fixed, and we have seen that this is attained when the difference scheme is stable.

Now we shall examine the matter when we have an analogous linear difference equation in which the coefficients are all special tridiagonal matrices.

$$P_n^{(0)} = T^{-1} \tilde{P}_n^{(0)} T = \frac{2}{M+1} T \tilde{P}_n^{(0)} T, \quad (4.24)$$

and in it by (3.6) $\|T\| \leq M$, so that

$$\|P_n^{(0)}\| \leq \frac{2M^2}{M+1} \|\tilde{P}_n^{(0)}\|, \quad (4.25)$$

and we wish to propose that the condition

$$|\rho_{j,s}| \leq 1 \quad (j=1, \dots, M; s=1, \dots, m+1) \quad (4.26)$$

along with the already assumed condition that the roots $\rho_{j,s}$ ($s=1, \dots, m+1$) of each of the j -th equation (3.19) are all distinct is sufficient for the stability. Here we shall illustrate this in two familiar schemes.

In the explicit formula (1.13) we have

$$A^{(0)} = a_0 I + \alpha_0 J, \quad a_0 = 2(1-r^2), \quad \alpha_0 = r^2, \\ A^{(1)} = a_1 I + \alpha_1 J, \quad a_1 = -1, \quad \alpha_1 = 0,$$

so that by (3.10)

$$x_j^{(0)} = 2(1-r^2) + 2r^2 \cos \frac{j\pi}{M+1} = 2 - 4r^2 \sin^2 \frac{j\pi}{2(M+1)},$$

$$x_j^{(1)} = -1,$$

and the j -th characteristic equation is

$$\rho^2 - 2 \left(1 - 2r^2 \sin^2 \frac{j\pi}{2(M+1)} \right) \rho + 1 = 0. \quad (4.27)$$

In order that the above condition may be satisfied the two roots $\rho_{j,1}$ and $\rho_{j,2}$ must be conjugate complex, i.e. the inequality

$$r^2 \sin^2 \frac{j\pi}{2(M+1)} < 1 \quad (4.28)$$

must hold, so that we must have

$$0 < r \leq 1 \quad (4.29)$$

as the condition for stability.

In the implicit scheme (1.18) it is simpler to diagonalize it as it stands so that we have

$$\tilde{F}v_{n+1} = -2Iv_n + Iv_{n-1}, \quad (4.30)$$

where

$$\tilde{F} = \text{Diag} \left[-(1+2r^2) + 2r^2 \cos \frac{\pi}{M+1}, \dots, -(1+2r^2) + 2r^2 \cos \frac{M\pi}{M+1} \right],$$

and the j -th characteristic equation has the form:

$$\left(1 + 4r^2 \sin^2 \frac{j\pi}{2(M+1)} \right) \rho^2 - 2\rho + 1 = 0. \quad (4.31)$$

Here the discriminant being equal to

$$-4r^2 \sin^2 \frac{j\pi}{2(M+1)},$$

the stability condition (4.26) is satisfied for

$$0 < r < \infty, \quad (4.32)$$

which is a well known fact.

Now let us return to general discussions. If the proposed assumption for stability is satisfied for positive parameter values of r lying in some closed

interval I_r , the $m+1$ distinct roots $\rho_{j,s}$ ($s=1, \dots, m+1$) lying in the unit circle will remain respectively in some closed domains that are disjoint each other, so that the quantities $\sum |\gamma_{1,s}|, \dots, \sum |\gamma_{M,s}|$ in (3.26) will be bounded by a positive constant. Then we shall have uniform boundedness

$$\|\tilde{P}_n^{(0)}\| \leq \max \left\{ \sum_{s=1}^{m+1} |\gamma_{1,s} \rho_{1,s}^n|, \dots, \sum_{s=1}^{m+1} |\gamma_{M,s} \rho_{M,s}^n| \right\} \leq A \quad (4.33)$$

for $r \in I_r$, $0 < k \leq k_0$, $n=0, 1, \dots, N \leq \tau/k$, and from (4.25) the expected behavior for $P_n^{(0)}$:

$$\|P_n^{(0)}\| = O\left(\frac{1}{k}\right). \quad (4.34)$$

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