ON A PROBLEM PROPOSED BY EELLS-SAMPSON

By

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Dedicated to Professor Tosio Kitagawa on his 60-th birthday

(Received May 4, 1970)

§ 1. Introduction.

J. Eells, Jr. and J. H. Sampson proposed the following problem in [2], § 6, #5.

Let M be a compact oriented Riemannian manifold with a metric g . We define an inner product on $C^\infty(M)$ by

$$(1.1) \quad \langle \varphi, \psi \rangle = \int_M \varphi(x) \psi(x) dx + \int_M \langle d\varphi, d\psi \rangle dx$$

and denote by $L_1^2(M)$ the Hilbert space obtained by completion with respect to the above norm. The Dirichlet integral D is a continuous quadratic form $D: L_1^2(M) \rightarrow \mathbb{R}$ defined on $C^\infty(M)$ by

$$(1.2) \quad D(\varphi) = \frac{1}{2} \int_M \langle d\varphi, d\varphi \rangle dx.$$

Let M' be another such manifold with Dirichlet integral D' . J. Eells, Jr. and J. H. Sampson say that two pairs $(L_1^2(M), D)$ and $(L_1^2(M'), D')$ are equivalent if there exists a mapping f from $L_1^2(M)$ onto $L_1^2(M')$ satisfying

$$(1.3) \quad D' \circ f = D \quad \text{on } L_1^2(M)$$

and

$$(1.4) \quad \langle \varphi, \psi \rangle = \langle f\varphi, f\psi \rangle' \quad \text{for } \forall \varphi, \psi \in L_1^2(M).$$

Now the problem is stated as follows:

PROBLEM. *Is it true that M and M' are isometric if and only if $(L_1^2(M), D)$ and $(L_1^2(M'), D')$ are equivalent?*

In this paper, we shall prove two theorems and give an answer to this problem. The only if part of this problem is obvious. But the if part is negative. This follows from our Theorem 1 and J. Milnor's example [3].

THEOREM 1. *The following two propositions are equivalent:*

- (1) $S_p(M, g) = S_p(M', g')$.
- (2) $(L_1^2(M), D)$ and $(L_1^2(M'), D')$ are equivalent.

In the above Theorem, $S_p(M, g)$ (resp. $S_p(M', g')$) denotes the set of all eigenvalues (with multiplicity) of the Laplacian with respect to g (resp. g').

J. Milnor showed that there exist 16-dimensional tori T and T' with canonical Riemannian metrics g and g' which satisfy $S_p(T, g) = S_p(T', g')$ but are not isometric. Therefore, as remarked, the if part of the problem is negative. Nevertheless, if we impose certain conditions on f , the if part is also true, that is,

THEOREM 2. *Assume $(L_1^2(M), D)$ and $(L_1^2(M'), D')$ are equivalent under f . Then f maps $C^\infty(M)$ onto $C^\infty(M')$. Furthermore, if f , which is restricted to $C^\infty(M)$, and f^{-1} , which is restricted to $C^\infty(M')$, map non-negative functions on non-negative functions, then M and M' are isometric.*

We devote §2 to the proof of Theorem 1 and §3 to that of Theorem 2.

§ 2. Proof of Theorem 1.

Assume that $(L_1^2(M), D)$ and $(L_1^2(M'), D')$ are equivalent under f . It follows from (1.1) that $C^\infty(M) \subset L_1^2(M) \subset L^2(M)$. If we set $\langle \varphi, \psi \rangle = (\varphi, \psi) + \delta(\varphi, \psi)$ for any $\varphi, \psi \in L_1^2(M)$, where $(\cdot, \cdot)$ is an inner product in $L^2(M)$, then δ is a positive bilinear form on $L_1^2(M)$ and $\delta(\varphi, \varphi) = 2D(\varphi)$. It is easy to see that f is linear and (1.3) is equivalent to $\delta(\varphi_1, \varphi_2) = \delta'(f\varphi_1, f\varphi_2)$ for any $\varphi_1, \varphi_2 \in L_1^2(M)$. Therefore (1.3) and (1.4) are equivalent to

$$(2.1) \quad (\varphi, \psi) = (f\varphi, f\psi),$$

and

$$(2.2) \quad \delta(\varphi, \psi) = \delta'(f\varphi, f\psi).$$

For any $\varphi \in C^\infty(M)$,

$$\Delta\varphi^2 = 2\varphi\Delta\varphi + 2\langle d\varphi, d\varphi \rangle \quad (\Delta: \text{Laplacian}).$$

Integrating this equation, we have

$$0 = \int_M \Delta\varphi^2 dx = 2 \int_M \varphi \Delta\varphi dx + 2 \int_M \langle d\varphi, d\varphi \rangle dx.$$

Thus we have

$$(2.3) \quad - \int_M \varphi \Delta\varphi dx = \int_M \langle d\varphi, d\varphi \rangle dx \quad (= 2D(\varphi)).$$

It is well-known that the eigenvalues $\lambda_j (j=1, 2, \dots)$ are all real and non-positive and of finite multiplicity. Moreover, they have no finite point of accumulation point on the real line, the corresponding eigen-functions $\varphi_j (j=1, 2, \dots)$ belong to $C^\infty(M)$ and form a complete system in $L^2(M)$. So we can order them as follows:

$$0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_k \geq \dots \rightarrow -\infty.$$

We normalize φ_j with respect to $(\cdot, \cdot)$ and denote them by the same φ_j . The Fourier coefficients of $\Delta\varphi$ with respect to $\{\varphi_j\}$ are

$$(\varphi_j, \Delta\varphi) = (\Delta\varphi_j, \varphi) = \lambda_j(\varphi_j, \varphi) = \lambda_j c_j,$$

where $\varphi \in C^\infty(M)$ and $\varphi = \sum_{j=1}^{\infty} c_j \varphi_j$ (in $L^2(M)$). Thus we have

$$\Delta\varphi = \sum_{j=1}^{\infty} \lambda_j c_j \varphi_j \quad (\text{in } L^2(M)).$$

Therefore we have

$$\inf_{\varphi} B_1(\varphi) = \inf_{\varphi} \frac{2D(\varphi)}{(\varphi, \varphi)} = -\lambda_1 = 0 \quad (\varphi \in C^\infty(M)),$$

$$\text{since } 0 \leq \frac{2D(\varphi)}{(\varphi, \varphi)} = \frac{-(\varphi, \Delta\varphi)}{(\varphi, \varphi)} = \frac{\sum_{j=1}^{\infty} (-\lambda_j) c_j^2}{\sum_{j=1}^{\infty} c_j^2}.$$

It is easy to see that a function φ in $C^\infty(M)$ satisfying $B_1(\varphi) = 0$ is a

constant multiple of φ_1 . Similarly we consider

$$\inf_{\varphi} B_2(\varphi) \equiv \inf_{\varphi} \frac{2D(\varphi)}{(\varphi, \varphi)} \quad (\varphi \in C^\infty(M) \setminus [\varphi_1])$$

where we denote by $[f_1, \dots, f_k]$ the linear space spanned by $f_1, \dots, f_k$ and by $C^\infty(M) \setminus [f_1, \dots, f_k]$ the orthogonal complement of $[f_1, \dots, f_k]$ in $C^\infty(M)$ with respect to $(\cdot, \cdot)$. Then as above, it is easy to see that $\inf B_2(\varphi) = -\lambda_2$

since $0 \leq \frac{2D(\varphi)}{(\varphi, \varphi)} = \frac{\sum_{j=2}^{\infty} (-\lambda_j) c_j^2}{\sum_{j=2}^{\infty} c_j^2}$ and that the set of functions of $C^\infty(M) \setminus [\varphi_1]$

satisfying $B_2(\varphi) = -\lambda_2$ is just the space $[\varphi_2, \varphi_3, \dots, \varphi_{i_2}]$, where $i_2 - 1$ is the multiplicity of λ_2 . We choose an element ψ_2 satisfying $B_2(\psi_2) = -\lambda_2$, which is an eigenfunction of Δ corresponding to λ_2 and consider $C^\infty(M) \setminus [\varphi_1, \psi_2]$ and so on. In this way, we can obtain all the eigenvalues of Δ and the corresponding eigenfunctions which are linearly independent.

Next we consider the above procedure in $L_1^2(M)$. For any $\varphi \in L_1^2(M)$ there exists a sequence $\{\xi_n\} \subset C^\infty(M)$ such that $\langle \xi_n - \varphi, \xi_n - \varphi \rangle \rightarrow 0 (n \rightarrow \infty)$. This sequence $\{\xi_n\}$ satisfies $(\xi_n, \xi_n) \rightarrow (\varphi, \varphi)$ and $D(\xi_n) \rightarrow D(\varphi) (n \rightarrow \infty)$. So we have

$$\begin{aligned} \inf_{\varphi} C_1(\varphi) &\equiv \inf_{\varphi} \frac{2D(\varphi)}{(\varphi, \varphi)} \quad (\varphi \in L_1^2(M)) \\ &= \inf_{\varphi} \frac{2D(\varphi)}{(\varphi, \varphi)} \quad (\varphi \in C^\infty(M)) \\ &= -\lambda_1 = 0. \end{aligned}$$

Here, we remark that $\{\varphi_i\}$ also forms the orthogonal complete system in $L_1^2(M)$.

In fact,

$$\Delta(\varphi_i \varphi_j) = 2 \langle d\varphi_i, d\varphi_j \rangle + \Delta\varphi_i \cdot \varphi_j + \varphi_i \cdot \Delta\varphi_j.$$

Integrating this equation,

$$0 = \int_M \Delta(\varphi_i \varphi_j) dx = 2 \int_M \varphi_i \Delta\varphi_j dx + 2 \int_M \langle d\varphi_i, d\varphi_j \rangle dx.$$

Hence,

$$\int_M \langle d\varphi_i, d\varphi_j \rangle dx = - \int_M \varphi_i \cdot \Delta \varphi_j dx = -\lambda_j \int_M \varphi_i \cdot \varphi_j dx = 0.$$

This shows $\langle \varphi_i, \varphi_j \rangle = 0$. Also, it is easy to see that $\{\varphi_i\}$ forms complete system in $L_1^2(M)$.

By the above remark, we know the orthogonal complement space $L_1^2(M) \setminus [\varphi_1, \dots, \varphi_k]$ of $[\varphi_1, \dots, \varphi_k]$ in $L_1^2(M)$ is the same as the orthogonal complement in $L_1^2(M)$ of $[\varphi_1, \dots, \varphi_k]$ with respect to $(,)$.

Note that any $\varphi \in L_1^2(M) \setminus [\varphi_1]$ can be approximated arbitrarily by functions belonging to $C^\infty(M) \setminus [\varphi_1] \subset L_1^2(M) \setminus [\varphi_1]$.

Hence we see that

$$\begin{aligned} \inf_{\varphi} C_2(\varphi) &= \inf_{\varphi} \frac{2D(\varphi)}{(\varphi, \varphi)} \quad (\varphi \in L_1^2(M) \setminus [\varphi_1]) \\ &= \inf_{\varphi} \frac{2D(\varphi)}{(\varphi, \varphi)} \quad (\varphi \in C^\infty(M) \setminus [\varphi_1]) \\ &= -\lambda_2. \end{aligned}$$

The function $\varphi \in L_1^2(M)$ satisfying $C_1(\varphi) = 0$ belongs to $[\varphi_1]$. For, if we write $\varphi = \sum_{j=1}^{\infty} c_j \varphi_j$ in $L^2(M)$, by the above result, we see $c_1 \neq 0$. If there exists $c_i (i > 1)$ such that $c_i \neq 0$, we choose the minimum index $i_0, c_{i_0} \neq 0$. There exists a sequence $\{\xi_n\} \subset C^\infty(M)$ such that $\xi_n \rightarrow \varphi$ in $L_1^2(M)$. Then it is easy to see $(\xi_n, \xi_n) \rightarrow (\varphi, \varphi)$, $D(\xi_n) \rightarrow D(\varphi)$ and if we write $\xi_n = \sum_{i=1}^{\infty} c_{ni} \varphi_i$, $c_{ni_0} \rightarrow c_{i_0}$. Then

$$\begin{aligned} C_1(\varphi) &= \frac{2D(\varphi)}{(\varphi, \varphi)} \geq \frac{-\frac{1}{4} \lambda_{i_0} c_{i_0}^2}{2(\varphi, \varphi)} > 0 \quad \text{since} \\ \frac{2D(\xi_n)}{(\xi_n, \xi_n)} &= \frac{\sum_{i=1}^{\infty} (-\lambda_i) (c_{ni})^2}{\sum_{i=1}^{\infty} (c_{ni})^2} \geq \frac{(-\lambda_{i_0}) \left(\frac{1}{2} c_{i_0}\right)^2}{\sum_{i=1}^{\infty} (c_{ni})^2} \geq \frac{-\frac{1}{4} \lambda_{i_0} c_{i_0}^2}{2(\varphi, \varphi)} \neq 0. \end{aligned}$$

This is a contradiction. Hence $c_1 \neq 0$, $c_i = 0 (i > 1)$ i.e. $\varphi \in [\varphi_1]$. Similarly we see that the function $\varphi \in L_1^2(M)$ satisfying $C_2(\varphi) = -\lambda_2$ is an eigenfunction corresponding to the eigenvalue λ_2 .

Continuing this procedure, we can obtain all the eigenvalues of Δ and the corresponding eigenfunctions. It is easily seen from the above consideration that $S_p(M, g) = S_p(M', g')$ and f maps the eigenfunctions of Δ on the eigenfunctions of Δ' corresponding to the same eigenvalues.

Hence (2) implies (1).

Now we shall show that f maps $C^\infty(M)$ onto $C^\infty(M')$. For any $\varphi \in L^2_1(M)$ if we write $\varphi = \sum_{j=1}^{\infty} c_j \varphi_j$ in $L^2(M)$, then we have $f\varphi = \sum_{j=1}^{\infty} c_j f\varphi_j$ in $L^2(M')$. Furthermore if $\varphi \in C^\infty(M)$, the convergence of $\sum_{i=1}^{\infty} c_i \varphi_i$ on M and that of $\sum_{i=1}^{\infty} c_i f\varphi_i$ on M' are both uniform. Therefore we can regard $f\varphi$ as a continuous function.

In fact, it is well-known that

$$(2.4) \quad \Delta \varphi_i = \lambda_i \varphi_i \quad (i \geq 2)$$

and

$$(2.5) \quad \varphi_i = \lambda_i \Omega \varphi_i \quad (i \geq 2)$$

are equivalent, where Ω is the integral operator with Green function ω as its kernel. ω is of index $n-2$ on $D = \{(x, y) \in M \times M \mid x = y\}$ and $\omega \in C^\infty(M \times M \setminus D)$, ($n = \dim M$). We construct kernels $\omega^{(k)}(x, y)$ ($k=1, 2, 3, \dots$) as follows:

We set

$$\omega^{(1)}(x, y) = \omega(x, y)$$

and

$$\omega^{(k+1)}(x, y) = \int_M \omega^{(k)}(x, z) \omega(z, y) dz.$$

Then $\omega^{(k)}(x, y)$ is of index $n-2k$ on D , particularly it is a continuous function of (x, y) for $k > \left\lceil \frac{n}{2} \right\rceil$ and

$$(2.6) \quad \varphi_i = \lambda_i^j \Omega^j \varphi_i \quad (i \geq 2)$$

holds. Also, (2.5) is equivalent to (2.6). The above results also hold about Δ' and f maps the eigenfunctions of Δ on the eigenfunctions of Δ'

corresponding to the same eigenvalues. Therefore, $\varphi_i = \lambda_i^m \mathcal{Q}^m \varphi_i$ and $f\varphi_i = \lambda_i^m \mathcal{Q}'^m f\varphi_i$ for $i \geq 2$, where $m = \max\left(\left[\frac{n}{2}\right] + 1, \left[\frac{n'}{2}\right] + 1\right)$, $n' = \dim M'$. If we set $(\varphi, \varphi_1) = c_1$ for $\varphi \in C^\infty(M)$, then $\varphi - c_1\varphi_1 = \mathcal{Q}^m \beta$ for some $\beta \in C^\infty(M)$. Since $\mathcal{Q}^m$ is a symmetric integral operator with a continuous kernel, its eigenfunctions are $\varphi_2, \varphi_3, \dots$. So the convergence of $\varphi - c_1\varphi_1 = \sum_{i=2}^{\infty} c_i \varphi_i$ is uniform on M , that is, the convergence of $\varphi = \sum_{i=1}^{\infty} c_i \varphi_i$ is uniform on M . If we set $c'_i = (\mathcal{Q}'^m f\beta, f\varphi_i)$, then

$$\begin{aligned} c'_i &= (f\beta, \mathcal{Q}'^m f\varphi_i) = \left(f\beta, -\frac{1}{\lambda_i^m} f\varphi_i\right) \\ &= \left(\beta, -\frac{1}{\lambda_i^m} \varphi_i\right) = (\beta, \mathcal{Q}^m \varphi_i) = (\mathcal{Q}^m \beta, \varphi_i) = c_i. \end{aligned}$$

Hence we have $\mathcal{Q}'^m f\beta = \sum_{i=2}^{\infty} c_i f\varphi_i$ (in $L^2(M)$).

Since the convergence of $\sum_{i=2}^{\infty} c_i f\varphi_i$ is uniform on M' , $f\varphi$ is a continuous function. Now we shall show that $f\varphi \in C^\infty(M')$ for any $\varphi \in C^\infty(M)$. $\varphi - c_1\varphi_1 = \mathcal{Q}\beta$ ($\beta \in C^\infty(M)$) for any $\varphi \in C^\infty(M)$ and $\mathcal{Q}'f\beta \in C^1(M')$ since $f\beta \in C^0(M')$ and the kernel $\omega'(x', y')$ is of index $n' - 2$ on D' . Investigating Fourier coefficients, we see that $f\varphi = c_1 f\varphi_1 + \mathcal{Q}'f\beta$ in $L^2(M')$. This implies that $f\varphi = c_1 f\varphi_1 + \mathcal{Q}'f\beta$ on M' , for $f\varphi$ and $\mathcal{Q}'f\beta$ are continuous and $f\varphi_1$ is constant. Also since $\mathcal{Q}'f\beta \in C^1(M')$, we obtain $f\varphi \in C^1(M')$ for any $\varphi \in C^\infty(M)$. As $\beta \in C^\infty(M)$, also we have $f\beta \in C^1(M')$. Similarly, using the fact that $\omega'(x', y')$ is of index $n' - 2$, we see that $\mathcal{Q}'f\beta \in C^2(M')$ and consequently $f\varphi \in C^2(M')$. Using recursively this method, we see that $f\varphi \in C^\infty(M')$ for any $\varphi \in C^\infty(M)$.

Now we shall prove that (1) implies (2). Assume that $S_p(M, g) = S_p(M', g')$. Let

$$0 = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_k \geq \dots \rightarrow -\infty$$

be the eigenvalues of Δ and Δ' and $\{\varphi_1, \varphi_2, \dots\}$, $\{\psi_1, \psi_2, \dots\}$ the corresponding eigenfunctions on M and M' which are orthonormal in $L^2(M)$ and $L^2(M')$ respectively. We construct a mapping $f: L_1^2(M) \rightarrow L_1^2(M')$ which satisfies the desired properties. We define $f\varphi_i = \psi_i$ for φ_i . For any $\varphi \in C^\infty(M)$, we write $\varphi = \sum_{i=1}^{\infty} c_i \varphi_i$ (where $c_i = (\varphi, \varphi_i)$) and define $f\varphi$ by

$$f\varphi = \sum_{i=1}^{\infty} c_i f\varphi_i = \sum_{i=1}^{\infty} c_i \psi_i.$$

As we remarked above, the convergence of the right hand side of this equation is uniform, and also $f\varphi$ belongs to $C^\infty(M')$. Conversely any element of $C^\infty(M')$ is expressed as $f\varphi$ for some $\varphi \in C^\infty(M)$. We have only to show that this mapping f satisfies (1.3) and (1.4) or equivalently (2.1) and (2.2). But we can easily see that f satisfies (2.1) and (2.2).

Since f is continuous on $C^\infty(M)$ which is a dense subset of $L_1^2(M)$, we can extend f on the whole $L_1^2(M)$ and the extended mapping f also satisfies (2.1) and (2.2). Therefore (1) implies (2).

§ 3. Proof of Theorem 2.

Since we have proved in §2 that f (resp. f^{-1}) maps $C^\infty(M)$ (resp. $C^\infty(M')$) onto $C^\infty(M')$ (resp. $C^\infty(M)$), throughout this section, we restrict f (resp. f^{-1}) on $C^\infty(M)$ (resp. $C^\infty(M')$). And we assume that f, f^{-1} map non-negative functions on non-negative functions.

It is well-known that $\text{vol}(M) = \text{vol}(M')$ (e.g. [1]). Hence we have

$$\varphi_1 = (\text{vol}(M))^{-\frac{1}{2}} = (\text{vol}(M'))^{-\frac{1}{2}} = f\varphi_1.$$

Since f is linear, f maps a constant function c on M to a constant function c on M' . If we set $a = \text{Min}_{x \in M} \varphi(x)$ for $\varphi \in C^\infty(M)$, $\varphi - a \geq 0$ implies $f(\varphi - a) = f\varphi - a \geq 0$. Also if we set $b = \text{Min}_{x' \in M'} (f\varphi(x') - a) \geq 0$, then $f\varphi - a - b \geq 0$. Hence we have $\varphi - a - b \geq 0$ and so $b = 0$, that is,

$$\text{Min}_{x \in M} \varphi(x) = \text{Min}_{x' \in M'} (f\varphi)(x').$$

Similarly, using that f and f^{-1} map non-positive functions on non-positive functions, we have

$$\text{Max}_{x \in M} \varphi(x) = \text{Max}_{x' \in M'} (f\varphi)(x').$$

Therefore, if we set $F_x = \{\varphi \in C^\infty(M) \mid \varphi(x) = \text{Max}_{y \in M} \varphi(y)\}$, $\varphi(x) = a$ and $\psi(x) = b$ for $\varphi, \psi \in F_x$, it follows that $\text{Max} f\varphi = a$, $\text{Max} f\psi = b$ and $\text{Max} f(\varphi + \psi) = a + b = \text{Max} f\varphi + \text{Max} f\psi$ since $\text{Max}(\varphi + \psi) = \varphi(x) + \psi(x) = a + b$. This implies that the intersection of the sets $X_x^{(i)} (i=1, 2)$ on which

$f\varphi_i \in f \cdot F_x (i=1, 2)$ attain their maximum values is not empty and this property also holds for any finite $f\varphi_i \in f \cdot F_x$. So there exists a non-empty common subset X'_x of M' on which all the functions of $f \cdot F_x$ attain their maximum values, for M' is compact and a family $\{X'^{(\alpha)}_x\}$ of closed subsets has a finite intersection property, where we denote by $X'^{(\alpha)}_x$ the set of all points in M' at which $f\varphi_\alpha \in f \cdot F_x$ attains its maximum value. Also set $F'_x = \{\varphi \mid -\varphi \in F_x\} = \{\varphi \in C^\infty(M) \mid \varphi(x) = \min_{y \in M} \varphi(y)\}$, all the functions of $f \cdot F'_x$ attain their minimum values on X'_x . Next we shall show that X'_x is an only one point. At first, if x and y are distinct points of M , then the intersection of X'_x and X'_y is empty. Indeed, there exists a function $\varphi \in C^\infty(M)$ such that

$$\max_{z \in M} \varphi(z) = \varphi(x), \quad \min_{z \in M} \varphi(z) = \varphi(y) \quad \text{and} \quad \varphi(x) \neq \varphi(y).$$

Then, since $\max_{z' \in M'} f\varphi(z') = \varphi(x)$ and $\min_{z' \in M'} f\varphi(z') = \varphi(y)$, $f\varphi$ attains respectively maximum value and minimum value on distinct points. This implies the intersection of X'_x and X'_y is empty. Now, note there exists a function φ' such that φ' attains maximum value at only one point in M' , we know X'_x is an only one point. From now on, we denote X'_x by x' . Hence we can define a mapping $\Phi: M \rightarrow M'$ by $\Phi(x) = x'$. Furthermore, note f maps $C^\infty(M)$ onto $C^\infty(M')$, this mapping Φ is clearly onto.

At first, we show that Φ is a homeomorphism. For any fixed x of M , let U be an arbitrary open neighborhood of x . Then there exist an open set V and $\varphi \in C^\infty(M)$ such that $U \supset \bar{V} \supset V \ni x$ and $0 \leq \varphi \leq 1$ on M , $\varphi = 1$ on V and $\varphi = 0$ on $M \setminus U$. Then

$$\varphi(y) = (f\varphi)(\Phi(y)) = \begin{cases} 1 & \text{on } V, \\ 0 & \text{on } M \setminus U, \end{cases}$$

and $0 \leq f\varphi \leq 1 \quad \text{on } M'.$

If we set $U' = \{y' \in M' \mid (f\varphi)(y') > 0\}$, U' is an open set which contains $\Phi(x)$ since $(f\varphi)(\Phi(x)) = 1$.

And we have $\Phi(U) \supset U'$ i.e. $U \supset \Phi^{-1}(U')$. This shows that Φ^{-1} is continuous. Similarly we can show that Φ is also continuous.

Next we shall prove that Φ is a diffeomorphism. To prove this, we show that $\varphi = (f\varphi)\Phi$ for any $\varphi \in C^\infty(M)$. For any $x \in M$, we denote by $S(r)$ the sphere of radius r with center x . Choose $\varphi_n \in C^\infty(M)$ ($n=1, 2, \dots$)

such that

$$\varphi_n(y) = \begin{cases} 1 & \text{on } S(1/n), \\ 0 & \text{on } M \setminus S(2/n), \end{cases}$$

and

$$0 \leq \varphi_n \leq 1 \quad \text{on } M.$$

Then we have

$$\varphi_n(y) = f\varphi_n(\Phi(y)) = \begin{cases} 1 & \text{on } S(1/n), \\ 0 & \text{on } M \setminus S(2/n), \end{cases}$$

and

$$0 \leq f\varphi_n \leq 1 \quad \text{on } M'.$$

And

$$\begin{aligned} \int_M \varphi \varphi_n dx &= \int_{M'} f\varphi \cdot f\varphi_n dx', \\ \int_M \varphi_n dx &= \int_{M'} f\varphi_n dx', \\ \varphi_n &\geq 0, \quad f\varphi_n \geq 0. \end{aligned}$$

Using the mean value theorem, we have

$$\int_M \varphi \varphi_n dx = \varphi(x'_n) \int_M \varphi_n dx \quad (x'_n \in S(2/n))$$

and

$$\int_{M'} f\varphi \cdot f\varphi_n dx' = f\varphi(\Phi(x''_n)) \int_{M'} f\varphi_n dx' \quad (x''_n \in S(2/n)).$$

Hence, we have

$$\varphi(x'_n) = \frac{\int_M \varphi \varphi_n dx}{\int_M \varphi_n dx} = \frac{\int_{M'} f\varphi \cdot f\varphi_n dx'}{\int_{M'} f\varphi_n dx'} = f\varphi(\Phi(x''_n))$$

and $\varphi(x) = f\varphi(\varnothing(x))$ as $n \rightarrow \infty$, for $\varphi, f\varphi\varnothing$ are continuous functions and x'_n, x''_n converge to x . Similarly, it is shown that $\varphi' = (f^{-1}\varphi')\varnothing^{-1}$ for any $\varphi' \in C^\infty(M')$. Therefore $\varnothing$ is a diffeomorphism. By virtue of this, we can assume that $M = M'$ as C^∞ -manifold and there are given two Riemannian structures g and g' on M .

$\int_M \varphi \psi dx = \int_M \varphi \psi dx'$ for any $\varphi, \psi \in C^\infty(M)$. Since we can consider that φ, ψ are C^∞ -functions on a coordinate neighborhood U , this implies $\sqrt{g} = \sqrt{g'}$ on U .

Also $\int_M \langle d\varphi, d\psi \rangle dx = \int_M \langle d\varphi, d\psi \rangle' dx'$ for any $\varphi, \psi \in C^\infty(M)$.

The local expression of $\langle d\varphi, d\psi \rangle$ is $g^{ij} \frac{\partial \varphi}{\partial x^i} \frac{\partial \psi}{\partial x^j}$.

Hence we have $g^{ij} \sqrt{g} = g'^{ij} \sqrt{g'}$ on U . Consequently we have $g^{ij} = g'^{ij}$ on U . And so $\varnothing$ is an isometry.

References

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Appendix 4-1. Automatic M-S translation program

544	TUC,	X,FMSTR	.		639	EOR,	STGWS	.	
545	ADD,	STGWD	.		640	SUB,	SPACE1	.	
546	SUB,	DT	+	7.	641	JUZ,	READ	+	1.
547	JUZ,	STJ2	.		642	TUC,	UA.	.	
548	TUC,	UA.	.		643	EOR,	STGWS	.	
549	ADD,	STGWD	.		644	SUB,	+	73D	
550	ADD,	DT	.		645	JUZ,	READ	+	1.
551	TUC,	STGWD	.		646	SUB,	DT	.	
552	JMP,	STRING	+	32.	647	JUZ,	READ	+	1.
553	TUC,	UA.	.		648	TUC,	UA.	.	
554	ADD,	STLN	.		649	BAD,	+777777777700D		
555	TUC,	FMSTR	+	30.	650	BAD,	+888888888800D		
556	CAL,	ALLOUT	.		651	BSR,	48.	.	
557	ADD,	FMSTR	+	30.	652	EOR,	STGWS	.	
558	TUC,	STLN	.		653	LOI,	DT	+	12.
559	ADD,	FMSTR	+	20.	654	SEQ,	TABL	.	
560	TUC,	FMSTR	.		655	JIN,	READ	+	18.
561	JMP,	STRING	+	32.	656	LOI,	DT	+	8.
562	STJ2 :	LOI,	UA.		657	SEQ,	TABL	+	36.
563	TUC,	UA.	.		658	JIN,	READ	+	21.
564	EOR,	STGWS	.		659	TUC,	+	UA.	
565	SHL,	X,	.		660	ADD,	DT	+	1.
566	EOR,	STWS	.		661	TUC,	LETIND	.	
567	TUC,	STWS	.		662	NDE,	.	.	
568	ADD,	STGCUT	.		663	JMP,	READ	+	23.
569	ADD,	STLN	.		664	TUC,	UA.	.	
570	TUC,	STLN	.		665	ADD,	DT	.	
571	ADD,	SPACE6	.		666	TUC,	LETIND	.	
572	LOI,	STLN	.		667	EOR,	X,	36.	
573	SHR,	X,	.		668	TUC,	STGWS	.	
574	EOR,	STWS	.		669	JMP,	READ	+	23.
575	LOI,	STGWD	.		670	TUC,	UA.	.	
576	TUC,	X,FMSTR	.		671	ADD,	-	10	
577	NDE,	.	.		672	TUC,	LETIND	.	
578	TUC,	UA.	.		673	TUC,	UA.	.	
579	LOI,	UA.	.		674	ADD,	STGCUT	.	
580	JRS,	.	.		675	ADD,	DT	.	
581	JMP,	STRING	.		676	TUC,	STGCUT	.	
582	JMP,	ALLOUT	+	1.	677	TUC,	SPACE	.	
583	JMP,	.	.		678	LOI,	UA.	.	
584	TUC,	UA.	.		679	JRS,	.	.	
585	ADD,	DT	+	22.	680	JMP,	READ	.	
586	LOI,	DT	+	7.	681	LO	(	700	)
587	TBL,	FMSTR	.		682	STKPT :	+	.	D
588	LOI,	.	.	D	683	LETIND:	+	.	D
589	SEL,	32.	.		684	STGWS :	+	.	D
590	LPR,	9000.	.		685	STGCUT:	+	.	D
591	TUC,	UA.	.		686	WS :	+	.	D
					687	SPACE :	+	.	D
592	ADD,	SPACE6	.		688	DT :	+	.	1D
593	LOI,	DT	+	7.	689	.	+	.	2D
594	SBI,	1.	.		690	.	+	.	6D
595	TUA,	X,FMSTR	.		691	.	+	.	10D
596	JIN,	ALLOUT	+	6.	692	.	+	.	12D
597	TUC,	UA.	.		693	.	+	.	18D
598	TUC,	STGWD	.		694	.	+	.	19D
599	TUC,	STLN	.		695	.	+	.	20D
600	JRS,	.	.		696	.	+	.	26D
601	JMP,	ALLOUT	.		697	.	+	.	27D
602	JMP,	PSDN	+	1.	698	.	+	.	28D
603	JMP,	.	.		699	BLANK :	+	.	30D
604	LOI,	STKPT	.		700	.	+	.	36D
605	TUC,	X,STACK	.		701	FTM :	+000000=	.	
606	ADD,	STKPT	.		702	RPART :	+000000	.	
607	ADD,	DT	.		703	COMMA :	+000000	.	
608	TUC,	STKPT	.		704	STAR :	+000000	.	
609	JRS,	.	.		705	LPART :	+000000	.	
610	JMP,	PSDN	.		706	SPACE1:	+	.	72D
611	POPUP :	JMP,	POPUP	+	707	LBKRT :	+000000	.	
612	JMP,	.	.	2.	708	RBKRT :	+000000	.	
613	TOPSTK:	+	.	D	709	.	+	.	100D
614	TUC,	UA.	.		710	.	+	.	120D
615	LOI,	STKPT	.		711	.	+	.	3000D
616	SBI,	1.	.		712	DOBPAR:	+000000	.	
617	TRI,	STKPT	.		713	RPSLP :	+0000	.	
618	TUC,	X,STACK	.		714	SPACE6:	+303030303030D	.	
619	SBI,	1.	.		715	ARROW :	+000000	.	
620	ADD,	X,STACK	.		716	STARTM:	+000000	.	
621	TUC,	TOPSTK	.		717	ENOM :	+000000	.	
622	LOI,	UA.	.		718	LANDM :	+000000	.	
623	JRS,	.	.		719	HEXP :	+	.	10D
624	JMP,	POPUP	.		720	FUNCT :	+	.	50D
625	JMPTOP:	TUC,	UA.		721	LAMDA :	+	.	100D
626	LOI,	TOPSTK	.		722	LABL :	+	.	150D
627	JMP,	X, 1000.	.		723	IDENT :	+	.	200D
628	READ :	JMP,	READ	+	724	FORMS :	+	.	230D
629	JMP,	.	.	1.	725	VARLIS:	+	.	300D
630	TUC,	UA.	.		726	IDENTI:	+	.	350D
631	TUC,	STGCUT	.		727	FUCNOT:	+	.	400D
632	NDE,	.	.		728	VARIL :	+	.	450D
633	SEL,	50.	.		729	LABELM:	+000000/	.	
634	RCH,	1.	.		730	COND :	+	.	500D
635	TUC,	STGWS	.		731	CONS :	+	.	550D
636	ADD,	SPACE	.		732	VARI :	+	.	600D
637	JUM,	READ	+	7.	733	TABL :	+000000	.	
638	TUC,	UA.	.		734	.	+000000	.	
					735	.	+000000	.	
					736	.	+000000	.	
					737	.	+000000	.	

```

738      *00000f
739      *00000a
740      *00000h
741      *00000i
742      *00000j
743      *00000k
744      *00000l
745      *00000m
746      *00000n
747      *00000o
748      *00000p
749      *00000q
750      *00000r
751      *00000s
752      *00000t
753      *00000u
754      *00000v
755      *00000w
756      *00000x
757      *00000y
758      *00000z
759      *000000
760      *000001
761      *000002
762      *000003
763      *000004
764      *000005
765      *000006
766      *000007
767      *000008
768      *000009
769      *00000A
770      *00000B
771      *00000C
772      *00000D
773      *00000E
774      *00000F
775      *00000G
776      *00000H
777      *00000I
778      *00000J
779      *00000K
780      *00000L
781      *00000M
782      *00000N
783      *00000O
784      *00000P
785      *00000Q
786      *00000R
787      *00000S

```

```

838      +      D
839      +      D
840      +      D
841      +      D
842      +      D
843      +      D
844      +      D
845      +      D
846      +      D
847      +      D
848      +      D
849      +      D
850      +      D
851      +      D
852      +      D
853      +      D
854      +      D
855      +      D
856      +      D
857      +      D
858      +      D
859      +      D
860      +      D
861      +      D
862      +      D
863      +      D
864      +      D
865      +      D
866      +      D
867      +      D
868      +      D
869      +      D
870      +      D
871      +      D
872      +      D
873      +      D
874      +      D
875      +      D
876      +      D
877      +      D
878      +      D
879      +      D
880      +      D
881      +      D
882      +      D
883      +      D
884      +      D
885      +      D
886      +      D
887      +      D

```

```

788      *00000T
789      *00000U
790      *00000V
791      *00000W
792      *00000X
793      *00000Y
794      *00000Z
795      *000000
796      *000001
797      *000002
798      *000003
799      *000004
800      *000005
801      *000006
802      *000007
803      *000008
804      *000009
805      (      300      )
806      STACK I      +      D
807      +      D
808      +      D
809      +      D
810      +      D
811      +      D
812      +      D
813      +      D
814      +      D
815      +      D
816      +      D
817      +      D
818      +      D
819      +      D
820      +      D
821      +      D
822      +      D
823      +      D
824      +      D
825      +      D
826      +      D
827      +      D
828      +      D
829      +      D
830      +      D
831      +      D
832      +      D
833      +      D
834      +      D
835      +      D
836      +      D
837      +      D

```

```

888      +      D
889      +      D
890      +      D
891      +      D
892      +      D
893      +      D
894      +      D
895      +      D
896      +      D
897      +      D
898      +      D
899      +      D
900      +      D
901      +      D
902      +      D
903      +      D
904      +      D
905      +      D
906      +      D
907      +      D
908      +      D
909      +      D
910      +      D
911      +      D
912      +      D
913      +      D
914      +      D
915      +      D
916      +      D
917      +      D
918      +      D
919      +      D
920      +      D
921      +      D
922      +      D
923      +      D
924      +      D
925      +      D
926      +      D
927      +      D
928      +      D
929      +      D
930      +      D
931      +      D

```

FHSTR :

START (200)

Appendix 5. Instruction table of OKITAC 5090-C used in the program

Code	Instruction	Remarks
TUC , n.	Transfer UA Clear	$[UA] \rightarrow n + x, 0 \rightarrow UA \sim LA$
ADD , n.	Add	$[UA] + [n + x] \rightarrow UA$
SUB , n.	Subtract	$[UA] - [n + x] \rightarrow UA$
BAD , n.	Binary add	$[UA]_b + [n + x]_b \rightarrow UA$
SHR , N.	Shift Right	$[UA \sim LA] \times 10^{-(N+x)} \rightarrow UA \sim LA$
BSR , N.	Binary Shift Right	$[UA \sim LA] \times 2^{-(N+x)} \rightarrow UA \sim LA$
EOR , n.	Exclusive or	$[UA]_i \oplus [n + x]_i \rightarrow UA_i$
SEQ , n.	Search equal	$u + x \rightarrow x \text{ words}$
LOI , n.	Load index	$[n + x] \rightarrow X$
SBI , N.	Subtract index	$X - N \rightarrow X$
JIN , n.	Jump if index non zero.	
JUZ , n.	Jump if UA zero.	
JUM , n.	Jump if UA minus	
JAH , n.	Jump and Halt	
JRS , n.	Jump right set	
ÖCH , N.	Out character	
SEL , N.	Select	
LPR , n.	Line print	
CAL , n.	Subroutine call	
NOE , .	Non effect	