**A REMARK ON THE TRACE FORMULA FOR AN
INSEPARABLE CORRESPONDENCE IN
AN ALGEBRAIC FUNCTION FIELD**

By

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Dedicated to Professor Tosio Kitagawa on his 60-th birthday

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1. Let k be an algebraic function field of one variable with a perfect constant field k_0 of the characteristic $p > 0$. H. Kappus [2] has constructed the non-trivial linear transformations of the vector space of differentials of the first kind in k for inseparable correspondences and proved that the trace formula of their representations is equal to the fixed point formula.

If k_0 is especially a finite field $GF(p^h)$ and k is an elliptic function field, we know well that it is possible to determine by the calculation of the number N_1 of the prime divisors of degree 1 in k whether the Hasse invariant A of k is 0 or not. (M. Yamaguchi [3]; p. 33).

The purpose of this note is to prove that the relation between A and N_1 is explicitly given by the formula

$$A^{1+p+\cdots+p^{h-1}} = \overline{1-N_1},$$

by means of the results of H. Kappus, where $\overline{1-N_1}$ denotes the residue class of $1-N_1 \bmod p$ and it is identified with the element of k_0 .

2. We shall quote here and in the next section the several well-known results. We denote by g the genus of k and assume that $g > 0$. Moreover, in this section, we assume that there exists a system of non-special prime divisors of k ; i.e., there exists a system $\{\mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_g\}$ of g different prime divisors of degree 1 of k such that the divisor class of $\mathfrak{P}_1\mathfrak{P}_2\cdots\mathfrak{P}_g$ is of dimension 1.

Let t_i be a prime element of $\mathfrak{P}_i$ for $1 \leq i \leq g$. The following proposition is due to Hasse ([1]; Satz 4).

PROPOSITION 1. *There exist the elements $v_1, v_2, \dots, v_g$ in k satisfying the following conditions.*

$$(i) \quad v_j \equiv \frac{e_{ij}}{t_i^p} - \frac{a_{ij}}{t_i} \pmod{\mathfrak{P}_i^0} \text{ for } 1 \leq i, j \leq g,$$

$$\text{where } e_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{and } a_{ij} \in k_0.$$

$$(ii) \quad \text{ord}_{\mathfrak{P}} v_j \geq \begin{cases} -p & \text{if } \mathfrak{P} = \mathfrak{P}_i \\ 0 & \text{if } \mathfrak{P} \neq \mathfrak{P}_i \end{cases}$$

for a prime divisor $\mathfrak{P}$ of k , where $\text{ord}_{\mathfrak{P}} v_j$ means the order of v_j at $\mathfrak{P}$.

The g -rowed square matrix $A = (a_{ij})$ generated by a_{ij} in Proposition 1 is the so-called Hasse-Witt matrix associated with $\{\mathfrak{P}_i, t_i\}$.

Now we take a basis of the differentials of the first kind in k and denote by $\{du_i\}$, $(1 \leq i \leq g)$, it. Then we can obtain the following representations for du_i , $(1 \leq i \leq g)$,

$$\frac{du_i}{dt_j} \equiv \sum_{\nu=0}^{p-1} b_{ij}^{(\nu)} t_j^{\nu} \pmod{\mathfrak{P}_j^p},$$

where $b_{ij}^{(\nu)} \in k_0$ and it is uniquely determined by du_i , $\mathfrak{P}_j$ and t_j for $1 \leq i, j \leq g$.

We denote by B_0 and B_{p-1} g -rowed square matrices $(b_{ij}^{(0)})$ and $(b_{ij}^{(p-1)})$ respectively. Then we have the well-known fact. (Hasse [1]; Satz 6).

PROPOSITION 2. *The matrix B_0 is regular and the equality $A = B_0^{-1} B_{p-1}$ holds for the Hasse-Witt matrix A of k .*

3. We assume that k_0 is a finite field $GF(p^h)$. Let k' be a field which is isomorphic with k over k_0 and algebraically disjoint with k over k_0 . Then we may assume that k and k' are given by the forms $k = k_0(x, y)$ and $k' = k_0(x', y')$ satisfying the same equation $f(x, y) = f(x', y') = 0$, where the coefficients of f belong to k_0 .

The Frobenius correspondence F of k is associated with the prime ideal

$$F_{k \times k'} = k \times k'(x - x'^q, y - y'^q)$$

in the integral domain $k \times k'$, where $q = p^h$.

If we denote by K and k^π the residue class fields modulo $F_{k \times k'}$ of $k \times k'$ and k' respectively, the equality $K = k^\pi = k^{q^{-1}}$ holds and so K is purely inseparable and of finite degree q over k . (Figure).

Fig.

$k^{q^{-1}}$ means the set of q -th roots of all elements of k .

Now, let ω be a differential of k . For the Frobenius correspondence F of k , H. Kappus [2] has defined the k_0 -linear transformation of the space of differentials of k to itself by the rule

$$\omega F = C^h(\omega^\pi)$$

where C denotes the Cartier-operator and C^h means to act C h times. ω^π means the image of ω for π and it is a differential of $K = k^\pi = k^{q^{-1}}$.

If ω is of the first kind, ωF is so, since the mapping π is the k_0 -linear transformation of differentials of the first kind of k into those of $k^\pi = K$ and the operator C^h is the k_0 -linear transformation of differentials of the first kind of K into those of k .

Hence we can obtain a matrix representation P of F in the vector space of differentials of the first kind of k . The following proposition is due to H. Kappus ([2]; p. 382).

PROPOSITION 3. *Let k be a field of algebraic functions of one variable; assume that the field of constants of k is a finite field $GF(p^h)$. Denote by N_1 the number of prime divisors of degree 1 in k and denote by g the genus*

of k ; assume that $g > 0$.

Then, for a matrix representation P of the Frobenius correspondence of k in the vector space of differentials of the first kind of k , the equality

$$\text{Trace } P = \overline{1 - N_1}$$

holds.

4. We assume that there exists a system of non-special prime divisors of k . We shall show that a representation of Frobenius correspondence F of k is explicitly obtained by the Hasse-Witt matrix. Let $\{du_i\}$, $(1 \leq i \leq g)$, be a basis of differentials of the first kind of k ; fix it. Let $\{\mathfrak{P}_j\}$, $(1 \leq j \leq g)$, be a system of non-special prime divisors of k and let t_j be a prime element of $\mathfrak{P}_j$, $(1 \leq j \leq g)$; fix them.

We may assume that $k = k_0(x, y)$, where x and y are elements of k which satisfy some irreducible equation $f(x, y) = 0$ with the coefficients in k_0 .

Then, for each integer l , $(1 \leq l \leq h)$, the field $k^{p^{-l}}$ coincides with $k_0(x^{p^{-l}}, y^{p^{-l}})$, where $x^{p^{-l}}$ and $y^{p^{-l}}$ satisfy the irreducible equation $f^{p^{-l}}(x^{p^{-l}}, y^{p^{-l}}) = 0$. $f^{p^{-l}}$ means the polynomial whose coefficients are the p^l -th roots of the coefficients of the polynomial f .

For every element z of k , the correspondence $z \rightarrow z^{p^{-l}}$ gives an isomorphism of k onto $k^{p^{-l}}$. This isomorphism gives naturally the transformations of divisors and differentials of k into those of $k^{p^{-l}}$.

It follows that the image $\{\mathfrak{P}_j^{(l)}\}$ of $\{\mathfrak{P}_j\}$ is a system of non-special prime divisors of $k^{p^{-l}}$ and the image $\{du_i^{(h-l)}\}$ of $\{du_i\}$ is a basis of differentials of the first kind of $k^{p^{-l}}$. Obviously the image $t_j^{(l)}$ of t_j coincides with $t_j^{p^{-l}}$ and it is a prime element of $\mathfrak{P}_j^{(l)}$.

Moreover, we see easily that the image $A^{p^{-l}} = (a_{ij}^{p^{-l}})$ of the Hasse-Witt matrix $A = (a_{ij})$ associated with $\{\mathfrak{P}_j, t_j\}$ in k is also the Hasse-Witt matrix associated with $\{\mathfrak{P}_j^{(l)}, t_j^{p^{-l}}\}$ in $k^{p^{-l}}$. If we denote by du_i^π , $\mathfrak{P}_j^\pi$ and t_j^π the images of du_i , $\mathfrak{P}_j$ and t_j for F respectively, the equalities $du_i = du_i^{(0)}$, $\mathfrak{P}_j^\pi = \mathfrak{P}_j^{(h)}$ and $t_j^\pi = t_j^{p^{-h}}$ hold.

Now we shall find the matrix representation P of F with respect to the basis $\{du_i\}$. We may assume that the differential du_i has the representation

$$du_i = \sum_{\nu=0}^{p-1} b_{ij}^{(\nu)} t_j^\nu dt_j \pmod{\mathfrak{P}_j^p},$$

where $b_{ij}^{(\nu)} \in k_0$ for $1 \leq i, j \leq g$, $0 \leq \nu \leq p-1$.

Then, on the one hand, the congruence

$$du_i^{(h-l)} \equiv \sum_{\nu=0}^{p-1} (b_{ij}^{(\nu)})^{p-l} t_j^{p-l-\nu} dt_j^{p-l} \pmod{\mathfrak{P}_j^{(l)p}}$$

holds for $1 \leq l \leq h$, $1 \leq i, j \leq g$.

It follows obviously that for the Cartier-operator C the congruence

$$C(du_i^{(h-l)}) \equiv (b_{ij}^{(p-1)})^{p-l} dt_j^{p-(l-1)} \pmod{\mathfrak{P}_j^{(l-1)}}$$

holds for $1 \leq i, j \leq g$.

On the other hand, there exist the elements $s_{in}^{(h-(l-1))}$, $(1 \leq i, j \leq g)$, in k_0 such that

$$C(du_i^{(h-l)}) = \sum_{n=1}^g s_{in}^{(h-(l-1))} du_n^{(h-(l-1))},$$

since $C(du_i^{(h-l)})$ is a differential of the first kind of $k^{p^{-(l-1)}}$ and $du_n^{(h-(l-1))}$, $(1 \leq n \leq g)$, is a basis of such differentials.

So the other congruence

$$C(du_i^{(h-l)}) \equiv \sum_{n=1}^g s_{in}^{(h-(l-1))} (b_{nj}^{(0)})^{p-(l-1)} dt_j^{p-(l-1)} \pmod{\mathfrak{P}_j^{(l-1)}}$$

holds for $1 \leq i, j \leq g$. Thus we obtain

$$(b_{ij}^{(p-1)})^{p-l} = \sum_{n=1}^g s_{in}^{(h-(l-1))} (b_{nj}^{(0)})^{p-(l-1)}; \quad \text{i. e.,}$$

if we denote by $S_{h-(l-1)}$, B_0 and B_{p-1} the matrices $(s_{ij}^{(h-(l-1))})$, $(b_{ij}^{(0)})$ and $(b_{ij}^{(p-1)})$ respectively, the equality

$$B_{p-1}^{p-l} = S_{h-(l-1)} B_0^{p-(l-1)}$$

holds, where B^{p-l} means the matrix (b_{ij}^{p-l}) .

It is rewritten by

$$S_{h-(l-1)} = B_{p-1}^{p-l} (B_0^{-1})^{p-(l-1)},$$

since B_0 is a regular matrix from Proposition 2. So we obtain inductively

$$P \begin{bmatrix} du_1 \\ \vdots \\ du_g \end{bmatrix} = \begin{bmatrix} C^h(du_1) \\ \vdots \\ C^h(du_g) \end{bmatrix} = \{B_{p-1}^{p-h}(B_0^{-1})^{p-(h-1)}\} \cdots \\ \cdots \{B_{p-1}^{p-l}(B_0^{-1})^{p-(l-1)}\} \cdots \{B_{p-1}^{p-1}B_0^{-1}\} \begin{bmatrix} du_1 \\ \vdots \\ du_g \end{bmatrix}.$$

Thus we have

$$\begin{aligned} P &= \{B_0^{p-h} A^{p-h} (B_0^{-1})^{p-(h-1)}\} \cdots \\ &\quad \cdots \{B_0^{p-l} A^{p-l} (B_0^{-1})^{p-(l-1)}\} \cdots \{B_0^{p-1} A^{p-1} B_0^{-1}\} \\ &= B_0^{p-h} A^{p-h} \cdots A^{p-l} \cdots A^{p-1} B_0^{-1} \\ &= B_0 A A^p \cdots A^{p^{h-1}} B_0^{-1}, \end{aligned}$$

since $B_{p-1} = B_0 A$ holds from Proposition 2.

Therefore we obtain the equality

$$\text{Trace } P = \text{Trace } (A A^p \cdots A^{p^{h-1}}).$$

This means that $\text{Trace } (A A^p \cdots A^{p^{h-1}})$ is independent on the choice of $\{\mathfrak{P}_j, t_j\}$.

Hence, by making use of Proposition 3, we have following results.

THEOREM. *Let k be a field of algebraic functions of one variable; assume that the field of constants of k is a finite field $GF(p^h)$. Denote by g the genus of k ; assume that $g > 0$. Moreover assume that there exists a system of non-special prime divisors of k .*

Then the equality

$$\text{Trace } (A A^p \cdots A^{p^{h-1}}) = \overline{1 - N_1}$$

holds for the number N_1 of prime divisors of degree 1 and the Hasse-Witt matrix A .

COROLLARY. *Let notations and assumptions be as in the above Theorem. If k is an elliptic function field, the equality*

$$A^{1+p+\cdots+p^{h-1}} = \overline{1 - N_1}$$

holds; i.e., a necessary and sufficient condition for $A=0$ is $N_1 \equiv 1 \pmod{p}$.

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