MULTI-DIMENSIONAL CARDINAL SPLINE FUNCTION AND ITS APPLICATIONS

By

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Dedicated to Professor Tosio Kitagawa on his 60-th birthday

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Recently W. Bickley has considered a method for solving two-point boundary value problems by using one dimensional cubic spline functions ([1]). There seem to be few literatures on an application of multidimensional spline functions to partial differential equations, for example, elliptic and parabolic ones.

The object of this note is to introduce the suitable multidimensional cardinal spline functions and to give some consistency relations, by which we are able to get an approximate solution for partial differential equation even with the variable coefficients. In the section 1, we shall show that the error estimation of the spline interpolation with respect to the Sobolev's norm satisfies the inequality of the same type as that of the Hermite interpolation. In the sections 2 and 3 we shall treat partial differential equations, i. e., elliptic and parabolic ones. These results of the sections 2 and 3 show that the spline interpolation method is as "universal" as the difference method and gives the continuous approximate solutions, while the latter furnishes the approximate ones at equidistant points.

1. Multi-dimensional cardinal spline function.

First we notice that B -spline $Q_{m+1}(x)$ can be characterized by the convolution of the uniform distribution $\chi(x)$ on $[0, 1]$, namely

$$(1) \quad Q_{m+1}(x) \equiv \chi_{m+1}(x) \equiv (\chi * \cdots * \chi)(x) \quad (\text{See [5; p. 17]}).$$

From this fact the formulae $(-1)^{(j)} Q_{m+1}^{(j)}(x+1) = Q_{m+1}^{(j)}(m-x)$, $0 \leq j \leq m-1$, $Q'_{m+1}(x+1) = Q_m(x+1) - Q_m(x)$ and etc. are easily proved.

We use the following notation: $m=(m_1, \dots, m_n)$, $p=(p_1, p_2, \dots, p_n)$, $w(x)=w(x_1, x_2, \dots, x_n)$, $\chi_{m+1}\left(\frac{x}{h}-\alpha\right)=\chi_{m_1+1}\left(\frac{x_1}{h_1}-\alpha_1\right)\cdots\chi_{m_n+1}\left(\frac{x_n}{h_n}-\alpha_n\right)$ and etc. There is the very important result about the approximation of Sobolev space $W^{p,r}(\mathcal{Q})$, which is a slight extension of [2]:

If $\mathcal{Q}$ is a bounded domain in R^n and sufficiently regular, and $\phi(x) \in W^{p,r}(\mathcal{Q}) \cap W^{p,q}(\mathcal{Q})$, then we have the estimation such that

$$\inf_{w \in \text{Span}\left\{\chi_{m+1}\left(\frac{x}{h}-\alpha\right)\right\}_{\alpha=-\infty}^{\infty}} \|w - \phi\|_{k,r} \leq ch^{p-k-\frac{1}{q}+\frac{1}{r}} \|D^p \phi\|_q \text{ for } 1 \leq q \leq r, 0 \leq k \leq p-1$$

and

$$\inf_{w \in \text{Span}\left\{\chi_{m+1}\left(\frac{x}{h}-\alpha\right)\right\}_{\alpha=-\infty}^{\infty}} \|w - \phi\|_{k,r} \leq ch^{p-k} \|D^p \phi\|_q \text{ for } r \leq q \leq +\infty, 0 \leq k \leq p-1.$$

(See [2] for the proof and definitions.)

Under the above important result, we shall introduce the multi-dimensional cardinal spline function $Q_{m+1}\left(\frac{x}{h}-\alpha\right)$, and define the vector space $S^m = \text{Span}\left\{Q_{m+1}\left(\frac{x}{h}-\alpha\right)\right\}_{\alpha=-\infty}^{\infty}$.

2. Elliptic differential equation.

In considering Laplace equations $\Delta u(x) = f(x, u(x))$, $u|_r$ being given, the following consistency relation is available:

If $w(x) \in S^m$, then we have the recurrence formula

$$(2) \quad \sum_{i=0}^m \Delta Q_{m+1}\left(m - \frac{x}{h}\right) \Big|_{x=ih} w_i = \sum_{i=0}^m Q_{m+1}(m-i) \Delta w_i,$$

where $w_i = w(ih) = w(i_1 h_1, \dots, i_n h_n)$.

In the case when $n=2$, $m=(3, 3)$ and $h_1=h_2=h$, (2) can be written as follows:

$$\begin{aligned} & \frac{1}{h^2} \left\{ (w_{0,0} + w_{1,0} + w_{2,0} + w_{0,1} + w_{2,1} + w_{0,2} + w_{1,2} + w_{2,2}) - 8w_{11} \right\} \\ & = \frac{1}{6} \left\{ (\Delta w_{0,0} + \Delta w_{2,0} + \Delta w_{0,2} + \Delta w_{2,2}) \right\} \end{aligned}$$

$$+4(\Delta w_{1,0} + \Delta w_{0,1} + \Delta w_{2,1} + \Delta w_{1,2}) + 16\Delta w_{11}\}.$$

For example, we shall consider the equation $\Delta u(x) = f(x)$, and then get a nine-point difference scheme which is the same as that of [3: p. 254].

3. Parabolic differential equation.

Next, we shall consider the parabolic differential equation, $\frac{\partial u(x)}{\partial x_1} = \Delta u(x) + f(x, u(x))$, $u(x_1, 0)$, $u(x_1, 1)$, $u(0, x_2)$ being given, where $\Delta = \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$.

In this case, the following consistency relation holds:

$$(3) \quad \sum_{i=0}^m \left\{ \frac{\partial}{\partial x_1} Q_{m+1} \left(m - \frac{x}{h} \right) - \Delta Q_{m+1} \left(m - \frac{x}{h} \right) \right\} \Big|_{x=ih} w_i \\ = \sum_{i=0}^m Q_{m+1}(m-i) (w_{x_1} - \Delta w)_{x=ih} \quad \text{if } w(x) \in S^m.$$

We notice that (3) is a (m_1-1) -step implicit difference scheme. In the case when $n=2$, $m=(2, 3)$, $h_1=k$ and $h_2=h$, (3) can be written as follows:

$$\left(\frac{1}{6k} - \frac{1}{2h^2} \right) (w_{1,0} + w_{1,2}) + \left(\frac{4}{6k} + \frac{1}{h^2} \right) w_{1,1} \\ - \left(\frac{1}{6k} + \frac{1}{2h^2} \right) (w_{0,0} + w_{0,2}) + \left(-\frac{4}{6k} + \frac{1}{h^2} \right) w_{0,1} \\ = \frac{1}{12} ((w_{x_1} - \Delta w)_{0,0} + (w_{x_1} - \Delta w)_{0,2} + (w_{x_1} - \Delta w)_{1,0} + (w_{x_1} - \Delta w)_{2,1} \\ + 4(w_{x_1} - \Delta w)_{1,0} + (w_{x_1} - \Delta w)_{1,1}), \quad \text{if } w \in S^{(2,3)}.$$

In considering the equation $\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial x_2^2}$, we get the following implicit difference scheme

$$(1-3\lambda) (w_{1,0} + w_{1,2}) + (4+6\lambda) w_{1,1}$$

$$=(1+3\lambda)(w_{0,0}+w_{0,2})+(4-6\lambda)w_{0,1}, \quad \text{where } \lambda=k/h^2.$$

This difference scheme is unconditionally stable and is almost the same as that of Crank-Nicolson for relatively large λ .

Next, we shall consider the following parabolic equation in one space dimension.

$$\frac{\partial u}{\partial x_1} = a_0(x_1, x_2) \frac{\partial^2 u}{\partial x_2^2} + a_1(x_1, x_2) \frac{\partial u}{\partial x_2} + a_2(x_1, x_2) u.$$

(i) The case when $a_1(x_1, x_2) = a_2(x_1, x_2) = 0$.

We note that the consistency relation holds, i. e.,

$$\begin{aligned} & \frac{1}{h^2} \sum_{i,j=0}^{m-1,n-1} Q_{m+1}(m-i) Q''_{n+1}(n-j) w_{ij}^{(x_1)} \\ &= \frac{1}{k} \sum_{i,j=0}^{m-1,n-1} Q'_{m+1}(m-i) Q_{n+1}(n-j) w_{ij}^{(x_2 x_2)} \quad \text{if } w(x_1, x_2) \in S^{(m,n)}. \end{aligned}$$

In the case when $m=2$ and $n=3$, we have

$$\begin{aligned} & \left(a_{ij-1} \frac{1}{2h^2} + \frac{1}{6k} \right) w_{i,j-1}^{(x_2 x_2)} + \left(-a_{i,j} \frac{1}{h^2} + \frac{2}{3k} \right) w_{i,j}^{(x_2 x_2)} \\ &+ \left(a_{i,j+1} \frac{1}{2h^2} + \frac{1}{6k} \right) w_{i,j+1}^{(x_2 x_2)} \\ &= \left(-a_{i+1,j-1} \frac{1}{2h^2} + \frac{1}{6k} \right) w_{i+1,j-1}^{(x_2 x_2)} + \left(-a_{i+1,j+1} \frac{1}{2h^2} + \frac{1}{6k} \right) w_{i+1,j+1}^{(x_2 x_2)} \\ &+ \left(a_{i+1,j} \frac{1}{h^2} + \frac{2}{3k} \right) w_{i+1,j}^{(x_2 x_2)}. \end{aligned}$$

If we write the above difference scheme in the form $A_{i+1} u^{i+1} = B_i u^i$, A_{i+1} is nonsingular by Taussky's recurrence theorem. By the energy method we can show that this difference scheme satisfies the stability condition if $\beta \geq a(t, x) \geq \alpha > 0$.

The following relations between $w_{i,j}$ and $w_{i,j}^{(x_2 x_2)}$ are available for calculating $w_{i,j}$,

$$w_{i,j-1} - 2w_{i,j} + w_{i,j+1} = \frac{h^2}{6} (w''_{i,j-1} + 4w''_{i,j} + w''_{i,j+1}).$$

If $a(t, x) = 1$, we have the explicit difference scheme by setting $3k = h^2$.

(ii) The case when $a_1(x_1, x_2) \neq 0$ and $a_2(x_1, x_2) \neq 0$.

For any two dimensional spline function $w(x_1, x_2)$, we can regard $w(x_1, x_2)$ as one dimensional spline function with respect to x_1 .

Therefore, we obtain

$$-\frac{1}{k} \sum_{i=0}^{m-1} Q'_{m+1}(m-i)w(ik, x_2) = \sum_{i=0}^{m-1} Q_{m+1}(m-i) \cdot w^{(x_1)}(ik, x_2).$$

The object of this argument is to get an approximate solution by the collocation method, and so we have a next relation

$$\begin{aligned} & -\frac{1}{k} \sum_{i=0}^{m-1} Q'_{m+1}(m-i)w(ik, x_2) \\ &= \sum_{i=0}^{m-1} Q_{m+1}(m-i) \left(a_0(ik, x_2) \frac{\partial^2 w}{\partial x_2^2} + a_1(ik, x_2) \frac{\partial w}{\partial x_2} + a_2(ik, x_2) w(ik, x_2) \right). \end{aligned}$$

From the above, the following differential equation with respect to x_2 is obtained,

$$p((m-1)k, x_2) \frac{\partial^2 w}{\partial x_2^2} + q((m-1)k, x_2) \frac{\partial w}{\partial x_2} + r((m-1)k, x_2) w((m-1)k, x_2)$$

$$= S((m-1)k, \dots, 0; x_2), \quad \text{where}$$

$$p((m-1)k, x_2) = \frac{1}{m!} a_0((m-1)k, x_2),$$

$$q((m-1)k, x_2) = \frac{1}{m!} a_1((m-1)k, x_2),$$

$$r((m-1)k, x_2) = \frac{1}{m!} a_2((m-1)k, x_2) - \frac{1}{(m-1)!} \cdot \frac{1}{k},$$

and $S((m-1)k, \dots, 0; x_2)$ depends upon the data at $x_1 = 0, \dots, (m-2)k$.

As $w(ik, x_2)$ is a spline function with respect to x_2 , we obtain the three-terms recurrence formula by referring to [4] when $n=3$. The similar considerations can be done for the hyperbolic differential equations.

4. Numerical results.

In this section we discuss the numerical results we have obtained by using FACOM 230-60. We now consider the numerical solution of

$$\frac{\partial u}{\partial x_1} = \frac{\partial^2 u}{\partial x_2^2}, \quad u(0, x_2) = \sin \pi x_2, \quad u(x_1, 0) = u(x_1, 1) = 0.$$

The exact solution of this boundary problem is

$$u(x_1, x_2) = \sin \pi x_2 \cdot e^{-\pi^2 x_1}.$$

Table I shows the approximate values and their errors at $x_1 = 1$.

Table I. (mesh ratio $\lambda = k/h^2 = 1$)

	$w(1, i/20)$	absolute error	relative absolute error
1	$8.105\ 385 \times 10^{-6}$	1.4×10^{-8}	1.7×10^{-3}
2	$1.602\ 930 \times 10^{-5}$	3.6×10^{-8}	2.5×10^{-3}
3	$2.355\ 811 \times 10^{-5}$	7.7×10^{-8}	3.3×10^{-3}
4	$3.050\ 645 \times 10^{-5}$	10×10^{-8}	3.3×10^{-3}
5	$3.670\ 326 \times 10^{-5}$	13×10^{-8}	3.5×10^{-3}
6	$4.199\ 597 \times 10^{-5}$	15×10^{-8}	3.5×10^{-3}
7	$4.625\ 428 \times 10^{-5}$	17×10^{-8}	3.7×10^{-3}
8	$4.937\ 335 \times 10^{-5}$	18×10^{-8}	3.7×10^{-3}
9	$5.127\ 642 \times 10^{-5}$	19×10^{-8}	3.7×10^{-3}
10	$5.191\ 664 \times 10^{-5}$	19×10^{-8}	3.7×10^{-3}

We then consider the following parabolic linear problem with variable coefficient,

$$\frac{\partial u}{\partial x_1} = -\frac{1}{2}(1 + x_2 - x_2^2) \cdot \frac{\partial^2 u}{\partial x_2^2}, \quad u(0, x_2) = (1 + x_2 - x_2^2),$$

$u(x_1, 0) = u(x_1, 1) = e^{-x_1}$. The exact solution is $u(x_1, x_2) = (1 + x_2 - x_2^2)e^{-x_1}$.

The numerical results are summarized in the following table II,

Table II. (mesh ratio $\lambda = k/h^2 = 1$)

	$ w(1, i/20) - u(1, i/20) $	$ w_{xx}(1, i/20) - u_{xx}(1, i/20) $	$\frac{ w(1, i/20) - u(1, i/20) }{ u(1, i/20) }$
1	1.2×10^{-4}	4.0×10^{-5}	3.0×10^{-4}
2	2.4×10^{-4}	1.7×10^{-3}	6.0×10^{-4}
3	3.6×10^{-4}	4.4×10^{-3}	9.0×10^{-4}
4	1.4×10^{-4}	4.5×10^{-3}	3.3×10^{-4}
5	5.6×10^{-4}	4.6×10^{-3}	12.0×10^{-4}
6	6.7×10^{-4}	5.4×10^{-3}	14.9×10^{-4}
7	7.1×10^{-4}	7.1×10^{-3}	15.8×10^{-4}
8	7.6×10^{-4}	7.6×10^{-3}	16.5×10^{-4}
9	7.8×10^{-4}	7.9×10^{-3}	17.0×10^{-4}
10	7.0×10^{-4}	8.0×10^{-3}	15.2×10^{-4}

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