ON A PSEUDO-DIFFERENT IN AN ALGEBRAIC FUNCTION FIELD OF ONE VARIABLE

By

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The purpose of this note is to simplify the results of Nering [4], making no use of the reduction theory, and to prove that the conductor just coincides with an inverse $\tilde{s}$ -different in a constant field extension of an algebraic function field of one variable under the assumption II in §2.

§1. Pseudo-different.

Let K be an algebraic function field of one variable with constant field k_0 . Let l_0 be a finite extension of k_0 and then $Kl_0 = L$ is a constant field extension of K . Let x be an arbitrarily fixed transcendental element of K over k_0 . Denote by $\mathfrak{L}$ a non-trivial k -linear mapping of K into $k_0(x) = k$ and $\tilde{s}$ a non-trivial K -linear mapping of L into K . Let s_0 be a non-trivial k_0 -linear mapping of l_0 into k_0 . Let s denote a k -linear mapping of $l_0(x) = l$ into k satisfying the following assumption:

I s restricted to l coincides with s_0 .

The following lemma is well known. (Lamprecht [2], Tate [5]).

LEMMA 1. *There exists a uniquely determined l -linear mapping $\tilde{\mathfrak{L}}$ of L into l such that $\mathfrak{L} \circ \tilde{s} = s \circ \tilde{\mathfrak{L}}$, i.e., $\mathfrak{L}(\tilde{s}(\alpha)) = s(\tilde{\mathfrak{L}}(\alpha))$ for all $\alpha \in L$. If $\mathfrak{d}_{\tilde{s}}(L/K)$ is the $\tilde{s}$ -different of L with respect to K , $\mathfrak{d}_{\mathfrak{L}}(K/k)$ is the $\mathfrak{L}$ -different of K with respect to k , and $\mathfrak{d}_{\tilde{\mathfrak{L}}}(L/l)$ is the $\tilde{\mathfrak{L}}$ -different of L with respect to l , then*

$$(1) \quad \mathfrak{d}_{\mathfrak{L}}(K/k) \mathfrak{d}_{\tilde{s}}(L/K) = \mathfrak{d}_{\tilde{\mathfrak{L}}}(L/l).$$

PROOF. $\mathfrak{L} \circ \tilde{s}$ is the non-trivial k -linear mapping of L into k , and the first half of our assertion is obvious. Thus we have

$$\mathfrak{d}_{\mathfrak{L}}(K/k) \mathfrak{d}_{\tilde{s}}(L/K) = \mathfrak{d}_{s \circ \tilde{\mathfrak{L}}}(L/k) = \mathfrak{d}_s(l/k) \mathfrak{d}_{\tilde{\mathfrak{L}}}(L/l).$$

From the assumption I, we see readily that $\mathfrak{d}_s(l/k)$ is the unit divisor of l . Hence we obtain our lemma.

REMARK. It is the well-known fact that Lemma 1 holds also in the case where L is an arbitrary finite extension of K and l_0 is the constant field of L . (Lamprecht [2], Tate [5]). If we shall denote by g and G respectively the genus of K/k_0 and of L/l_0 , then in view of the formula (1) and the equality

$$d_L(\mathfrak{d}_s(K/k)) = \lambda_{L/K}^{-1}(2g - 2 + 2[K:k]),$$

we have easily Hurwitz's formula

$$G - 1 = \lambda_{L/K}^{-1}(g - 1) + \frac{1}{2} d_L(\mathfrak{d}_s(L/K)),$$

where d_L denotes the degree in L and $\lambda_{L/K} = \frac{[l_0:k_0]}{[L:K]}$ (e.g., Deuring [1]; pp. 117, 118).

§2. Conductor.

The notations and assumptions being the same as in §1, for the constant field extension L of K we assume the following

$$\text{II} \quad [L:K] = [l_0:k_0], \text{ i.e., } [K:k] = [L:l] = n,$$

and for the K -linear mapping $\tilde{s}$ of L into K we assume the following

$$\text{III} \quad \tilde{s} \text{ restricted to } l_0 \text{ coincides with } s_0.$$

Under these assumptions the l -linear mapping $\tilde{\mathfrak{L}}$ of L into l in Lemma 1 coincides also with the k -linear mapping $\mathfrak{L}$ of K into k on K . We put the rings $k_0[x] = \mathfrak{o}_1$, $k_0\left[\frac{1}{x}\right] = \mathfrak{o}_2$, $l_0[x] = \tilde{\mathfrak{o}}_1$, and $l_0\left[\frac{1}{x}\right] = \tilde{\mathfrak{o}}_2$. Let $(\mathfrak{O}_K)_i$ be the ring of integers of K with respect to $\mathfrak{o}_i$ for $i=1, 2$ and let $(\mathfrak{O}_L)_i$ be the ring of integers of L with respect to $\tilde{\mathfrak{o}}_i$. Whenever no confusion occurs, we shall omit the subindex i . We shall denote by $\overline{\mathfrak{O}_K}$ the $\tilde{\mathfrak{o}}$ -module generated by $\mathfrak{O}_K$, i.e., $\overline{\mathfrak{O}_K} = \mathfrak{O}_K \tilde{\mathfrak{o}}$. Then it is clear that $\mathfrak{O}_L$ contains $\overline{\mathfrak{O}_K}$, and that $\overline{\mathfrak{O}_K}$ is an n -dimensional module and so an order in $\mathfrak{O}_L$ because of the assumption II. Let $\mathfrak{F}_i$ for $i=1, 2$ be the conductor of $(\overline{\mathfrak{O}_K})_i$ with

respect to $(\mathfrak{O}_L)_i$ i.e., $\mathfrak{F}_i = \overline{(\mathfrak{O}_K)_i} : (\mathfrak{O}_L)_i = \{\alpha \in L; \alpha\beta \in \overline{(\mathfrak{O}_K)_i} \text{ for all } \beta \in (\mathfrak{O}_L)_i\}$. Since $1 \in (\mathfrak{O}_L)_i$, $\mathfrak{F}_i$ is obviously an integral ideal of $(\mathfrak{O}_L)_i$ for $i=1, 2$. Let $\mathfrak{f}_i$ be a divisor of L associated with $\mathfrak{F}_i$ for $i=1, 2$ and define a divisor $\mathfrak{f}$ of L by the rule

$$\text{ord}_{\mathfrak{P}} \mathfrak{f} = \begin{cases} \text{ord}_{\mathfrak{P}} \mathfrak{f}_1 & \text{for } \mathfrak{P} \nmid \mathfrak{n}_x, \\ \text{ord}_{\mathfrak{P}} \mathfrak{f}_2 & \text{for } \mathfrak{P} \mid \mathfrak{n}_x, \end{cases}$$

where $\mathfrak{n}_x$ is the denominator of x in L , $\mathfrak{P}$ is a prime divisor of L , and $\text{ord}_{\mathfrak{P}}$ denotes the order at $\mathfrak{P}$. Since $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are integral ideals, $\mathfrak{f}$ is an integral divisor of L . $\mathfrak{f}$ is called the **conductor** of L to K . Now let $\mathfrak{O}'_K$ be the complementary ideal of $\mathfrak{O}_K$ with respect to $\mathfrak{g}$ and $\mathfrak{o}$, i.e., $\mathfrak{O}'_K = \{\alpha \in K; \mathfrak{g}(\alpha\beta) \in \mathfrak{o} \text{ for all } \beta \in \mathfrak{O}_K\}$. Let $\overline{\mathfrak{O}'_K}$ be the complementary module of $\overline{\mathfrak{O}_K}$ with respect to $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{o}}$, i.e., $\overline{\mathfrak{O}'_K} = \{\alpha \in L; \tilde{\mathfrak{g}}(\alpha\beta) \in \tilde{\mathfrak{o}} \text{ for all } \beta \in \overline{\mathfrak{O}_K}\}$. We shall denote by $\overline{\mathfrak{O}'_K}$ the $\tilde{\mathfrak{o}}$ -module generated by $\mathfrak{O}'_K$, i.e., $\overline{\mathfrak{O}'_K} = \mathfrak{O}'_K \tilde{\mathfrak{o}}$. From the assumption II $\overline{\mathfrak{O}'_K}$ is an n -dimensional $\tilde{\mathfrak{o}}$ -module. Since $\tilde{\mathfrak{g}}$ coincides with $\mathfrak{g}$ on K , we see immediately

$$\text{LEMMA 2. } \overline{\mathfrak{O}'_K} = \overline{\mathfrak{O}'_K}.$$

The following lemma is well known. (Nering [3]).

$$\text{LEMMA 3. } (\overline{\mathfrak{O}'_K} \mathfrak{O}_L)^{-1} = (\mathfrak{O}'_L)^{-1} \mathfrak{F}.$$

PROOF. Here we shall mean by the prime, $(\prime)$, the complement with respect to $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{o}}$. Because of the fact that for any n -dimensional $\tilde{\mathfrak{o}}$ -module $\mathfrak{A}$ in L $(\mathfrak{A})' = \mathfrak{A}$, $(\overline{\mathfrak{O}'_K} \mathfrak{O}_L)' = (\overline{\mathfrak{O}'_K})' : \mathfrak{O}_L = \overline{\mathfrak{O}_K} : \mathfrak{O}_L = \mathfrak{F}$. Hence

$$\overline{\mathfrak{O}'_K} \mathfrak{O}_L = ((\overline{\mathfrak{O}'_K} \mathfrak{O}_L))' = \mathfrak{F}' = (\mathfrak{O}_L \mathfrak{F})' = \mathfrak{O}'_L : \mathfrak{F} = \mathfrak{O}'_L \mathfrak{F}^{-1}.$$

This completes the proof of the lemma.

THEOREM 1. *We have*

$$\mathfrak{d}_{\mathfrak{g}}(K/k) = \mathfrak{d}_{\tilde{\mathfrak{g}}}(L/l) \mathfrak{f}.$$

PROOF. Because $\overline{(\mathfrak{O}'_K)_1}(\mathfrak{O}_L)_1$ is the ideal of $(\mathfrak{O}_L)_1$ associated with the finite part of $(\mathfrak{d}_{\mathfrak{g}}(K/k))^{-1}$ with respect to the denominator $\mathfrak{n}_x$ of x , $\overline{(\mathfrak{O}_K)_1}(\mathfrak{O}_L)_1^{-1}$ is the ideal of $(\mathfrak{O}_L)_1$ associated with the finite part of

$\mathfrak{d}_{\mathfrak{g}}(K/k)$ with respect to $\mathfrak{n}_x$ from Lemma 2. On the other hand $(((\mathfrak{D}_L)_1)')^{-1}\mathfrak{F}_1$ is the ideal of $(\mathfrak{D}_L)_1$ associated with the finite part of $\mathfrak{d}_{\mathfrak{g}}(L/l)_{\mathfrak{f}_1}$ with respect to $\mathfrak{n}_x$. Therefore from Lemma 3 we see

$$\text{ord}_{\mathfrak{P}} \mathfrak{d}_{\mathfrak{g}}(K/k) = \text{ord}_{\mathfrak{P}} \mathfrak{d}_{\mathfrak{g}}(L/l)_{\mathfrak{f}_1} \quad \text{for } \mathfrak{P} \nmid \mathfrak{n}_x,$$

where $\mathfrak{P}$ is a prime divisor of L . In a similar fashion, we can obtain

$$\text{ord}_{\mathfrak{P}} \mathfrak{d}_{\mathfrak{g}}(K/k) = \text{ord}_{\mathfrak{P}} \mathfrak{d}_{\mathfrak{g}}(L/l)_{\mathfrak{f}_2} \quad \text{for } \mathfrak{P} \mid \mathfrak{n}_x.$$

Thus the theorem is completely proved.

By making use of Theorem 1 and Lemma 1, we obtain the following

$$\text{COROLLARY 1.} \quad \mathfrak{f} = (\mathfrak{d}_{\tilde{s}}(L/K))^{-1},$$

where $\tilde{s}$ is a K -linear mapping of L into K , whose restriction on l_0 coincides with s_0 .

Finally we shall consider the genus change in the constant field extension satisfying the assumption II. We shall denote by g and G the genus of K and of L respectively. Let $\bar{l}_0$ be the constant field of L . Let λ_0 be a non-trivial l_0 -linear mapping of $\bar{l}_0$ into l_0 . Let λ denote an l -linear mapping of $\bar{l}_0(x) = \bar{l}$ into $l_0(x) = l$ which coincides with λ_0 on $\bar{l}_0$. Then, as is well known, there exists a uniquely determined $\bar{l}$ -linear mapping $\bar{\mathfrak{g}}$ of L into $\bar{l}$ such that $\tilde{\mathfrak{g}} = \lambda \circ \bar{\mathfrak{g}}$, and we have $\mathfrak{d}_{\mathfrak{g}}(L/l) = \mathfrak{d}_{\bar{\mathfrak{g}}}(L/\bar{l})$ where $\mathfrak{d}_{\bar{\mathfrak{g}}}$ is the $\bar{\mathfrak{g}}$ -different of L with respect to $\bar{l}$. Therefore from Theorem 1

$$\begin{aligned} \lambda_{L/K}^{-1}(2g - 2 + 2n) &= d_L(\mathfrak{d}_{\mathfrak{g}}(K/k)) = d_L(\mathfrak{d}_{\bar{\mathfrak{g}}}(L/\bar{l})) + d_L(\mathfrak{f}) \\ &= 2G - 2 + 2[L : \bar{l}] + d_L(\mathfrak{f}) \quad \text{with} \quad \lambda_{L/K} = \frac{[\bar{l}_0 : k_0]}{[L : K]}. \end{aligned}$$

Thus we easily obtain the following

COROLLARY 2. *Under the assumption II,*

$$g - 1 = \lambda_{L/K}(G - 1) + \rho,$$

where ρ is the non-negative integer defined by $2\rho = \lambda_{L/K} d_L(\mathfrak{f})$.

This is a generalization of the formula of Nering [4].

Bibliography

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