NOTE ON THE RECURRENCE OF A MARKOV CHAIN

By

Toshihiro HAMACHI

Dedicated to Professor Wasao Sibagaki on his 60-th birthday

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1. The object of this paper is to give a set of the necessary and sufficient conditions for a state in a denumerable Markov chain, defined by the transition matrix (p_{ij}) ($i, j=1, 2, \dots$), to be recurrent-not-null and to be ergodic respectively. These notions are defined by means of a set of the transition probabilities $\{p_{ij}^{(n)}, n \geq 1\}$ and of the first entrance time distributions $\{f_{ij}^{(n)}, n \geq 1\}$ in the following. We say a state i is

- 1) transient, if $\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$,
 - 2) recurrent, if $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$,
 - 3) recurrent-null, if it is recurrent and $\sum_{n=1}^{\infty} n f_{ii}^{(n)} = \infty$,
and
 - 4) recurrent-not-null, if it is recurrent and $\sum_{n=1}^{\infty} n f_{ii}^{(n)} < \infty$ respectively.
- In particular, a recurrent-not-null state i is called to be ergodic if the greatest common divisor of the set of integers n for which $p_{ii}^{(n)} > 0$ equals one.

In order to enunciate our results, it is convenient to introduce the two sets $\mathcal{W}$ and $\mathcal{W}'$. Let $\mathcal{W}$ be the set of all (not necessary strictly) monotone decreasing sequences $\omega = \{\omega_n, n \geq 1\}$ for which $\lim_{n \rightarrow \infty} \omega_n = 0$ and $\sum_{n=1}^{\infty} \omega_n = \infty$, and let $\mathcal{W}'$ be the set of all sequences for which $\sum_{n=1}^{\infty} \omega_n = \infty$ with $\omega_n \geq 0$ ($n=1, 2, \dots$). Under these preparations we have the following theorems.

THEOREM 1. The necessary and sufficient condition that a state i is recurrent-not-null is

$$(A) \quad \sum_{n=1}^{\infty} \omega_n p_{ii}^{(n)} = \infty \quad \text{for every } \omega = \{\omega_n, n \geq 1\} \in \mathcal{W}.$$

THEOREM 2. A state i is ergodic if and only if

$$(B) \quad \sum_{n=1}^{\infty} \omega_n p_{ii}^{(n)} = \infty \text{ for every } \omega = \{\omega_n, n \geq 1\} \in W'.$$

COROLLARY. A state i is ergodic if and only if $\sum_{k=1}^{\infty} p_{ii}^{(n_k)} = \infty$ for every subsequence $\{n_k, k \geq 1\}$ of the sequence of all positive integers.

2. Proof of Theorems. In order to prove our theorems, we shall appeal to the following well-known results, which we enunciate in the following lemma.

LEMMA. (See K. L. Chung [2], p. 27 and p. 31.) If a state i is transient or recurrent-null, then $\lim_{n \rightarrow \infty} p_{ii}^{(n)} = 0$. If a state i is recurrent-not-null, then $\lim_{n \rightarrow \infty} p_{ii}^{(nd_i)} = \frac{d_i}{\sum_{n=1}^{\infty} n f_{ii}^{(n)}} > 0$, where d_i is the greatest common divisor of $\{n \mid p_{ii}^{(n)} > 0\}$.

PROOF OF THEOREM 1. Let a state i be recurrent-not-null. Then there exists an integer N so large that $|p_{ii}^{(nd_i)} - c| < c/2$ for every $n \geq N$ where c is the limit of a sequence $\{p_{ii}^{(nd_i)}, n \geq 1\}$ which exists and is positive from the preceding Lemma. Hence we have, for every $\omega = \{\omega_n, n \geq 1\} \in W$,

$$\begin{aligned} \sum_{n=1}^{\infty} \omega_n p_{ii}^{(n)} &= \sum_{n=1}^{\infty} \omega_{nd_i} p_{ii}^{(nd_i)} \geq \sum_{n=1}^{N-1} \omega_{nd_i} p_{ii}^{(nd_i)} + c/2 \sum_{n=N}^{\infty} \omega_{nd_i}, \\ \sum_{n=d_i}^{\infty} \omega_n &= \sum_{k=0}^{d_i-1} \sum_{n=1}^{\infty} \omega_{nd_i+k} \leq \sum_{n=1}^{\infty} \omega_{nd_i}, \end{aligned}$$

and consequently $\sum_{n=1}^{\infty} \omega_n p_{ii}^{(n)} = \infty$, which shows the necessity of the condition (A).

In order to prove the sufficiency of the condition (A), let us suppose that the state i is recurrent-null or transient. Since $\lim_{n \rightarrow \infty} p_{ii}^{(n)} = 0$ from the preceding Lemma, we can choose a subsequence $\{n_k, k \geq 0\}$ of integers such that $n_0 = 1$, $n_1 > 1$, $n_k - n_{k-1} > n_{k-1} - n_{k-2}$, and $p_{ii}^{(n_k)} < 1/2^k$ for every $n > n_k$. Let us define a sequence $\omega = \{\omega_n, n \geq 1\} \in W$ in the following that

$$\omega_n = \frac{1}{n_k - n_{k-1}} \quad \text{for } n_{k-1} \leq n < n_k.$$

Then we have $\sum_{n=1}^{\infty} \omega_n p_{ii}^{(n)} < \infty$, which contradicts to the condition (A), proving its sufficiency.

PROOF OF THEOREM 2. If a state i is ergodic, the limit of a sequence $\{p_{ii}^{(n)}, n \geq 1\}$ exists and is positive from the preceding Lemma. Then it is trivial that the state i satisfies the condition (B). Conversely, if the state i which satisfies the condition (B) is not ergodic, then there exists a subsequence $\{n_k, k \geq 1\}$ of positive integers such that $\lim_{k \rightarrow \infty} p_{ii}^{(n_k)} = 0$. Here

we may and shall assume the $\sum_{k=1}^{\infty} p_{ii}^{(n_k)} < \infty$. Now let us define a sequence $\omega = \{\omega_n, n \geq 1\} \in W$,

$$\omega_n = \begin{cases} 1 & \text{if } n = n_k \quad k=1, 2, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $\sum_{n=1}^{\infty} \omega_n p_{ii}^{(n)} < \infty$ which contradicts to the condition (B), completing the proof of Theorem 2.

The Corollary is now immediate from Theorem 2.

References

- [1] Ulrich KRENGEL, *Classification of States for Operators*, Proc. Fifth Berkeley Sympos. Math. Statist. Probability II, (1966), 415-429.
- [2] K. L. CHUNG, *Markov chains with stationary transition probabilities*, Springer (1960).