**AN APPLICATION OF SHIFT-COMMUTATIVE LINEAR
OPERATORS TO LINEAR AUTOMATIC
CONTROL SYSTEMS WITH RANDOMNESS
—A THEORETICAL FORMULATION—**

By

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Dedicated to Professor Tosio Kitagawa on his 60-th birthday

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1. Introduction.

Recently, with the developments of the theory of automatic control systems, the applications of the theory of stochastic processes related to the stochastic functional equations have been investigated by many research workers. In spite of many remarkable results having been obtained in the branch of automatic control systems, it seems to us as if the results on the automatic control systems with randomness were simply the formal analogue of those of non-random automatic control systems.

In this note, the author will try to give the theoretical foundations for the investigations of the linear automatic control systems with randomness, applying the results of T. Seguchi [1] and Appendixes in this note about the shift-commutative linear operators and the related stochastic processes. In the body of this note, to make clear the structure of the formulation, we shall explain the course of theory of automatic control systems with randomness, using only the results for shift-commutative linear operators and related stochastic or non-stochastic functional equations.

In the theory of automatic control systems, Professor T. Kitagawa has proposed the question that it is whether appropriate or not, from the Figure 3.1, to induce formally the equation

$$(*) \quad (A^{-1} + I) y(t) = x(t)$$

of the automatic control system. In spite of the above question being

very important, in this note, we shall make no reference to the above question.

2. Preliminaries.

Let us consider the *linear systems* of non-random cases. In the most primitive system is given by Fig. 2.1. As on p. 99 in N. Wiener [2], if the input and the output are $x(t)$ and $y(t)$ respectively, then the characteristics of this system are (1) *independency of the shift of time parameter* and (2) *linearity*.

FIG. 2.1.

The one of the most simple system among the various systems such as Fig. 2.1 is given by the following equation:

$$(2.1) \quad y(t) = \int_0^{\infty} \alpha(s) x(t-s) ds.$$

Because in a (*physical*) system the effect (=output) $y(t)$ can not precede the cause (=input) $x(t)$, the integral domain in (2.1) must be included in $[0, +\infty)$, as was mentioned on pp. 98–99 in N. Wiener [2].

As the function which represents the character of such system, there is the *transfer function* (t. f.) which is given by

$$(2.2) \quad A(z) \equiv \int_0^{\infty} \alpha(s) e^{-zs} ds, \quad z \in C.^{1)}$$

And if we put formally

$$(2.3) \quad X(z) \equiv \int_{-\infty}^{\infty} e^{-zt} x(t) dt \quad \text{and} \quad Y(z) \equiv \int_{-\infty}^{\infty} e^{-zt} y(t) dt,$$

then the following relation

$$(2.4) \quad X(z)A(z) = Y(z)$$

is satisfied.

1) C is the set of all complex numbers.

This result can be obtained by the quite formal calculations, but is limited to the very special types of systems which are represented by integral operators such as (2.1), the differential operators or difference operators etc.. In the non-random cases, though the above result is sufficient in practical uses, but it is questionable in general or theoretical, whether the right hand sides of (2.3) will be meaningful or not. And in the random cases, for example in the stationary input case (see Section 4), the corresponding formulas to (2.3) can not generally be defined. Henceforth, we shall try to generalize and theorize the theory of linear automatic control systems.

Let the relation corresponding to (2.1) be given by

$$(2.5) \quad y(t) = Ax(t),$$

where A may be a shift-commutative linear operator (S-C. L. Op.) defined in T. Seguchi [1]²⁾. As the corresponding function to the transfer function of the system, which is expressed by (2.5), is given by

$$(2.6) \quad C_A(z) \equiv Ae^{zt} |_{t=0},$$

and is called the *characterizing function* (c. f.) of the S-C. L. Op. A . And the function $G_A(\lambda) \equiv C_A(i\lambda)$ is called the *generating function* (g. f.) of the S-C. L. Op. A . It is easily seen that the c. f. $C_A(z)$ and the t. f. $A(z)$ are the same functions, if they are meaningful.

Throughout the body of this note, let us assume for simplicity that (A) for any S-C. L. Op. A , there exists the strictly inverse operator A^{-1} .³⁾

Let $x(t)$ and $y(t)$ be⁴⁾

$$x(t) \equiv \int_{-\infty}^{\infty} e^{i\lambda t} dZ_x(\lambda) \quad \text{and} \quad y(t) \equiv \int_{-\infty}^{\infty} e^{i\lambda t} dZ_y(\lambda),$$

then the following relation is obtained theoretically:

$$(2.7) \quad C_A(i\lambda) dZ_x(\lambda) = G_A(\lambda) dZ_x(\lambda) = dZ_y(\lambda).$$

From the assumption (A), the relation (2.7) is equivalent to the relation (2.4).

2, 3) See Appendix I.

4) If $x(t)$ and $y(t) \in L^1(m, 0)$ or $L^2(m, 0)$, these representations are meaningful. The notation $L^2(m; 0)$ will be defined in Appendixes.

Next, we shall consider the combinations of some systems. The fundamental combinations are the *series* and the *parallel*, which are given by Fig. 2.2 and Fig. 2.3. And in these cases, we have the following well known relations between C_d , C_A and C_r or A , A_1 and A_2 :

FIG. 2.2.

FIG. 2.3.

$$\text{series: } C_d(z) = C_A(z)C_r(z) \quad \text{or} \quad A(z) = A_1(z)A_2(z),$$

$$\text{parallel: } C_d(z) = C_A(z) + C_r(z) \quad \text{or} \quad A(z) = A_1(z) + A_2(z).$$

As the all systems are characterized by the c.f.s of operators, we shall represent the systems by c.f.s or operators corresponding to them. And the all complicated systems are obtained by finite arithmetical operations of series and parallels of unit systems. Hence, it may be considered that C_d (or A) is the rational function of C_{A_1} , C_{A_2} , ..., C_{A_n} (or operators A_1 , A_2 , ..., A_n), regarding as the above functions (or operators) as the variables.

3. Automatic control systems.

DEFINITION 3.1. The system which includes the feedback system (F.S.) is called the *automatic control system* (A.C.S.), and the feedback system will be represented by $C_r(z)$ or Γ .

The most primitive A.C.S. is given by the following Fig. 3.1:

FIG. 3.1.

Then the A.C.S. C_d is represented by well known relation as follow:

$$(3.1) \quad 1/C_d = 1/C_A + C_r, \quad \text{or} \quad C_d = C_A/(1 + C_A C_r).$$

Let the input $x(t)$ and the output $y(t)$ be given by the spectral representations

$$x(t) \equiv \int_{-\infty}^{\infty} e^{i\lambda t} dZ_x(\lambda) \quad \text{and} \quad y(t) \equiv \int_{-\infty}^{\infty} e^{i\lambda t} dZ_y(\lambda),$$

respectively. Then we have the relation

$$(3.2) \quad dZ_y(\lambda) = C_d(i\lambda) dZ_x(\lambda) \equiv G_d(\lambda) dZ_x(\lambda),$$

where C_d and G_d are that of (3.1) and $G_d(\lambda) \equiv C_d(i\lambda)$ respectively. Hence if the S-C.L.Op. $\mathcal{A}$ is the operator whose c.f. is C_d , then the relation

$$(3.3) \quad y(t) = \mathcal{A}x(t) \quad \text{or} \quad \mathcal{A}^{-1}y(t) = x(t)$$

holds. Here we wish to point out that the two equations of (3.3) have different meanings; that is, *in the left hand side equation, the output $y(t)$ is uniquely determined by the input $x(t)$* , but *in the right hand side one, $y(t)$ is not necessarily uniquely determined by $x(t)$* (consider the case that $\mathcal{A}^{-1}y(t) = 0$ has non-zero solutions). In the physical systems, there is the case that it appears, in the output, the oscillation which is independent of input $x(t)$, and it is well known that such case appears mainly in the automatic control system and that there are very intimate relations to the stability problems. The equation of right hand side in (3.3) shows the fact that we have just pointed out. Hence it is better to describe the relation between the input and the output in the form of right hand side.

Some examples of A.C.S.s in N. Wiener [2].

EXAMPLE 3.1. Let us consider the A.C.S. such as Fig. 3.2.

FIG. 3.2.

Then the relations

$$\begin{aligned} dZ_{\tilde{x}}(\lambda) &= G_{A_1}(\lambda) dZ_x(\lambda), & G_{A_2}(\lambda) [dZ_{\tilde{x}}(\lambda) - dZ_{\tilde{y}}(\lambda)] &= dZ_y(\lambda), \\ dZ_{\tilde{y}}(\lambda) &= G_r(\lambda) dZ_y(\lambda) \end{aligned}$$

hold. Hence, we obtain the following relation:

$$G_{A_1}(\lambda) dZ_x(\lambda) = [1/G_{A_2}(\lambda) + G_r(\lambda)] dZ_y(\lambda).$$

This equation means that between the input $x(t)$ and the output $y(t)$ the relation

$$(3.4) \quad A_1^{-1} [A_2^{-1} + I] y(t) = x(t)$$

holds. Especially, when $A_1 = I$, i. e. $C_{A_1}(z)$, $G_{A_1}(\lambda) \equiv 1$ for every $z \in C$ and $\lambda \in R^5$, then we have

$$[A_2^{-1} + I] y(t) = x(t).$$

EXAMPLE 3.2. Let us consider the A.C.S. such as Fig. 3.3.

FIG. 3.3.

Then in the same way as in Example 3.1, we have the following relation:

$$\begin{aligned} dZ_x(\lambda) &= [1 + G_{A_1}(\lambda) G_{A_2}(\lambda) G_r(\lambda)] dZ_y(\lambda) / [G_{A_1}(\lambda) G_{A_2}(\lambda)] \\ &= [1 / (G_{A_1}(\lambda) G_{A_2}(\lambda)) + G_r(\lambda)] dZ_y(\lambda). \end{aligned}$$

This equation means that between the input $x(t)$ and the output $y(t)$ the relation

$$(3.5) \quad [A_1^{-1} A_2^{-1} + I] y(t) = x(t)$$

holds.

It follows that all above examples are essentially represented in the form of right hand side in (3.3).

5) R is the set of all real numbers, i. e. $R \equiv (-\infty, +\infty)$.

4. A.C.S.s with stationary random input.

In this section and the next section, we shall concerned with the typical cases of A.C.S.s with randomness, i. e. the case of stationary input and the case with random (white) noise.

Let us consider the A.C.S.s such as Fig. 3.1, but let us assume that the input $x(t)$ and the output $y(t)$ are certain regular stationary processes. Then $x(t)$ and $y(t)$ can be represented as the followings:

$$\begin{aligned} x(t) &\equiv \int_{-\infty}^{\infty} e^{i\lambda t} dZ_x(\lambda), & r_x(t) &\equiv \int_{-\infty}^{\infty} e^{i\lambda t} \mu_x(d\lambda), \\ y(t) &\equiv \int_{-\infty}^{\infty} e^{i\lambda t} dZ_y(\lambda), & r_y(t) &\equiv \int_{-\infty}^{\infty} e^{i\lambda t} \mu_y(d\lambda), \end{aligned}$$

where $dZ_x(\lambda) \equiv dZ_x(\lambda, \omega)$, $dZ_y(\lambda) \equiv dZ_y(\lambda, \omega)$ and μ_x , μ_y are *orthogonal random measures* and *spectral measures* of $x(t)$ and $y(t)$ respectively. And between the input $x(t)$ and the output $y(t)$, the following relation holds:

$$(4.1) \quad \mathcal{A}^{-1}y(t) \equiv [\mathcal{A}^{-1} + \Gamma]y(t) = x(t),$$

where $\mathcal{A}^{-1}$ is a S-C.L.Op. whose c.f. is $C_{\mathcal{A}^{-1}} \equiv 1/C_{\mathcal{A}} = 1/C_{\mathcal{A}} + C_{\Gamma}$ and the interpretation of the equation (4.1) is such that; for every $\lambda \in R$

$$\begin{aligned} 1/C_{\mathcal{A}}(i\lambda) dZ_y(\lambda) &\equiv 1/G_{\mathcal{A}}(\lambda) dZ_y(\lambda) \equiv [1/G_{\mathcal{A}}(\lambda) + G_{\Gamma}(\lambda)] dZ_y(\lambda) \\ &\equiv [1/C_{\mathcal{A}}(i\lambda) + C_{\Gamma}(i\lambda)] dZ_y(\lambda) = dZ_x(\lambda) \end{aligned}$$

holds. Hence, if $1/C_{\mathcal{A}}(i\lambda) \neq 0$ for any $\lambda \in R$, then the relation

$$\begin{aligned} (4.2) \quad V[y(t, \omega)] &= \int_{-\infty}^{\infty} \mu_y(d\lambda) = \int_{-\infty}^{\infty} |C_{\mathcal{A}}(i\lambda)|^2 \mu_x(d\lambda) \\ &= \int_{-\infty}^{\infty} |C_{\mathcal{A}}(i\lambda)/[1 + C_{\mathcal{A}}(i\lambda)C_{\Gamma}(i\lambda)]|^2 \mu_x(d\lambda)^{6)} \end{aligned}$$

holds.

On the other hand, from Appendix II, if $C_{\mathcal{A}^{-1}}$ has the zeros on the imaginary axis, then the relation (4.2) does not necessarily holds. In this case, there exist stationary processes which satisfy the homogeneous equation

6) $V[*]$ means the variance of $*$.

$$(4.3) \quad \mathcal{A}^{-1}y(t) = 0$$

and are orthogonal with $y_s(t) \equiv \mathcal{A}x(t)$.⁷⁾ Furthermore, from Appendix III, if $C_{\mathcal{A}^{-1}}$ has the zeros in C , then the homogeneous equation (4.3) has the non-zero and non-stationary solution $y_0(t) \in C_c^0(R)$ or $C_c^0(R)$, i. e. if the zeros of $C_{\mathcal{A}^{-1}}$ are $z_1, z_2, \dots$, and whose multiplicities are $k_1, k_2, \dots$, then the solution of (4.3) is given by the following form: If $y_0 \in C_c^0(R)$

$$(4.4) \quad y_0(t) = \sum_m \sum_{j=0}^{k_m-1} e^{z_m t} t^j K_{mj} \equiv \sum_m P_m(t) e^{z_m t},$$

where $P_m(t) \equiv \sum_{j=0}^{k_m-1} K_{mj} t^j$, $m=1, 2, \dots$, are certain polynomials with constant coefficients. If $y_0 \in C_c^0(R)$

$$(4.5) \quad y_0(t, \omega) = \sum_m \sum_{j=0}^{k_m-1} e^{z_m t} t^j Z_{mj}(\omega) \equiv \sum_m P_m(t, \omega) e^{z_m t},$$

where $P_m(t, \omega) \equiv \sum_{j=0}^{k_m-1} Z_{mj}(\omega) t^j$, $m=1, 2, \dots$, are certain polynomials with random valued coefficients.

5. A.C.S.s with random noise.

In this section, we shall be concerned with the A.C.S. such as Fig. 5.1.

FIG. 5.1.

In this case, by the same ways as preceding cases, the relation

$$(5.1) \quad C_A(i\lambda) dZ_x(\lambda) + dZ_{\xi}(\lambda) \equiv G_A(\lambda) dZ_x(\lambda) + dZ_{\xi}(\lambda)$$

7) Here we consider the assumption (A). These results shall be mentioned in Appendix II, Proposition II. 1.

$$\begin{aligned}
&= [1 + G_A(\lambda)G_R(\lambda)]dZ_y(\lambda) \\
&\equiv [1 + C_A(i\lambda)C_R(i\lambda)]dZ_y(\lambda)
\end{aligned}$$

holds. Hence, we have the equation

$$(5.2) \quad Ax(t) + \dot{\xi}(t) = (I + A\Gamma)y(t).^{8)}$$

Now, let us consider the two equations such as

$$(5.3) \quad Ax(t) = (I + A\Gamma)y(t)$$

and

$$(5.4) \quad \dot{\xi}(t) = (I + A\Gamma)y(t),$$

where $x(t)$ and $y(t)$ are stationary processes belonging to $\mathfrak{S}^\circ$ or $\mathfrak{S}_0$. Then, under certain conditions, the stationary solution of (5.3) is given by

$$y_x(t) = (I + A\Gamma)^{-1}Ax(t) = \int_{-\infty}^{\infty} e^{i\lambda t} C_A(i\lambda) / [1 + C_A(i\lambda)C_R(i\lambda)] dZ_x(\lambda)$$

and the stationary solution of (5.4) is given by

$$y_{\dot{\xi}}(t) = (I + A\Gamma)^{-1}\dot{\xi}(t) = \int_{-\infty}^{\infty} e^{i\lambda t} / [1 + C_A(i\lambda)C_R(i\lambda)] dZ_{\dot{\xi}}(\lambda),$$

where $dZ_{\dot{\xi}}$ is such as $\dot{\xi}(\varphi) \equiv \int_{-\infty}^{\infty} (\mathcal{F}\varphi)(\lambda) dZ_{\dot{\xi}}(\lambda)$, and $\mathbf{E}[|dZ_{\dot{\xi}}(\lambda)|^2] \equiv d\lambda$.

Because $y_x(t)$ and $y_{\dot{\xi}}(t)$ satisfy the equations (5.3) and (5.4) respectively, $y_x(t) + y_{\dot{\xi}}(t)$ satisfies the equation (5.2). In fact

$$\begin{aligned}
(I + A\Gamma)y_x(t) + (I + A\Gamma)y_{\dot{\xi}}(t) &= (I + A\Gamma)(y_x(t) + y_{\dot{\xi}}(t)) \\
&= Ax(t) + \dot{\xi}(t).
\end{aligned}$$

For the equation (5.4) is meaningful, $(1 + C_A(z)C_R(z))$ must have not zeros on the imaginary axis.

In the present case, when we consider the non-stationary solution, we must consider the homogeneous equation

8) This and following (5.4) are the formal equations. In this case, we must write, in theoretical, that for every $\varphi \in \mathfrak{S}$ in the sense of random distributions $Ax(\varphi) + \dot{\xi}(\varphi) = (I + A\Gamma)y(\varphi)$, (see Appendixes).

$$(5.5) \quad (I + A\Gamma) y(t) = 0,$$

and solve the equation (5.5) such as the way mentioned in the last part of preceding section.

Such as in this case, there are the cases that we must consider the original equation dividing into several sub-equations. Furthermore, each original system (effector, compensator, feedback system, etc.) must belongs to the class $\tilde{A}_-$, A_- , etc. i.e. depend only upon the past, and each sub-system must satisfies certain suitable conditions.

6. Optimum A.C.S.s with randomness.

As were treated in Sections 4 and 5, the *automatic control systems with randomness* are considered the following types: (1) *The input $x(t)$ is the stationary process*, (2) *the A.C.S. with random (white) noise* and (3) *the mixed type of (1) and (2)*. Furthermore, in the practical cases, inputs, outputs and noises are not necessarily limited only one respectively. Hence, in general, we must consider the multi-dimensional cases. On the other hand, as were seen in the preceding examples, the problems of automatic control systems are very difficult and there are many uncultivated grounds. Hence we shall here be concerned with the at most only one-dimensional case, and try to construct the theoretical foundation to investigate the problem of automatical control system.

The cases concerned with shall be the next three equations:

$$(T I) \quad \Delta y(t) \equiv \Phi(A, \Gamma) y(t) = A_1 x(t) + A_2 \dot{\xi}(t).$$

$$(T II) \quad \Delta y(t) \equiv \Phi(A, \Gamma) y(t) = x(t).$$

$$(T III) \quad \Delta y(t) \equiv \Phi(A, \Gamma) y(t) = \dot{\xi}(t).$$

Where $x(t)$ is the input, $y(t)$ is the output and $\dot{\xi}(t)$ is the white noise, and (T III) must be considered that the input $x(t) \equiv \text{const.} = 0$. Furthermore, we shall assume that $x(t)$ is the stationary process, each A , Γ , A_1 and A_2 are all S-C.L.Op.s belonging to the class $\tilde{A}_-$, A_- , etc., and $\Delta \equiv \Phi(A, \Gamma)$ is the rational function when A and Γ are regarded as the variables, and Γ is the feedback system.

In general, the *problems of optimum automatic control systems* may be defined that: *Under the preassigned $\Phi(A, *)$, A_1 and A_2 , to seek for the optimum Γ_0 for certain preassigned criterion, within the certain restricted class of feedback systems.* (Though, in the wide sense, there may be the

cases of changing $\Phi(A, *)$ or A_1 or A_2 or etc., we shall use here the narrow definition.)

As the *restrictive requirements*, it should be considered that

(R 1) *the output $y(t)$ should not be dependent on the future of the input $x(t)$ and the noise $\xi(t)$,*

(R 2) *the output $y(t)$ must be stationary process or satisfy certain type of stability condition.*

As the *criteria*, it is natural to choose the *minimum-varianceness of stationary output $y(t)$* ,⁹⁾ or the *speed of convergence to the stable state*.

From the requirement (R 1), in all three cases, the condition

(S 1) *there exists the (strictly) inverse operator $\Delta^{-1} \equiv \Phi(A, \Gamma)^{-1}$ of the S-C.L.Op.¹⁰⁾ $\Delta \equiv \Phi(A, \Gamma)$, and Δ^{-1} should be dependent only on the past,* must be satisfied. And from the requirement (R 2) the conditions

(S 2) *the equations (T I), (T II) and (T III) are all meaningful,*

and the general solution $y_0(t)$ of the homogeneous equation

$$(6.1) \quad \Delta y(t) \equiv \Phi(A, \Gamma) y(t) = 0$$

satisfies the condition

$$(S 3) \quad y_0(t) \rightarrow 0, \quad \text{as } t \rightarrow \infty,$$

or

$$(S 3') \quad y_0(t) \rightarrow \text{certain stationary process, as } t \rightarrow \infty.$$

However, Proposition II. 2 and other results in Appendixes show that *if the condition (S 1) is satisfied then the condition (G 1)¹¹⁾ " $N_A = \phi$ " is satisfied*, and if the condition (G 1) is satisfied then there does not exist the stationary solution of homogeneous equation (6.1). Hence *the condition (S 3') is contradictory to the condition (S 1), and we shall omit the condition (S 3')*.

Conversely, if the conditions (S 1), (S 2) and (S 3) are satisfied, then the A.C.S. which is represented by (T I) or (T II) or (T III) is satisfies the requirements (R 1) and (R 2).

9) See N. Wiener [2].

10) From Proposition I. 5, we can see that $\Phi(A, \Gamma)$ and $\Phi(A, \Gamma)^{-1}$ are the S-C. L. Op. s..

11) These numbers of conditions were used in T. Seguchi [1].

Now, let us consider, as the criterion, the minimum-varianceness of stationary output $y(t, \omega)$. The values of $V[y(t, \omega)]$ are obtained by the following formulas:

$$(T I.1) \quad V[y(t, \omega)] = \int_{-\infty}^{\infty} [|C_{A_1}(i\lambda)|^2 \mu_x(d\lambda) + |C_{A_2}(i\lambda)|^2 d\lambda] / |C_d(i\lambda)|^2,$$

$$(T II.1) \quad V[y(t, \omega)] = \int_{-\infty}^{\infty} \mu_x(d\lambda) / |C_d(i\lambda)|^2,$$

$$(T III.1) \quad V[y(t, \omega)] = \int_{-\infty}^{\infty} d\lambda / |C_d(i\lambda)|^2,$$

where $C_d(z) = \Phi(C_A(z), C_F(z))$, and especially in (T I) we assume that x and ξ are mutually orthogonal.¹²⁾ Hence, the problem is reduced as the following: *To determine the feedback system Γ_0 so as to minimize the value (T I.1) or (T II.1) or (T III.1), within certain restricted class of feedback systems under the conditions (S 1), (S 2) and (S 3).* If we shall take the speed of convergence to stable state into the considerations, the problem becomes much more complicated.¹³⁾ But we shall have no regard for this in this note.

However, to solve the above mentioned problem which obtain the optimum A.C.S., in general, is very difficult but except certain special cases. The above mentioned formulation is obtained quite analogously in the cases of discrete time parameter. Mr. T. Naritomi [3] is solving, in the very special cases, the problem to obtain the optimum A.C.S. with discrete time parameter.

Here, let us state a little more in the matter of fact which are rough but by all means discuss to.

CONDITIONS OF (S 1).

Now, let C_d satisfy the condition (G 2) (Paley-Wiener's condition), i. e.

$$(G 2) \quad \int_{-\infty}^{\infty} |\log |C_d(i\lambda)| | / (1 + \lambda^2) d\lambda = \int_{-\infty}^{\infty} |\log |G_d(\lambda)| | / (1 + \lambda^2) d\lambda < \infty.$$

12) This assumption is natural, but we can calculate easily the value $V[y(t, \omega)]$ without this assumption. However it is necessary to assume somethings between x and ξ .

13) In this case, it will be occur the problem of relations between the value of $V[y(t, \omega)]$ and the speed of convergence. But there may be the possibility that the optimum conditions, between these two different kinds of criterions, are coincident.

Then from [4.1.IV] in T. Seguchi [1], the condition (G 1) holds. Furthermore, if we assume the condition

(C 3) C_A is an entire function,
then the conditions (G 2) and

(C 2) if $z \in C$ is any zero of C_A , then $\Re z < 0$
are equivalent. And from Theorem 5.2.4 in T. Seguchi [1], we can obtain the following result: If C_A satisfies the conditions (C 2) (or (G 2)) and (C 3), there exists a strictly and weakly inverse operator $\Delta^{-1} \equiv \Phi(A, \Gamma)^{-1}$ is belonging to the class $\tilde{A}_-$ or A_- .

The author is unsatisfactory to this result, because it seems to us that the condition (C 3) is too strong to obtain the result.

CONDITIONS OF (S 2).

From Proposition II. 4

(1°) the relation (T III) holds if and only if C_A satisfies the conditions (G 1), (G 3°) and (G 4°).

From the definition of various kinds of inverse operators and the results of T. Seguchi [1], Sections 4 and 5, we can easily see that

(2°) for the relation (T II) holds, it will be required that the conditions $\mathcal{B}(\Delta^{-1}) \supset \text{car. } (\mu_x)$ and

$$(G 4') \quad 1/G_A \in L^2(\mu_x; 0)$$

hold.

From the results (1°) and (2°),

(3°) for the relation (T I) holds, it will be required that the conditions $A_1, A_2 \in \tilde{A}_-$ or A_- , $\mathcal{B}(\Delta^{-1}) \supset [\text{car. } (\mu_x) \cap \mathcal{B}(A_1)] \cup \mathcal{B}(A_2)$ and

$$(G 4'') \quad 1/G_A \in L^2(|C_{A_1}|^2 \mu_x + |C_{A_2}|^2 m; 0)$$

hold.

But in the theory of automatic control systems, it will be required that there exists a (strictly) inverse operator Δ^{-1} and is belonging to the class $\tilde{A}_{e-}$ or A_{e-} .¹⁴⁾

If we assume the condition (C 3), the conditions of (S 2) are completely obtained, that is, under the assumption (C 3), the followings are the conditions of (S 2):

14) The classes $\tilde{A}_{e-}$ and A_{e-} are analogous notions as the classes $\tilde{A}_-$ and A_- .

(T III.2) (C 2), (G 3°) and (G 4°).

(T II.2) (C 2), (G 4') and $\mathcal{B}(\mathcal{A}^{-1}) \supset \text{car.}(\mu_x)$.

(T I.2) (G 4°), $A_1, A_2 \in \tilde{\mathcal{A}}_-$ or $\mathcal{A}_-$, and $\mathcal{B}(\mathcal{A}^{-1}) \supset [\text{car.}(\mu_x) \cap \mathcal{B}(A_1)] \cup \mathcal{B}(A_2)$.

But the author is unsatisfactory to these results, because it seems to us that the condition (C 3) is too strong to obtain the results.

CONDITIONS OF (S 3).

Theorem III. 3, (1°) and (2°) shall give the one answer of the problem.

Furthermore, if we assume the condition (C 3), then the stability condition (S 3) will hold if and only if the condition (C 2) is satisfied.

APPENDIXES. Shift-commutative linear operators and related functional equations.

For the recommendations and annotations, we shall refer to the definitions and properties of S-C.L.Op.s and the stochastic or non-stochastic functional equations related S-C.L.Op.s, which are connected with the investigations of automatic control systems.

The proofs of results shall be omitted, but will be shown the other day.

Let us use, throughout this note, the following *notations*:

$\mathcal{D} \equiv$ "the space of all *rapidly decreasing functions* in the sense of L. Schwartz [4]".

$\mathbf{M} \equiv \left\{ \mu; \int_{-\infty}^{\infty} |\varphi(\lambda)|^2 \mu(d\lambda) < \infty, \text{ for every } \varphi \in \mathcal{D}, \text{ non-negative measures} \right\}.$

$\mathbf{M}_k \equiv \left\{ \mu; \int_{-\infty}^{\infty} \mu(d\lambda)/(1+\lambda^2)^k < \infty, \mu \in \mathbf{M} \right\}.$

$\mathbf{L}^2(\mu; \mathcal{D}) \equiv \left\{ f; \int_{-\infty}^{\infty} |f(\lambda)\varphi(\lambda)|^2 \mu(d\lambda) < \infty, \text{ for every } \varphi \in \mathcal{D} \right\}.$

$\mathbf{L}^2(\mu; k) \equiv \left\{ f; \int_{-\infty}^{\infty} |f(\lambda)|^2/(1+\lambda^2)^k \mu(d\lambda) < \infty \right\}.$

$\mathcal{Q}(\mathfrak{B}, P) \equiv$ "the basic *probability space*".

$\mathbf{L}^2(\mathcal{Q}) \equiv \{x; \mathbf{E}[|x(\omega)|^2] < \infty\}.$

$\mathbf{L}^2(X), \mathbf{L}^2(X; t)$, etc.: See T. Seguchi [1], p. 131.

$\mathfrak{S} \equiv$ "the totality of (*weakly*) *stationary distributions*"

$\mathfrak{S}^\circ \equiv$ "the totality of (*weakly*) *stationary processes*".

$\mathfrak{S}_k \equiv \{x \in \mathfrak{S}; \mu_x \in \mathbf{M}_k\}$, where μ_x means the spectral measure of $x \in \mathfrak{S}$.

$$\mathcal{F}\varphi: (\mathcal{F}\varphi)(\lambda) \equiv 1/\sqrt{2\pi} \int_{-\infty}^{\infty} e^{i\lambda t} \varphi(t) dt.$$

$$\mathcal{A}(A) \equiv \{\lambda; e^{i\lambda \cdot} \in \mathcal{D}(A), \lambda \in R\}, \mathcal{B}(A) \equiv \{z; e^{z \cdot} \in \mathcal{D}(A), z \in C\}.$$

$$\mathbf{A}_- \equiv \text{"S-C.L.Op.s which are dependent only on the past"}.$$

$$N_A \equiv \{\lambda; G_A(\lambda) = 0, \lambda \in R\}.$$

$$[1/(D - \alpha I)]f(t) \equiv - \int_0^0 e^{\alpha u} \tau_{-u} f(t) du \quad (\alpha \in C, \neq 0)$$

$$\int_0^0 e^{\alpha u} \tau_{-u} f(t) du \equiv - \int_t^0 e^{\alpha(t-s)} f(s) ds$$

$$\equiv \begin{cases} \int_{-\infty}^0 e^{\alpha u} \tau_{-u} f(t) du = \int_t^{\infty} e^{\alpha(t-s)} f(s) ds, & \text{if } \Re \alpha > 0, \\ - \int_0^{\infty} e^{\alpha u} \tau_{-u} f(t) du = - \int_{-\infty}^t e^{\alpha(t-s)} f(s) ds, & \text{if } \Re \alpha < 0. \end{cases}$$

APPENDIX I. *The definition of shift-commutative linear operators.*

[1°] In T. Seguchi [1] and this note, the shift (operator) $\{T_t\}_{t \in R}$ ¹⁵⁾ is used in the following sense: Let $\mathbf{F}$ be a complex (or real) linear space. Then $\{T_t\}_{t \in R}$ is called the *shift* (operator) on $\mathbf{F}$ if and only if the conditions

- (1) for any $t \in R$, T_t is an operator such that $\mathbf{F} \rightarrow \mathbf{F}$,
- (2) for any $t, s \in R$, relations

$$T_t T_s = T_{t+s} \quad \text{and} \quad T_0 = I \text{ (identity)}$$

hold,

are satisfied.

[2°] The definition of the shift-commutative linear operator (S-C.L. Op.) $A \in \tilde{\mathcal{A}}$ and $\mathbf{A}$ are given in T. Seguchi [1]. But it is more comprehensive to define as follows:

DEFINITION I.1. Let $\mathbf{F}$ be a complex (or real) linear space and let $(A, \mathbf{D}, \{T_t\}_{t \in R})$ be such that:

- (1) $\mathbf{D}$ is a non-trivial sub-space of $\mathbf{F}$, i.e. $\mathbf{D} \neq \{0\}$.
- (2) $\{T_t\}_{t \in R}$ is the shift defined on $\mathbf{F}$ and satisfies the condition

$$(L1) \quad T_t \mathbf{D} \subset \mathbf{D}, \quad \text{for any } t \in R.$$

15) The parameter space of shift is not necessarily R , but is an additive group, for example it may be $N \equiv$ "the set of all integers".

(3) A is a linear operator defined on $\mathbf{D}$ and satisfies the condition

$$(L\ 2) \quad A\mathbf{D} \subset \mathbf{F} \equiv \mathcal{D}(T_t).^{16)}$$

(4) Between the linear operator A and the shift $\{T_t\}$ the condition

(L 3) for any $f \in \mathbf{D}$ and for any $t \in R$

$$T_t(Af) = A(T_tf)$$

holds.

Then we shall call the linear operator A is the *shift-commutative linear operator* (S-C.L.Op.) defined on $\mathbf{D}$ w.r.t. T_t and we shall hereafter denote $\mathbf{D}$ by $\mathcal{D}(A)$.

The notation $\tilde{\mathcal{A}}$ is the class of all S-C.L.Op.s A , when $\mathbf{F}$ is the linear space of all real- or complex-valued functions defined on R , the shift $T_t \equiv \tau_t$ ($\tau_t f(\cdot) \equiv f(\cdot + t)$ for any $f \in \mathbf{F}$ and for every $t \in R$); and $\mathcal{D}(A)$ means the domain of $A \in \tilde{\mathcal{A}}$.

Let $A \in \tilde{\mathcal{A}}$, then $Af(t) \equiv (Af)(t)$ is the value of Af at t , and the following relations holds: There exist two functions G_A and C_A such that for any $\lambda \in \mathcal{A}(A)$ and $z \in \mathcal{B}(A)$

$$(I.1) \quad Ae^{i\lambda t} \equiv e^{i\lambda t} G_A(\lambda) \quad \text{and} \quad Ae^{zt} \equiv e^{zt} C_A(z), \quad \text{for every } t \in R.$$

The above two functions $G_A(\lambda)$ and $C_A(z)$ are called the *generating function* (g.f.) and the *characterizing function* (c.f.) of $A \in \tilde{\mathcal{A}}$, respectively.

DEFINITION I.2. If $x \in \mathfrak{S}$ has the orthogonal random measure and the spectral measure dZ and μ respectively, and if S-C.L.Op. $A \in \tilde{\mathcal{A}}$ has the g.f. $G_A \in L^2(\mu; \mathfrak{S})$, then the corresponding S-C.L.Op. in $\mathfrak{S}$ to $A \in \tilde{\mathcal{A}}$ is defined by, for every $\varphi \in \mathfrak{S}$

$$(I.2) \quad Ax(\varphi) \equiv \int_{-\infty}^{\infty} (\mathcal{F}\varphi)(\lambda) G_A(\lambda) dZ(\lambda).^{17)}$$

The domain of A in $\mathfrak{S}$ is denoted by $\mathcal{D}(A) \equiv \{x; G_A \in L^2(\mu_x; \mathfrak{S})\}$. The class of all operators A in $\mathfrak{S}$, defined above (I.2) and whose domain in $\mathfrak{S}$ is not empty, is called the *shift-commutative linear operators* (S-C.L.Op.s) in $\mathfrak{S}$ and denoted by $\mathcal{A}$.

PROPOSITION I.1. The covariance distribution of Ax defined by (I.2) is

16) $\mathcal{D}(*)$ means the domain of operator $*$.

17) See Remark on p. 126 in T. Seguchi [1].

given by

$$(I.3) \quad r_{Ax}(\varphi) \equiv \int_{-\infty}^{\infty} (\mathcal{F}\varphi)(\lambda) |G_A(\lambda)|^2 \mu(d\lambda)$$

for every $\varphi \in \mathcal{S}$.

PROPOSITION I.2. *Let $x \in \mathfrak{S}_0$ and $G_A \in L^2(\mu_x; 0)$. Then for the corresponding stationary process $x \in \mathfrak{S}^\circ$ to $x \in \mathfrak{S}_0$, the following expressions hold:*

$$(I.4) \quad Ax(t) = \int_{-\infty}^{\infty} e^{i\lambda t} G_A(\lambda) dZ_x(\lambda)$$

$$(I.5) \quad r_{Ax}(t) = \int_{-\infty}^{\infty} e^{i\lambda t} |G_A(\lambda)|^2 \mu_x(d\lambda),$$

where dZ_x and μ_x are the orthogonal random measure and the spectral measure of $x \in \mathfrak{S}_0$ respectively.

[3°] In T. Seguchi [1], the above definitions are sufficient to study the relations between S-C.L.Op.s and the stationary processes (include the stationary distributions). But to study the linear automatic control systems, we must refer to the additional various preparations.

(1) Let $S \rightrightarrows C$ mean the following convergence of sets: (a) $S \in \mathfrak{B}(C) \equiv$ “the Borel field on C in the sense of $\mathfrak{B}(R) \otimes \mathfrak{B}(R)$ ”, where $\mathfrak{B}(R) \otimes \mathfrak{B}(R)$ is the direct product-Borel field of $\mathfrak{B}(R)$ and $\mathfrak{B}(R)$, and $\mathfrak{B}(R)$ means the smallest Borel field on $R \equiv (-\infty, \infty)$. (b) $d(z_1, z_2) \equiv |z_1 - z_2|$ ($z_1, z_2 \in C$) and $d(S, z_0) \equiv \inf_{z \in S} d(z, z_0)$ and (c) S is increasing and converges to C , satisfying the conditions $S \in \mathfrak{B}(C)$ and $d(S^c, 0) \rightarrow \infty$, where S^c means the complementary set of $S \subset C$.

(2) Let $C_c^n(R)^{(18)}$ be the class of all complex- (or real-) valued functions f such that: (a) For any $t \in R$

$$(I.6) \quad f(t) \equiv \int_C \sum_j t^j e^{zt} dK_{fj}(z) \\ \equiv \lim_{S \rightrightarrows C} \int_S \sum_j t^j e^{zt} dK_{fj}(z) \quad (\text{uniform convergence})$$

18) The class $C_c^n(R)$ is hinted at *Cauchy's series* w.r.t. A (linear translatable operator) in T. Kitagawa [5].

i.e. for any $\varepsilon > 0$ there exists certain $(R \ni) l \equiv l(\mathcal{S}, t) > 0$ such that for any $S \in \mathfrak{B}(C)$, where $d(S^c, 0) \geq l$, the relation

$$\left| f(t) - \int_S \sum_j t^j e^{zt} dK_{fj}(z) \right| < \varepsilon$$

holds, (b) $f \in C^n(R)$ and (c) $C_e^n(R)$ is the metric space w.r.t. ρ defined by

$$(I.7) \quad \rho(f_1, f_2) \equiv \sup_{t \in R} |f_1(t) - f_2(t)|.$$

We can easily obtain the following:

PROPOSITION I.3. *The class $C_e^n(R)$ is a complex (or real) linear space of functions.*

(3) Let $C_e^n(R)^{(19)}$ be the class of all stochastic processes x such that:

(a) $x \in \mathfrak{R}$ (regular stochastic process), i.e. x satisfies the assumptions

(A 1) *any $x \in \mathfrak{R}$ is continuous in the mean at any time $t \in R$,*

(A 2) *any $x \in \mathfrak{R}$ is the one-dimensional (real- or) complex-valued process with the mean value 0 and the finite variance at any time $t \in R$.*

(b) For any $t \in R$

$$(I.8) \quad \begin{aligned} x(t) &\equiv x(t, \omega) \equiv \int_C \sum_j t^j e^{zt} dZ_{xj}(z, \omega) \\ &= \text{l.i.m.}_{S \rightrightarrows C} \int_S \sum_j t^j e^{zt} dZ_{xj}(z, \omega) \\ &\quad (\text{uniform convergence in the mean}), \end{aligned}$$

i.e. for any $\varepsilon > 0$ there exists certain $(R \ni) l \equiv l(\mathcal{S}, t) > 0$ such that for any $S \in \mathfrak{B}(C)$, where $d(S^c, 0) \geq l$ the relation

$$\mathbb{E} \left[\left| x(t, \omega) - \int_S \sum_j t^j e^{zt} dZ_{xj}(z, \omega) \right|^2 \right] < \varepsilon$$

holds, (c) $x \in C^n(R)$ (x has the continuous n -th derivative in the sense of mean square w.r.t. probability measure) and (d) $C_e^n(R)$ is the metric space whose metric ρ is defined by

$$(I.9) \quad \rho(x_1, x_2) \equiv \sup_{t \in R} (\mathbb{E} [|x_1(t, \omega) - x_2(t, \omega)|^2])^{1/2}.$$

19) The class $C_e^n(R)$ is hinted at *stochastic Cauchy's series* w.r.t. A in T. Kitagawa [6] and T. Onoyama [7] and [8].

We can easily obtain the followings:

PROPOSITION I.4. (1°) *The class $\mathbf{C}_c^n(R)$ is a complex (or real) linear space.*

(2°) *If $x \in \mathbf{C}_c^n(R)$ satisfies the conditions (a) $j \equiv 0$ only, (b) $\text{car.}(dZ_{x_0}) = \text{"imaginary axis"}$ and (c) dZ_{x_0} is an orthogonal random measure, then $x \in \mathfrak{S}^\circ$.*

As were shown in Propositions I.3 and I.4, $\mathbf{C}_c^n(R)$ and $\mathbf{C}_c^n(R)$ are both (topological) linear spaces. Hence, from definitions of $\mathbf{C}_c^n(R)$ and $\mathbf{C}_c^n(R)$, we can consider the S-C.L.Op.s defined in $\mathbf{C}_c^0(R) = \bigcup_{n \geq 0} \mathbf{C}_c^n(R)$ or $\mathbf{C}_c^0(R) = \bigcup_{n \geq 0} \mathbf{C}_c^n(R)$ w.r.t. the shift τ_t . Furthermore, from Proposition I.3 the relation $\mathbf{C}_c^0(R) \subset \mathbf{F} \equiv \text{"the totality of all (real- or) complex-valued functions on } R\text{"}$, holds, hence any S-C.L.Op. A defined in $\mathbf{C}_c^0(R)$ w.r.t. τ_t belongs to $\tilde{\mathbf{A}}$. Hence any A defined in $\mathbf{C}_c^0(R)$ characterized by the c.f. C_A (or the g.f. G_A).

As was defined the class $\mathbf{A}$ of S-C.L.Op.s, from the class $\tilde{\mathbf{A}}$ and $\mathbf{A}_s$, in T. Seguchi [1], we shall first define the class $\tilde{\mathbf{A}}_c$ of S-C.L.Op.s in $\mathbf{C}_c^0(R)$.

DEFINITION I.3. *The class $\tilde{\mathbf{A}}_c$ of S-C.L.Op.s in $\mathbf{C}_c^0(R)$ is such that:*

$$(1) \quad \tilde{\mathbf{A}}_c \subset \tilde{\mathbf{A}}.$$

$$(2) \quad \mathcal{B}(A) = C - \{\text{all poles of } C_A(z)\} \quad \text{and} \quad C_A \in \mathbf{C}^0(\mathcal{B}(A)).$$

$$(3) \quad \text{Any } A \in \tilde{\mathbf{A}}_c \text{ is commutative with } D_z \equiv d/dz \text{ and } \int_C \sum_j * dK_{fj}(z).$$

(4) *There exists at least one function $f \in \mathbf{C}_c^0(R)$, such that, if $f \in \mathbf{C}_c^0(R)$ is given in (I.6), then*

$$(I.10) \quad \begin{aligned} Af(t) &\equiv \int_C \sum_j A[t^j e^{zt}] dK_{fj}(z) = \int_C \sum_j D_z^j [e^{zt} C_A(z)] dK_{fj}(z) \\ &= \int_C \sum_j e^{zt} \sum_{h=0}^j \binom{j}{h} t^{j-h} C_A^{(h)}(z) dK_{fj}(z), \end{aligned}$$

where we assume the right hand side is meaningful in the same sense as in the formula (I.6).

By the same way as the definition of the class $\mathbf{A}$ of S-C.L.Op.s in $\mathfrak{S}$, we can define the class $\mathbf{A}_c$ of S-C.L.Op.s in $\mathbf{C}_c^0(R)$ as the following:

DEFINITION I.4. *The corresponding S-C.L.Op. $A \in \mathbf{A}_c$ to $A \in \tilde{\mathbf{A}}_c$, is such that; for certain $x \in \mathbf{C}_c^0(R)$ which is expressed in (I.8),*

$$\begin{aligned}
 \text{(I.11)} \quad Ax(t, \omega) &\equiv \int_C \sum_j A[t^j e^{zt}] dZ_{xj}(z, \omega) \\
 &\equiv \int_C \sum_j D_z^j [C_A(z) e^{zt}] dZ_{xj}(z, \omega) \\
 &= \int_C \sum_j e^{zt} \sum_{h=0}^j \binom{j}{h} t^{j-h} C_A^{(h)}(z) dZ_{xj}(z, \omega),
 \end{aligned}$$

where we assume the right hand side is meaningful in the same sense as in the formula (I.8).

The characterizations of the classes $\tilde{\mathcal{A}}_e$ and $\mathcal{A}_e$ are unknown as yet. But the results in Appendix III may give a key of the characterizations of the classes $\tilde{\mathcal{A}}_e$ and $\mathcal{A}_e$.

[4°] As was shown in T. Seguchi [1], Theorem 3.4.5, in the class $\mathcal{A}$, for a S-C.L.Op. $A \in \mathcal{A}$ there exists the inverse operator $A^{-1} \in \mathcal{A}$ if and only if (G 1) “ $N_A = \{\lambda; G_A(\lambda) = 0\} = \emptyset$ ” (and (G 4) “ $1/G_A \in L^2(\mu; \mathcal{C})$ for certain $\mu \in \mathcal{M}$ ”) hold. But, in the classes $\tilde{\mathcal{A}}_e$, $\mathcal{A}_e$, etc., the above conditions are not sufficient for the existence of the inverse S-C.L.Op. Hence we shall define three sense of inverse operator of $A \in \tilde{\mathcal{A}}$, etc..

DEFINITION I.5. For a S-C.L.Op. $A \in \tilde{\mathcal{A}}$ the S-C.L.Op. $\Gamma \in \tilde{\mathcal{A}}$ is called

(1°) the *inverse operator* if and only if the conditions (1) $\tilde{\mathcal{B}}(A) = \tilde{\mathcal{B}}(\Gamma)$, where $\tilde{\mathcal{B}}(A) \equiv \mathcal{B}(A) - \{\text{all zeros of } C_A(z)\}$, (2) any zero z_0 of $C_A(z)$ with multiplicity k_0 is the pole of $C_\Gamma(z)$ of order k_0 and any zero z_1 of $C_\Gamma(z)$ with multiplicity k_1 is the pole of $C_A(z)$ of order k_1 are satisfied. And in this case we shall denote $\Gamma \equiv \hat{A}^{-1}$.

(2°) the *strictly inverse operator* if and only if the conditions (1) $\overline{\mathcal{B}(A)} = \overline{\mathcal{B}(\Gamma)} = C$, where $\overline{\mathcal{B}(A)}$ means the closure of $\mathcal{B}(A)$, and (2) in (1°) are satisfied. And in this case we shall denote $\Gamma \equiv A^{-1}$.

(3°) the *weakly inverse operator* if and only if the condition $G_\Gamma(\lambda) = 1/G_A(\lambda)$ for every $\lambda \in R$ is satisfied. And in this case we shall denote $\Gamma \equiv \bar{A}^{-1}$.

The classes $\mathcal{A}_{es} \equiv \mathcal{A}_s$ and $\mathcal{A}_e$ can be defined by analogy of the definitions of the classes $\mathcal{A}_s$ and $\mathcal{A}$. According to these definitions, the inverse operator of a S-C.L.Op. in $\mathcal{C}$ is the weakly inverse operator of the corresponding S-C.L.Op. in $\mathcal{F}$.

PROPOSITION I.5 (Theorem 3.4.4).²⁰⁾ *Let A, Γ , and Δ be the S-C.L.Op.s..*

Then

- (1°) *if $\Delta = aA + b\Gamma$ then $C_\Delta = aC_A + bC_\Gamma$, and $G_\Delta = aG_A + bG_\Gamma$,*
- (2°) *if $\Delta = A\Gamma$ then $C_\Delta = C_A C_\Gamma = C_\Gamma C_A$, and $G_\Delta = G_A G_\Gamma = G_\Gamma G_A$,*
- (3°) *if there exists the inverse operator A^{-1} then $C_{A^{-1}} = 1/C_A$ and $G_{A^{-1}} = 1/G_A$.*

APPENDIX II. *Two stationary processes connected by certain S-C.L.Op..*

Let x and y be two stationary processes connected by $A \in \mathbf{A}$ such that

$$(II.1) \quad Ay = x,$$

that is, if x and y are stationary processes belonging to $\mathfrak{S}$ or $\mathfrak{S}^\circ$

$$(II.2) \quad Ay(\varphi) = \int_{-\infty}^{\infty} (\mathcal{F}\varphi)(\lambda) G_A(\lambda) dZ_y(\lambda) = \int_{-\infty}^{\infty} (\mathcal{F}\varphi)(\lambda) dZ_x(\lambda) = x(\varphi)$$

or

$$(II.3) \quad Ay(t) = \int_{-\infty}^{\infty} e^{i\lambda t} G_A(\lambda) dZ_y(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t} dZ_x(\lambda) = x(t)$$

respectively.

PROPOSITION II.1. (1°) *If $N_A = \phi$, then the standard solution y_s of (II.1) is the unique solution and is denoted as $y = y_s = \bar{A}^{-1}x$.*

(2°) *If $N_A \neq \phi$, then any stationary process satisfying (II.1) is given as the following form;*

$$y = y_0 + y_s,$$

where y_0 is the solution of the homogeneous equation

$$(II.4) \quad Ay = 0,$$

y_s is given by standard solution of (II.1) such that:

$$(II.5) \quad y_s(\varphi) = \int_{R-N_A} (\mathcal{F}\varphi)(\lambda) / G_A(\lambda) dZ_x(\lambda), \quad \text{for any } \varphi \in \mathfrak{D},$$

or

20) The number in () is the result of T. Seguchi [1].

$$(II.6) \quad y_s(t) \equiv \int_{R-N_A} e^{i\lambda t} / G_A(\lambda) dZ_x(\lambda),$$

and y_0 and y_s are mutually orthogonal.

PROPOSITION II.2 (Theorem 4.2.3). (1°) If, in (II.1), $A \in \mathbf{A}_-$, then for every $t \in R$

$$\mathbf{L}^2(Y; t) \supset \mathbf{L}^2(X; t)$$

holds.

(2°) Furthermore, if there exists the (weakly) inverse operator $\bar{A}^{-1} \in \mathbf{A}_-$, then for every $t \in R$

$$\mathbf{L}^2(Y; t) = \mathbf{L}^2(X; t)$$

holds.

Where the notations $\mathbf{L}^2(X; t)$ and $\mathbf{L}^2(Y; t)$ are on p. 131 in T. Seguchi [1].

PROPOSITION II.3 (Theorem 4.2.4). Let two stationary processes x and y be connected by $A \in \mathbf{A}$ as (II.1). If y is purely non-deterministic and if G_A satisfies the condition (G 2) (Paley-Wiener's condition), then x is purely non-deterministic. Conversely, if x is purely non-deterministic and if G_A satisfies the condition (G 2), then y is purely non-deterministic.

PROPOSITION II.4 (Theorem 5.2.1). Let $x \in \mathfrak{S}_0$, $\dot{\xi}$ be a differentiation of a homogeneous orthogonal random measure $d\xi$ and $A \in \mathbf{A}$. Then

(1°) the relation

$$(II.7) \quad Ax = \dot{\xi}$$

holds if and only if the conditions (G 1) " $N_A = \phi$ ", (G 3°) " $G_A \in L^2(m; 1)$ " and (G 4°) " $1/G_A \in L^2(m; 0)$ " are satisfied, where m is the usual Lebesgue measure on R .

(2°) any $x \in \mathfrak{S}_0$ satisfying (II.1) is given by uniquely for every $\varphi \in \mathcal{O}$

$$x(\varphi) = \bar{A}^{-1} \dot{\xi}(\varphi) = \int_{-\infty}^{\infty} (\mathcal{F}\varphi)(\lambda) d\tilde{\xi}(\lambda) / G_A(\lambda),$$

where $d\tilde{\xi} \equiv dZ_{\dot{\xi}}$ is such as

$$\hat{\xi}(\varphi) \equiv \int_{-\infty}^{\infty} (\mathcal{F}\varphi)(\lambda) d\tilde{\xi}(\lambda).$$

(3°) $x \in \mathfrak{S}^\circ$, corresponding to $x \in \mathfrak{S}_0$, is represented by

$$x(t) = \int_{-\infty}^{\infty} e^{i\lambda t} d\tilde{\xi}(\lambda) / G_A(\lambda).$$

PROPOSITION II.5 (Corollary of Theorem 5.2.2). *Let $x \in \mathfrak{S}_0$ and $\hat{\xi}$ satisfy the relation (II.7), and C_A be an entire function. Then $x \in \mathfrak{S}$ is purely non-deterministic if and only if the condition (C 2) “if z is the zero of C_A then $\operatorname{Re} z < 0$ ” is satisfied. (Here, conditions (G 1), (G 3°) and (G 4°) are assumed.)*

APPENDIX III. *The homogeneous functional equation related S-C.L.Op..*

[1°] The next two lemmas are well known or will be easily proved.

LEMMA III.1. *Let $P(z) \equiv \sum_{j=0}^n A_j z^j$ be a polynomial of order n and $z_1, z_2, \dots, z_m$ be the zeros of $P(z)$ whose multiplicities are $k_1, k_2, \dots, k_m$ respectively. Then the general solution $y_0(t)$ of the differential equation*

$$(III.1) \quad P(D)y(t) \equiv \sum_{j=0}^n A_j y^{(j)}(t) \equiv A_0 \prod_{j=1}^m (D - z_j I)^{k_j} y(t) = 0$$

is given by the following form:

$$(III.2) \quad y_0(t) = \sum_{j=1}^m P_j(t) e^{z_j t},$$

where $P_j(t)$ is the polynomial of order $k_j - 1$, $j = 1, 2, \dots, m$, with constant coefficients.

LEMMA III.2. *Let A be a S-C.L.Op. whose c.f. is $C_A(z) = 1/(z - \alpha)$ ($\operatorname{Re} \alpha \neq 0$). Then the general solution $y_0(t)$ of the integral equation*

$$(III.3) \quad [1/(D - \alpha I)]y(t) \equiv - \int_0^t e^{\alpha u} \tau_{-u} y(t) du \equiv \int_t^0 e^{\alpha(t-s)} y(s) ds = 0$$

is identically zero, i.e. $y_0(t) \equiv 0$.

LEMMA III.3. *Let Γ be a S-C.L.Op. belonging to $\tilde{\mathcal{A}}$, and satisfies the following conditions:*

(1) $C_\Gamma \in \mathcal{C}^0(C)$ and $C_\Gamma(z) \neq 0$ for any $z \in C$.

(2) For any function $f(t) \equiv e_0(z, t) \equiv e^{zt} (z \in C, t \in R)$, is commutative with $D_z \equiv d/dz$ and $\int_C * dK(z)$.

(3) For any function $f(t) \equiv e_K(t) \in C_c^0(R)$, the relation

$$\Gamma e_K(t) = \int_C \Gamma e^{zt} dK(z) = \int_C e^{zt} C_\Gamma(z) dK(z)$$

holds, where $e_K(t) \equiv \int_C e^{zt} dK(z)$.

Then any solution $y_0(t) \in C_c^0(R)$ of the homogeneous equation

$$(III.4) \quad \Gamma y(t) = 0$$

is identically zero, i.e. $y_0(t) \equiv 0$.

The author is displeased with the conditions of this lemma, but can not find the suitable conditions.

THEOREM III.1. Let A be a S-C.L.Op. belonging to $\tilde{A}$, such that:

(1) $C_A(z) \equiv P(z)/Q(z) C_\Gamma(z)$, $z \in C$.

(2) $P(z) \equiv \sum_{j=0}^n A_j z^j$ is a polynomial of order n , whose zeros are $\alpha_1, \alpha_2, \dots, \alpha_m$ with multiplicities $k_1, k_2, \dots, k_m$ and any $\alpha_j \neq 0$.

(3) $Q(z) \equiv \sum_{h=0}^q B_h z^h$ is a polynomial of order q , whose zeros are $\beta_1, \beta_2, \dots, \beta_p$ with multiplicities $r_1, r_2, \dots, r_p$ and which satisfy for any h and j , $\operatorname{Re} \beta_h \neq 0$ and $\beta_h \neq \alpha_j$.

(4) Γ is a S-C.L.Op. which satisfies the conditions in Lemma III.3.

Then any solution $y_0(t) \in C_c^0(R)$ of the homogeneous equation

$$(III.5) \quad A y(t) = 0$$

is given by the following form:

$$(III.7) \quad y_0(t) = \sum_{j=1}^m P_j(t) e^{\alpha_j t},$$

where P_j is the polynomial of order $k_j - 1$, $j = 1, 2, \dots, m$, with constant coefficients.

This theorem shows that the S-C.L.Op. $A \in \tilde{A}$ such as in Theorem III.1 is belonging to the class $\tilde{A}_c$.

[2°] LEMMA III.4. Let $P(z) \equiv \sum_{j=0}^n A_j z^j$ be a polynomial of order n and $z_1, z_2, \dots, z_m$ be the zeros of $P(z)$ whose multiplicities are $k_1, k_2, \dots, k_m$ respectively. Then the general stochastic solution $y_0(t) \equiv y_0(t, \omega) \in \mathfrak{R}$ of the stochastic differential equation

$$(III.8) \quad P(D)y(t, \omega) \equiv A_0 \prod_{j=1}^m (D - z_j I)^{k_j} y(t, \omega) = 0$$

is given by the following form:

$$(III.9) \quad y_0(t) \equiv y_0(t, \omega) \equiv \sum_{j=1}^m P_j(t, \omega) e^{z_j t} \equiv \sum_{j=1}^m e^{z_j t} \sum_{h=0}^{k_j-1} Z_{jh}(\omega) t^h.$$

LEMMA III.5. Let A be a S-C.L.Op. whose c.f. is $C_A(z) \equiv 1/(z - \alpha)$ ($\operatorname{Re} \alpha \neq 0$). Then the general stochastic solution $y_0(t) \equiv y_0(t, \omega)$ of the integral equation

$$(III.10) \quad [1/(D - \alpha I)]y(t, \omega) = 0$$

is identically zero, i.e. $y_0(t, \omega) \equiv 0$.

THEOREM III.2. Let A be a S-C.L.Op. defined in $C_c^0(R)$ and satisfy such conditions as in Theorem III.1. Then the any stochastic solution $y_0(t) \equiv y_0(t, \omega) \in C_c^0(R)$ of the homogeneous stochastic equation

$$(III.11) \quad Ay(t, \omega) = 0$$

is given by the following form:

$$(III.12) \quad y_0(t) \equiv y_0(t, \omega) = \sum_{j=1}^m P_j(t, \omega) e^{\alpha_j t} \equiv \sum_{j=1}^m e^{\alpha_j t} \sum_{h=0}^{k_j-1} Z_{jh}(\omega) t^h.$$

THEOREM III.3. Let A be a S-C.L.Op. such as Theorem III.1 or III.2, and let us consider the homogeneous (stochastic or non-stochastic) equation

$$(III.13) \quad Ay(t) = 0.$$

Then the following results are obtained:

(1°) The S-C.L.Op. A is dependent only on the past if and only if the following conditions are satisfied;

- (1) any zero of $P(z)$ has the negative real part,
- (2) any zero of $Q(z)$ has the negative real part,

(3) *the S-C.L.Op. Γ is dependent only on the past.*

(2°) *If the conditions (1), (2) and (3) of (1°) are satisfied, then the any stochastic or non-stochastic solution $y_0(t) \in C_c^0(R)$ or $C_c^0(R)$ satisfies the following stability condition (S 3), i.e.*

$$y_0(t) \rightarrow 0, \quad \text{as} \quad t \rightarrow \infty.$$

(3°) *If $P(z)$ has the pure imaginary zero ($\neq 0$), then the stochastic homogeneous equation has the stationary solution*

$$y_0(t) \equiv y_0(t, \omega) = \sum_j e^{ib_j t} Z_j(\omega),$$

where $ib_j, j=1, 2, \dots$, are the pure imaginary zeros of $P(z)$, and $Z_j(\omega), j=1, 2, \dots$, are any mutually orthogonal random variables with the mean values 0 and the finite variances.

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ERRATA
of
“SHIFT-COMMUTATIVE LINEAR OPERATORS
AND STATIONARY PROCESSES

By
Tsunsetami SEGUCHI”
(Received Dec. 2, 1969)

Page	Line	Error	Correct
107	↓ 13	fined	find
—	↓ 20	territory	territory
110	↓ 10	$X_R \equiv (x(t, \omega), t \in R, \omega \in \Omega)$	$X_R \equiv (x(t, \omega), t \in R; \omega \in \Omega)$
—	—	$X_{\mathcal{S}} \equiv (x(\varphi_1, \omega), \varphi \in \mathcal{S}, \omega \in \Omega)$	$X_{\mathcal{S}} \equiv (x(\varphi, \omega), \varphi \in \mathcal{S}; \omega \in \Omega)$
111	↓ 16	<i>stationary distribution</i>	<i>stationary process</i>
—	↑ 3	<i>the convariance distribution</i>	<i>the covariance distribution</i>
114	↓ 2	$\{e^{i\lambda t}; -\infty < t < \infty\}$	$\{e^{it\cdot}; t \in R\}$
—	↓ 14	$\{e^{i\lambda t}; -\infty < t < \infty\}$	$\{e^{it\cdot}; t \in R\}$
115	↓ 12	introduced by T. Kawata	introduced by T. Kawata
116	↓ 2	$\mathfrak{U}(A) \equiv \{\lambda; e^{i\lambda t} \in \mathcal{D}(A), \text{ for every } t \in R, \lambda \in R\}$	$\mathfrak{U}(A) \equiv \{\lambda; e^{i\lambda\cdot} \in \mathcal{D}(A), \lambda \in R\}$
—	↓ 3	$\mathfrak{Z}(A) \equiv \{z; e^{zt} \in \mathcal{D}(A), \text{ for every } t \in R, z \in C\}$	$\mathfrak{Z}(A) \equiv \{z; e^{z\cdot} \in \mathcal{D}(A), z \in C\}$
119	↓ 16	types (2.3.4) are ...	types (2.3.3) and (2.3.4) are ...
—	↓ 20	, $G_- \subset \mathfrak{Z}(A_1) \subset C$, where $G_- \equiv \{z; \Re z < 0\}$	(omit)
120	↓ 1	, $G_- \subset \mathfrak{Z}(A_2) \subset C$	(omit)
126	↑ 2	... $G_T(\lambda) dZ(\lambda)$ is ...	... $G_T(\lambda) dZ(\lambda) = \Gamma x(\varphi)$ is ...
127	↓ 15	(2.3.2) or (2.3.3), then	(2.3.2), then
134	↓ 16	$\mathbf{A}_P$: The totality ... series on C .	(omit)
~ ↓ 17			
137	↑ 7	, y_s is given in	, y_s given in
139	↑ 9	If y is a purely ...	If y is purely ...
142	↓ 11	$\hat{\xi}(S) = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} \left(\dots \right)$	$\tilde{\xi}(S) = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} \left(\dots \right)$
147	↓ 1	$A = C_A(D) \in \mathbf{A}_P \subset \mathbf{A}_-$	$A = C_A(D) \in \mathbf{A}_E \subset \mathbf{A}_-$
148	↓ 7	$x = C_A(D)x = \dots$	$Ax = C_A(D)x = \dots$
149	↑ 1	which corresponding as	are regarded as
~150	~ ↓ 1		