CONTINUATION AND QUOTIENT REPRESENTATION OF MEROMORPHIC FUNCTIONS

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Introduction.

In [6] Kajiwar-Sakai proved that a domain of meromorphy of a meromorphic function f in a domain Ω over a Stein manifold is a Stein manifold and that f can be represented as a quotient g/h of holomorphic functions g and h in Ω . In [5] Kajiwar proved that either a domain of meromorphy of a meromorphic function f in a domain Ω over the complex projective n -space P_n coincides with P_n or is a Stein manifold and that either f can be meromorphically continued to P_n or f can be represented as a quotient g/h of holomorphic functions g and h in Ω . The main purpose of the present paper is to treat the problem of continuation and quotient representation of a meromorphic function in a domain over a product space $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_s} \times X$ of complex projective spaces $P_{n_1}, P_{n_2}, \cdots, P_{n_s}$ and a Stein manifold X .

§1. Envelope of meromorphy.

Let M be a complex manifold. If φ is a locally biholomorphic mapping of a complex manifold Ω in M , (Ω, φ) is called an *open set over M* . (Ω, φ) is said to be *pseudoconvex* if $\varphi^{-1}(V)$ is a Stein manifold for any Stein subdomain V of M . If Ω is connected, (Ω, φ) is called a *domain over M* . If φ is a biholomorphic mapping of Ω in M , (Ω, φ) is called a *univalent open set over M* and is identified with the open subset $\varphi(\Omega)$ of M . Let (Ω, φ) and (Ω', φ') be open sets over M . A holomorphic mapping λ of Ω in Ω' with $\varphi = \varphi' \circ \lambda$ is called a *mapping of (Ω, φ) in (Ω', φ')* . In this case (Ω, λ) is an open set over Ω' . If λ is a biholomorphic mapping of Ω onto Ω' , (Ω, φ) and (Ω', φ') are identified.

Consider domains (Ω, φ) and (Ω', φ') over M with a mapping λ of (Ω, φ) in (Ω', φ') . Let f be a meromorphic function in Ω . A meromorphic function f' in Ω' with $f = f' \circ \lambda$ is called a *meromorphic continuation* of f to $(\lambda, \Omega', \varphi')$, shortly to (Ω', φ') or more shortly to Ω' . Let $\mathcal{F}$ be a family of meromorphic functions in Ω . If any meromorphic function of $\mathcal{F}$ has a meromorphic continuation to $(\lambda, \Omega', \varphi')$, $(\lambda, \Omega', \varphi')$ or shortly (Ω', φ') is called a *meromorphic completion* of (Ω, φ) with respect to $\mathcal{F}$. A meromorphic completion $(\tilde{\lambda}_{\mathcal{F}}, \tilde{\Omega}_{\mathcal{F}}, \tilde{\varphi}_{\mathcal{F}})$ of (Ω, φ) with respect to $\mathcal{F}$, or shortly $(\tilde{\Omega}_{\mathcal{F}}, \tilde{\varphi}_{\mathcal{F}})$ is called an *envelope of meromorphy* of (Ω, φ) with respect to $\mathcal{F}$ if the following conditions are satisfied:

Let $(\lambda', \Omega', \varphi')$ be another meromorphic completion of (Ω, φ) with respect to $\mathcal{F}$ and $\mathcal{F}'$ be the set of meromorphic continuations of all meromorphic functions of $\mathcal{F}$ to $(\lambda', \Omega', \varphi')$. Then there exists a mapping ψ of (Ω', φ') in $(\tilde{\Omega}_{\mathcal{F}}, \tilde{\varphi}_{\mathcal{F}})$ with $\tilde{\lambda}_{\mathcal{F}} = \psi \circ \lambda'$ such that $(\psi, \tilde{\Omega}_{\mathcal{F}}, \tilde{\varphi}_{\mathcal{F}})$ is a meromorphic completion of (Ω', φ') with respect to $\mathcal{F}'$.

If $\mathcal{F}$ is the family of all meromorphic functions in Ω , an envelope of meromorphy of (Ω, φ) with respect to $\mathcal{F}$ is called shortly an *envelope of meromorphy* of (Ω, φ) . If $\mathcal{F}$ consists of only a meromorphic function f in Ω , an envelope of meromorphy of (Ω, φ) with respect to $\mathcal{F}$ is called shortly a *domain of meromorphy* of f .

A complex manifold X is called a manifold of *weak Poincaré type* if any meromorphic function f in X can be represented as a quotient g/h of holomorphic functions g and h in X . The following Propositions are given in [5] and [6].

PROPOSITION 1. *Let (Ω, φ) be a domain over a complex manifold M and $\mathcal{F}$ be a family of meromorphic functions in Ω . Then there exists uniquely an envelope of meromorphy of (Ω, φ) with respect to $\mathcal{F}$.*

PROPOSITION 2. *Let (Ω, φ) be a domain over a Stein manifold, $\mathcal{F}$ be a family of meromorphic functions in Ω and $(\tilde{\lambda}_{\mathcal{F}}, \tilde{\Omega}_{\mathcal{F}}, \tilde{\varphi}_{\mathcal{F}})$ be the envelope of meromorphy of (Ω, φ) with respect to $\mathcal{F}$. Then $\tilde{\Omega}_{\mathcal{F}}$ is a Stein manifold.*

PROPOSITION 3. *Let (Ω, φ) be a domain over a Stein manifold. Then Ω is a complex manifold of weak Poincaré type.*

PROPOSITION 4. *Let (Ω, φ) be a domain over P_n , $\mathcal{F}$ be a family of meromorphic functions in Ω and $(\tilde{\lambda}_{\mathcal{F}}, \tilde{\Omega}_{\mathcal{F}}, \tilde{\varphi}_{\mathcal{F}})$ be the envelope of mero-*

morphism of (Ω, φ) with respect to $\mathcal{F}$. If $\tilde{\varphi}_{\mathcal{F}}$ is not a biholomorphic mapping of $\tilde{\Omega}_{\mathcal{F}}$ onto P_n , $\tilde{\Omega}_{\mathcal{F}}$ is a Stein manifold and any meromorphic function f of $\mathcal{F}$ can be represented as a quotient g/h of holomorphic functions g and h in Ω .

PROPOSITION 5. *Let (Ω, φ) be a domain over P_n . Then the envelope of meromorphy of (Ω, φ) coincides with the envelope of holomorphy of (Ω, φ) .*

§2. Preliminaries.

LEMMA 6. *Let f and g be holomorphic mappings of a connected complex manifold X in a complex manifold Y such that $f=g$ in a non-empty subdomain X_0 of X . Then $f=g$ in X .*

PROOF. Let E be the set of all point x of X such that $f=g$ in a neighbourhood of x . E is a non-empty open subset of X . Let x_0 be a point of the closure of E . Let U be a neighbourhood of x_0 such that a biholomorphic mapping τ maps U onto a polydisc U' in a complex euclidean space. Suppose that $f(x_0) \neq g(x_0)$. There exist neighbourhood V_1 and V_2 of $f(x_0)$ and $g(x_0)$ such that $V_1 \cap V_2 = \emptyset$. We put $U_0 = f^{-1}(V_1) \cap g^{-1}(V_2) \cap U$. U_0 is a neighbourhood of x_0 . There exists a point x of $E \cap U_0$. Then we have $f(x) = g(x)$. Since $x \in U_0$, we have $f(x) \in V_1$ and $g(x) \in V_2$. But this is a contradiction. Hence we have $f(x_0) = g(x_0)$. There exists a biholomorphic mapping σ of a neighbourhood V of $f(x_0) = g(x_0)$ onto an open polydisc V' in a complex euclidean space. $\sigma \circ f \circ \tau^{-1}$ and $\sigma \circ g \circ \tau^{-1}$ are holomorphic mapping of a polydisc U' in a polydisc V' such that $\sigma \circ f \circ \tau^{-1} = \sigma \circ g \circ \tau^{-1}$ in an open subset of U' . From the theory of identity we have $\sigma \circ f \circ \tau^{-1} = \sigma \circ g \circ \tau^{-1}$ in U' . Hence we have $f=g$ in U . Thus we have proved that E is closed. Since E is a non-empty open and closed subset of a connected space X , we have $E=X$.

LEMMA 7. *Let P be a compact connected complex manifold and X be a connected complex manifold. Let (Ω, φ) be a domain over $Y=P \times X$. Let Q be a submanifold of Ω such that $\varphi|_Q$ is a biholomorphic mapping of Q onto $P \times \{x\}$ for a point $x \in X$. Then, there exist a neighbourhood U of Q in Ω and a neighbourhood W of x in X such that $\varphi|_U$ is a biholomorphic mapping of U onto $P \times W$.*

PROOF. Let $\eta: Y \rightarrow P$ and $\xi: Y \rightarrow X$ be the canonical projections. For any point ω of Q , there exist a neighbourhood $U'(\omega)$ of ω in Ω , a neighbourhood $V(\omega)$ of $\eta(\varphi(\omega))$ in P and a neighbourhood $W(\omega)$ of $\xi(\varphi(\omega))$ in X such that $\varphi|U'(\omega)$ is a biholomorphic mapping of $U'(\omega)$ onto $V(\omega) \times W(\omega)$. Since Q is compact, there exists a finite number of point $\omega_1, \dots, \omega_m$ in Q such that $\bigcup_{j=1}^m U'(\omega_j) \supset Q$. Let W be a connected neighbourhood of x contained in $\bigcap_{j=1}^m W(\omega_j)$. We put

$$U(\omega_j) = U'(\omega_j) \cap \varphi^{-1}(\xi^{-1}(W))$$

for $j=1, 2, \dots, m$. Then $\varphi|U(\omega_j)$ is a biholomorphic mapping of $U(\omega_j)$ onto $V(\omega_j) \times W$ for $j=1, 2, \dots, m$. If $V(\omega_j) \cap V(\omega_k) \neq \emptyset$, then we have

$$(\varphi|U(\omega_j))^{-1} = (\varphi|U(\omega_k))^{-1}$$

in a neighbourhood of $(V(\omega_j) \cap V(\omega_k)) \times \{x\}$. From Lemma 6 we have

$$(\varphi|U(\omega_j))^{-1} = (\varphi|U(\omega_k))^{-1}$$

in $(V(\omega_j) \cap V(\omega_k)) \times W$. If we put

$$f = (\varphi|U(\omega_j))^{-1}$$

in $V(\omega_j) \times W$ for $j=1, 2, \dots, m$, then f is a well-defined holomorphic mapping of $P \times W$ onto $U = \bigcup_{j=1}^m U(\omega_j)$ such that $\varphi \circ f$ is the identity of $P \times W$.

LEMMA 8. *Let P_1 and P_2 be compact connected complex manifolds and X be a connected complex manifold. Let (Ω, φ) be a domain over $Y = P_1 \times P_2 \times X$. Let Q_1 be a submanifold of Ω such that $\varphi|Q_1$ is a biholomorphic mapping of Q_1 onto $P_1 \times \{p_2\} \times \{x\}$ for a point p_2 of P_2 and a point x of X and that for any point ω of Q_1 there exists a submanifold $Q_2(\omega)$ of Ω containing ω such that $\varphi|Q_2(\omega)$ is a biholomorphic mapping of $Q_2(\omega)$ onto $\{\eta_1(\varphi(\omega))\} \times P_2 \times \{x\}$ where $\eta_1: Y \rightarrow P_1$, $\eta_2: Y \rightarrow P_2$, $\eta: Y \rightarrow P = P_1 \times P_2$, $\xi: Y \rightarrow X$ are the canonical projections. Then there exists a submanifold Q of Ω such that $\varphi|Q$ is a biholomorphic mapping of Q onto $P \times \{x\}$.*

PROOF. Let ω be any point of Q_1 . There exists a neighbourhood

$U'(\omega)$ of $Q_2(\omega)$, a connected neighbourhood $V_1(\omega)$ of $\eta_1(\varphi(\omega))$ in P_1 and a neighbourhood $W_1(\omega)$ of $\xi(\omega)$ in X such that $\varphi|U'(\omega)$ is a biholomorphic mapping of $U'(\omega)$ onto $V_1(\omega) \times P_2 \times W(\omega)$. Since Q_1 is compact, there exists a finite number of points $\omega_1, \dots, \omega_m$ of Q_1 such that $\bigcup_{j=1}^m U'(\omega_j) \supset Q_1$. Let W be a connected neighbourhood of x contained in $\bigcap_{j=1}^m W(\omega_j)$. We put

$$U(\omega_j) = U'(\omega_j) \cap \varphi^{-1}(V_1(\omega_j) \times P_2 \times W)$$

for $j=1, 2, \dots, m$. If we put

$$f = (\varphi|U(\omega_j))^{-1}$$

in $V_1(\omega_j) \times P_2 \times W$ for $j=1, 2, \dots, m$, then f is a well-defined holomorphic mapping of $P_1 \times P_2 \times W$ onto $U = \bigcup_{j=1}^m U(\omega_j)$ such that $\varphi \circ f$ is the identity of $P_1 \times P_2 \times W$.

LEMMA 9. *Let P_m be the complex projective space of dimension m . Let a and b be positive numbers with $a < b$. Let (Ω, φ) be a pseudoconvex domain over $P_m \times \mathbb{C}^n$ such that φ maps an open subset E of Ω biholomorphically onto*

$$D = P_m \times W_a \cup V \times W_b$$

where

$$W_a = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n; |z_j| < a \ (j=1, 2, \dots, n)\}$$

and V is a non-empty subdomain of P_m . Then there exists a subdomain F of Ω such that φ maps F biholomorphically onto $G = P_m \times W_b$.

PROOF. Let $p = (p_1, p_2, \dots, p_{m+1})$ be the homogeneous coordinates of P_m . We put

$$H_j = \{p \in P_m; p_j \neq 0\} \quad \text{for } j=1, 2, \dots, m+1.$$

Then $\{H_j \times \mathbb{C}^n; j=1, 2, \dots, m+1\}$ is a Stein covering of $P_m \times \mathbb{C}^n$. H_j is equivalent to $\mathbb{C}^m$. Since H_j is dense in P_m , we have $H_j \cap V \neq \emptyset$. We put

$$\begin{aligned}\Omega_j &= \varphi^{-1}(H_j \times \mathbf{C}^n), \quad \varphi_j = \varphi|_{\Omega_j}, \quad E_j = E \cap \Omega_j, \quad D_j = D \cap (H_j \times \mathbf{C}^n), \\ G_j &= G \cap (H_j \times \mathbf{C}^n) \quad \text{for } j=1, 2, \dots, m+1.\end{aligned}$$

Then (Ω_j, φ_j) is a pseudoconvex domain over $H_j \times \mathbf{C}^n = \mathbf{C}^{m+n}$ such that φ_j maps E_j biholomorphically onto D_j . Hence there exists a subdomain F_j of Ω containing E_j such that φ_j maps F_j biholomorphically onto G_j . If we put

$$f = (\varphi|_{F_j})^{-1}$$

in G_j for $j=1, 2, \dots, m+1$, then f is a well-defined holomorphic mapping of $G = \bigcup_{j=1}^{m+1} G_j$ onto $F = \bigcup_{j=1}^{m+1} F_j$ such that $\varphi \circ f$ is the identity of G .

Let α be a mapping of A in D and β be a mapping of B in E . We define a mapping $\alpha \times \beta$ of $A \times B$ in $D \times E$ by putting

$$(\alpha \times \beta)(x, y) = (\alpha(x), \beta(y)) \quad \text{for } (x, y) \in A \times B.$$

LEMMA. 10. *Let X and Y be complex manifolds. Let (Ω, φ) and (Δ, ψ) be, respectively, domains over $X \times Y$ and X such that there exists a biholomorphic mapping λ of Ω onto $\Delta \times Y$ with $\varphi = (\psi \times 1) \circ \lambda$. If (Ω, φ) is a pseudoconvex domain over $X \times Y$, (Δ, ψ) is a pseudoconvex domain over X .*

PROOF. Let U be a Stein subdomain of X and V be a Stein subdomain of Y . Since (Ω, φ) is a pseudoconvex domain over $X \times Y$, $\varphi^{-1}(U \times V)$ is a Stein manifold. Hence $\lambda(\varphi^{-1}(U \times V)) = (\psi \times 1)^{-1}(U \times V) = \psi^{-1}(U) \times V$ is a Stein manifold. Since $\psi^{-1}(U)$ is a submanifold of a Stein manifold $\psi^{-1}(U) \times V$, $\psi^{-1}(U)$ is a Stein manifold. Hence (Δ, ψ) is a pseudoconvex domain over X .

§3. Main results.

For a positive integer m , let P_m be the complex projective space of dimension m . Let n be a positive integer. Let $m_1, m_2, \dots$ and m_n be positive integers. Let

$$P = P_{m_1} \times P_{m_2} \times \dots \times P_{m_n}$$

be the product space of n complex projective spaces. Let X be a con-

nected Stein manifold. Consider the product space

$$Y = P \times X.$$

Let ξ be a projection of Y onto X , η_j be the projection of Y onto P_{m_j} and η be the projection of Y onto P . Let (Ω, φ) be a domain over Y . A submanifold Q of Ω is called a *univalent set equivalent to P_{m_j}* if $\varphi|_Q$ is a biholomorphic mapping of Q onto a submanifold

$$\{p_1\} \times \{p_2\} \times \cdots \times \{p_{j-1}\} \times P_{m_j} \times \{p_{j+1}\} \times \cdots \times \{p_n\} \times \{x\}$$

of Y of dimension m_j for some $p_k \in P_k$ ($k \neq j$) and $x \in X$. The following theorem is due to Docquier-Grauert [1] and Fujita [3].

THEOREM 11. *Let (Ω, φ) be a pseudoconvex domain over Y such that Ω does not contain a univalent set equivalent to P_{m_j} for $j=1, 2, \dots, n$. Then Ω is a Stein manifold.*

PROOF. From Docquier-Grauert [1], there exist a domain of holomorphy (G, ψ) over $\mathbb{C}^N$, a nonsingular analytic set H in G , a biholomorphic mapping λ of X onto H and a holomorphic mapping ρ of G onto H such that $\rho|_H$ is the identity mapping of H and that each fibre $\rho^{-1}(x)$ of ρ is a nonsingular analytic set for $x \in X$. We put

$$\Omega^* = \{(\omega, g) \in \Omega \times G; \lambda(\xi(\varphi(\omega))) = \rho(g)\}.$$

We put

$$\Phi(\omega, g) = (\eta(\varphi(\omega)), g)$$

for $(\omega, g) \in \Omega^*$. Then Φ is a mapping of Ω^* in $P \times G$. Let (ω, g) be any point of Ω^* . There exist, respectively, open neighbourhoods W , U and V of ω , $\eta(\omega)$ and $\xi(\omega)$ such that $\varphi|_W$ is a biholomorphic mapping of W onto $U \times V$. We put

$$W^* = \{(\omega, g) \in \Omega \times G; \eta(\varphi(\omega)) \in U, \rho(g) \in \lambda(V)\}.$$

Then $\Phi|_{W^*}$ is a bijective mapping of W^* onto $U \times \rho^{-1}(\lambda(V))$. We can induce a topology in Ω^* such that Φ is a local homeomorphism of the Hausdorff space Ω^* in $P \times G$. Then (Ω^*, Φ) is a domain over $P \times G$. Let

U and V be, respectively, Stein subdomains of P and X . Since Ω is pseudoconvex, $\varphi^{-1}(U \times V)$ is a Stein manifold. There exists a positive real analytic strongly plurisubharmonic function d_1 on $\varphi^{-1}(U \times V)$ such that $\{\omega \in \varphi^{-1}(U \times V); d_1(\omega) \leq a\}$ is a compact subset of $\varphi^{-1}(U \times V)$ for any real number a . Since V is a Stein manifold, $\rho^{-1}(\lambda(V))$ is a Stein manifold in G . There exists a positive real analytic strongly plurisubharmonic function d_2 on $\rho^{-1}(\lambda(V))$ such that $\{g \in \rho^{-1}(\lambda(V)); d_2(g) \leq a\}$ is a compact subset of $\rho^{-1}(\lambda(V))$ for any real number a . We put

$$d(\omega, g) = d_1(\omega) + d_2(g)$$

for $(\omega, g) \in \Omega^*$. Then d is a positive real analytic strongly plurisubharmonic function in $\Phi^{-1}(U \times \rho^{-1}(\lambda(V)))$ such that $\{(\omega, g) \in \Phi^{-1}(U \times \rho^{-1}(\lambda(V))); d(\omega, g) \leq a\}$ is compact for any real number a . Since the set of such domains $U \times \rho^{-1}(\lambda(V))$ forms a Stein covering of $P \times G$, (Ω^*, Φ) is a pseudoconvex domain over $P \times G$.

$\mathbb{C}^N$ is considered as a subdomain of P_N . Let μ be the canonical injection of $\mathbb{C}^N$ in P_N . We put

$$\Phi^*(\omega, g) = (\eta(\varphi(\omega)), \mu(\psi(g)))$$

for $(\omega, g) \in \Omega^*$. Then (Ω^*, Φ^*) is a pseudoconvex domain over the product space $P \times P_N$ of $(n+1)$ complex projective spaces such that Ω^* does not contain a univalent set equivalent to P_{m_j} ($j=1, 2, \dots, n$) or P_N . From Fujita [3], Ω^* is a Stein manifold. Then Ω is a Stein manifold.

LEMMA 12. *Let (Ω, φ) be a domain over Y . Let $\mathcal{F}$ be a family of meromorphic functions in Ω and $(\tilde{\lambda}_{\mathcal{F}}, \tilde{\Omega}_{\mathcal{F}}, \tilde{\varphi}_{\mathcal{F}})$ be the envelope of meromorphy of (Ω, φ) with respect to $\mathcal{F}$. If $\tilde{\Omega}_{\mathcal{F}}$ contains a univalent set equivalent to P_{m_j} , any point of $\tilde{\Omega}_{\mathcal{F}}$ is contained in a univalent set equivalent to P_{m_j} .*

PROOF. We may assume that $j=1$. We put $Z = P_{m_2} \times P_{m_3} \times \dots \times P_{m_n} \times X$. Let E be the set of all point ω of $\tilde{\Omega}_{\mathcal{F}}$ such that ω is contained in a univalent set equivalent to P_{m_1} . From Lemma 7, E is a non-empty open subset of $\tilde{\Omega}_{\mathcal{F}}$. Let ω be a point of the closure of E . There exist, respectively, open neighbourhoods W , U and V of ω , $\eta_1(\tilde{\varphi}_{\mathcal{F}}(\omega))$ and $\eta'(\tilde{\varphi}_{\mathcal{F}}(\omega))$ such that $\tilde{\varphi}_{\mathcal{F}}|_W$ is a biholomorphic mapping of W onto $U \times V$ and U and V are coordinate neighbourhoods where η_1 is the projection of Y onto P_{m_1} and

η' is the projection of Y onto Z . There exist a point $z \in V$ and a univalent set Q equivalent to P_{m_1} such that $\varphi_{\mathcal{F}}|Q$ is a biholomorphic mapping of Q onto $P_{m_1} \times \{z\}$. From Lemma 7, there exist, respectively, open neighbourhoods $Q^\#$ and V_0 of Q and z such that $\varphi_{\mathcal{F}}|Q^\#$ is a biholomorphic mapping of $Q^\#$ onto $P_{m_1} \times V_0$. We may assume that there exists a biholomorphic mapping μ of V onto a polydisc V' such that $\mu(V_0)$ and V' is a polydisc with centre the origin. Let f be any meromorphic function of $\mathcal{F}$ and $\tilde{f}$ be its meromorphic continuation to $(\tilde{\lambda}_{\mathcal{F}}, \tilde{\Omega}_{\mathcal{F}}, \tilde{\varphi}_{\mathcal{F}})$. Then, from Lemma 9, $\tilde{f} \circ (\tilde{\varphi}|W)^{-1} \circ (1 \times \mu^{-1})$ is meromorphically continued to $P_{m_1} \times V$. Since $(\tilde{\lambda}_{\mathcal{F}}, \tilde{\Omega}_{\mathcal{F}}, \tilde{\varphi}_{\mathcal{F}})$ is the envelope of meromorphy of (Ω, φ) with respect to $\mathcal{F}$, there exists a biholomorphic mapping ν of $P_{m_1} \times V$ in $\tilde{\Omega}_{\mathcal{F}}$ such that $\tilde{\varphi}_{\mathcal{F}} \circ \nu$ is the identity of $P_{m_1} \times V$. Any point of $\nu(P_{m_1} \times V)$ is contained in a univalent set equivalent to P_{m_1} . Since $\nu(P_{m_1} \times V) \supset W$, $\omega \in E$. Since E is a non-empty open and closed set in the connected set $\tilde{\Omega}_{\mathcal{F}}$, we have $E = \tilde{\Omega}_{\mathcal{F}}$.

LEMMA 13. *Let (Ω, φ) be a domain over Y . Let $\mathcal{F}$ be a family of meromorphic functions in Ω and $(\tilde{\lambda}_{\mathcal{F}}, \tilde{\Omega}_{\mathcal{F}}, \tilde{\varphi}_{\mathcal{F}})$ be the envelope of meromorphy of (Ω, φ) with respect to $\mathcal{F}$. Assume that $\tilde{\Omega}_{\mathcal{F}}$ contains univalent sets equivalent to $P_{m_1}, P_{m_2}, \dots$ and P_{m_s} and $\tilde{\Omega}_{\mathcal{F}}$ does not contain univalent sets equivalent to P_{m_j} for $s+1 \leq j \leq n$. We put $P = P_{m_1} \times P_{m_2} \times \dots \times P_{m_s}$, $Z = P_{m_{s+1}} \times \dots \times P_{m_n} \times X$. Then there exist a Stein manifold (Δ, ψ) over Z and a biholomorphic mapping ζ of $\tilde{\Omega}$ onto $P \times \Delta$ such that $\tilde{\varphi}_{\mathcal{F}} = (1 \times \psi) \circ \zeta$.*

PROOF. Let ω be a point of $\tilde{\Omega}_{\mathcal{F}}$. We put $(p, z) = \tilde{\varphi}_{\mathcal{F}}(\omega)$. From Lemmas 8 and 12 $\tilde{\varphi}_{\mathcal{F}}^{-1}(P \times \{z\})$ is a covering manifold of a simply connected manifold $P \times \{z\}$. Hence each $\tilde{\varphi}_{\mathcal{F}}$ maps each connected component of $\tilde{\varphi}_{\mathcal{F}}^{-1}(P \times \{z\})$ biholomorphically onto $P \times \{z\}$. We shall induce in $\tilde{\Omega}_{\mathcal{F}}$ an equivalence relation R as follows: $\omega_1 \sim \omega_2$ if and only if ω_1 and ω_2 belong to the same connected component of $\tilde{\varphi}_{\mathcal{F}}^{-1}(P \times \{z\})$ for some $z \in Z$. Then $\Delta = \tilde{\Omega}_{\mathcal{F}}/R$ is a complex manifold such that (Δ, ψ) is a domain over Z where μ is the canonical mapping of $\tilde{\Omega}_{\mathcal{F}}$ onto Δ and ψ is the canonical mapping of Δ in Z such that $\xi \circ \tilde{\varphi}_{\mathcal{F}} = \psi \circ \mu$. Then the mapping ζ defined by

$$\zeta(\omega) = (\eta(\tilde{\varphi}_{\mathcal{F}}(\omega)), \mu(\omega))$$

is a biholomorphic mapping of $\tilde{\Omega}_{\mathcal{F}}$ onto $P \times \Delta$ such that $\tilde{\varphi} = (1 \times \psi) \circ \zeta$.

Since $\tilde{\Omega}_{\mathcal{F}}$ is pseudoconvex, Δ is a pseudoconvex domain over Z which does not contain a set equivalent to P_{m_j} for $s+1 \leq j \leq n$ from Lemma 10. Hence from Theorem 11 Δ is a Stein manifold.

From Lemma 13 we have easily

THEOREM 14. *Let (Ω, φ) be a domain over Y . Let $\mathcal{F}$ be a family of meromorphic functions in Ω and $(\tilde{\lambda}_{\mathcal{F}}, \tilde{\Omega}_{\mathcal{F}}, \tilde{\varphi}_{\mathcal{F}})$ be the envelope of meromorphy of (Ω, φ) with respect to $\mathcal{F}$. Then, after a suitable permutation of $(m_1, m_2, \dots, m_n)$ either $\tilde{\Omega}_{\mathcal{F}}$ is a Stein manifold or there exist an integer s with $1 \leq s \leq n$, a Stein manifold (Δ, ψ) over $Z = P_{m_{s+1}} \times \dots \times P_{m_n} \times X$ and a biholomorphic mapping ζ of $\tilde{\Omega}_{\mathcal{F}}$ onto $P \times \Delta$ such that $\tilde{\varphi} = (1 \times \psi) \circ \zeta$ where $P = P_{m_1} \times P_{m_2} \times \dots \times P_{m_s}$.*

COROLLARY 1. *Let (Ω, φ) be a domain over Y . If the envelope $(\tilde{\lambda}, \tilde{\Omega}, \tilde{\varphi})$ of meromorphy of (Ω, φ) does not contain a univalent set equivalent to P_{m_j} ($j=1, 2, \dots, n$), then Ω is a complex manifold of weak Poincaré type.*

PROOF. From Theorem 14, $\tilde{\Omega}$ is a Stein manifold. Let f be a meromorphic function in Ω and $\tilde{f}$ be its meromorphic continuation to $(\tilde{\lambda}, \tilde{\Omega}, \tilde{\varphi})$. There exist holomorphic functions $\tilde{g}$ and $\tilde{h}$ in $\tilde{\Omega}$ such that $\tilde{f} = \tilde{g}/\tilde{h}$ in $\tilde{\Omega}$ from Hitotumatu-Kôta [4]. Then we have $f = g/h$ for holomorphic functions $g = \tilde{g} \circ \tilde{\lambda}$ and $h = \tilde{h} \circ \tilde{\lambda}$ in Ω .

COROLLARY 2. *Let (Ω, φ) be a domain over Y . Then the envelope $(\tilde{\lambda}_{\mathcal{X}}, \tilde{V}_{\mathcal{X}}, \tilde{\varphi}_{\mathcal{X}})$ of meromorphy of (Ω, φ) coincides with the envelope $(\tilde{\lambda}_0, \tilde{V}_0, \tilde{\varphi}_0)$ of holomorphy of (Ω, φ) .*

PROOF. As in [5], $(\tilde{\lambda}_{\mathcal{X}}, \tilde{V}_{\mathcal{X}}, \tilde{\varphi}_{\mathcal{X}})$ is a holomorphic completion of (V, φ) . From Theorem 14, $(\tilde{\lambda}_{\mathcal{X}}, \tilde{V}_{\mathcal{X}}, \tilde{\varphi}_{\mathcal{X}})$ is a domain of holomorphy over Y . Hence $(\tilde{\lambda}_{\mathcal{X}}, \tilde{V}_{\mathcal{X}}, \tilde{\varphi}_{\mathcal{X}})$ is a domain of holomorphy of (V, φ) .

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