UNIQUENESS CLASSES FOR CAUCHY PROBLEM

By

Yukiyoshi EBIHARA

(Received, Oct. 30, 1970)

§ 1. Introduction.

We treat some applications of the theory of generalized functions to general uniqueness problems for linear partial differential equations.

In this paper we restrict our attention to the investigation of the Cauchy problem, for equations with coefficients which do not depend on the spatial variables. We consider systems of partial differential equations of the form

$$(1) \quad \frac{\partial u_j(x, t)}{\partial t} = \sum_{k=1}^m P_{j,k} \left(i \frac{\partial}{\partial x} \right) u_k(x, t), \quad (j = 1, 2, \dots, m)$$

where $P_{j,k}(s)$ are entire functions in s with constant coefficients with some definite conditions on their behavior at infinity. To this form one can reduce the formally more complicated systems,

$$(2) \quad \frac{\partial^{\alpha_j} u_j(x, t)}{\partial t^{\alpha_j}} = \sum_{k=1}^m P_{j,k} \left(\frac{\partial}{\partial t}, i \frac{\partial}{\partial x} \right) u_k(x, t)$$

where the order in $\frac{\partial}{\partial t}$ of the functions $P_{j,k}$ is smaller than α_j . The Cauchy problem for the system (1) consists in determining a solution which satisfies the initial conditions,

$$(3) \quad u_j(x, 0) = u_j(x) \quad (j = 1, 2, \dots, m)$$

where $u_j(x)$ are some given functions.

A uniqueness class is a linear topological space of functions of the argument x , for which the uniqueness of the solution of the Cauchy problem is guaranteed for given initial conditions, provided the solution

exists.

We shall make use of generalized functions defined on test function spaces of infinitely differentiable or analytic functions with definite conditions on their behavior at infinity. The method of Fourier transforms plays an essential role in this theory, so we shall make use of spaces of type $\mathcal{W}$.

In I.M. Gel'fand and G.E. SHILOV [1] there are presented various uniqueness classes for several equations of type (1) for the polynomial coefficients $P_{j,k}$ in s . In this paper we construct some uniqueness classes for the entire function coefficients.

§2. The connection between the solutions of the Cauchy problem in a given Space and in the dual Space.

Let A_t be a given linear continuous operator which depends continuously on the parameter t , and which maps the topological vector space $\mathcal{O}$ into itself for each t ($0 \leq t \leq T$). The adjoint operator A_t^* is defined on the dual space $\mathcal{O}'$ and maps this space into itself. Consider the differential equation

$$(2.1) \quad \frac{du(t)}{dt} = -A_t^* u(t), \quad 0 \leq t \leq T,$$

$$(2.2) \quad u(0) = u_0,$$

where $u(t)$ is an unknown element of the space $\mathcal{O}'$ and u_0 is an arbitrary given initial element of $\mathcal{O}'$.

The solution of the Cauchy problem (2.1) (2.2) is closely connected with the solution of an analogous Cauchy problem in the space $\mathcal{O}$, that is, the problem of finding the solution of the equation

$$(2.3) \quad \frac{d\phi(t)}{dt} = A_t \phi(t),$$

under the initial condition,

$$(2.4) \quad \phi(t_0) = \phi_0 \quad 0 \leq t_0 \leq T.$$

For simplicity, we shall call that the Cauchy problem (2.3), (2.4) is *always solvable* if for each t_0 : $0 \leq t_0 \leq T$ and any element ϕ_0 in $\mathcal{O}$, there

exists a solution $\phi(t)$ defined for all $0 \leq t \leq T$ which for $t = t_0$ becomes equal to ϕ_0 and is such that $\phi(t)$ depends linearly and continuously on ϕ_0 .

THEOREM 1. ([1] Chap. II, 2.1). *If the Cauchy problem (2.3), (2.4) is always solvable in the space Φ , then the Cauchy problem (2.1), (2.2) in the dual space Φ' has a unique solution for any initial functional u_0 , and this solution depends continuously on the functional u_0 in the sense of the weak topology of the space Φ' .*

If we are interested only in the problem of uniqueness, the situation described above can be considerably generalized. We assume now that the space Φ is a part of a large space Φ_1 , which in turn is contained in a third topological vector space E , i.e.,

$$\Phi \subset \Phi_1 \subset E.$$

Each of these inclusions by assumptions preserves linear operations as well as convergence. We further assume that the space Φ is dense in E . Every continuous functional defined on E is thus at the same time a continuous linear functional on Φ and Φ_1 . Moreover we assume that the operator A_t can be extended from the space Φ , on which it is originally defined, to the space Φ_1 . Then the next theorem holds.

THEOREM 2. ([1]. Chap. II, 2.2). *If for any t_0 and $t(0 \leq t \leq t_0 \leq T)$ a linear continuous operator $Q_{t_0}^t$ is defined on the space Φ which maps this space into the space Φ_1 and, in addition, if applied to an arbitrary vector ϕ_0 of Φ , yields a solution of the Cauchy problem*

$$\frac{d\phi(t)}{dt} = A_t \phi(t), \quad \phi(t_0) = \phi_0, \quad \phi(t) \in \Phi_1,$$

then the Cauchy problem

$$\frac{du(t)}{dt} = -A_t^* u(t), \quad u(0) = u_0$$

admits in the interval $0 \leq t \leq T$ the unique solution $u(t) \in E'$.

Before proving our theorem, we must note some lemmas which are concerned with the space Type W and operations in it.

We first define a space of Type W .

DEFINITION 1. Let $\mu(\xi)$, $\omega(\eta)$ be increasing positive functions for

$\xi > 0$, $\eta > 0$, respectively, and define functions $M(x)$, $\Omega(y)$ as follows,

$$M(x) = \int_0^x \mu(\xi) d\xi, \quad x \geq 0,$$

$$\Omega(y) = \int_0^y \omega(\eta) d\eta, \quad y \geq 0.$$

Then, $M(x)$ and $\Omega(y)$ are positive, increasing and convex functions of x and y , respectively. For negative x , y , we put

$$M(x) = M(-x), \quad x < 0$$

$$\Omega(y) = \Omega(-y), \quad y < 0.$$

The space $W_{M_a}^{\Omega b}$ is the totality of the entire analytic functions of $z = x + iy$ which have properties

$$|\phi(x + iy)| \leq C_{\delta, \rho} \exp[-M\{(a - \delta)x\} + \Omega\{(b + \rho)y\}]$$

for every positive number δ , σ .

Then $W_{M_a}^{\Omega b}$ is a complete, perfect, countably normed space.

LEMMA 1. *Let the entire analytic function $f(z)$ satisfy the inequality*

$$|f(z)| \leq C \exp[M(a_0 x) + \Omega(b_0 y)]$$

where $M(x)$, $\Omega(y)$ are functions defining the space $W_{M_a}^{\Omega b}$, and $z = x + iy$. Then $f(z)$ is a multiplier in the space $W_{M_a}^{\Omega b}$ for $a > a_0$ and takes the space $W_{M_a}^{\Omega b}$ into the space $W_{M_{a-a_0}}^{\Omega b+b_0}$.

PROOF. Let $\phi(z)$ be in $W_{M_a}^{\Omega b}$. Then by definition of $W_{M_a}^{\Omega b}$, $\phi(z)$ satisfies the following inequality for any positive member δ and ρ ,

$$|\phi(z)| \leq C_{\delta, \rho} \exp[-M\{(a - \delta)x\} + \Omega\{(b + \rho)y\}].$$

Then we have

$$\begin{aligned} |f(z)\phi(z)| &\leq C \cdot C_{\delta, \rho} \exp[-M\{(a - \delta)x\} + M(a_0 x) \\ &\quad + \Omega\{(b + \rho)y\} + \Omega(b_0 y)] \end{aligned}$$

by the convexity of $M(x)$, $\Omega(y)$, and we obtain

$$M(a_0 x) - M\{(a - \varrho)x\} \leq -M\{(a - a_0 - \delta)x\}$$

$$\mathcal{Q}(b_0 y) + \mathcal{Q}\{(b + \rho)y\} \leq \mathcal{Q}\{(b + b_0 + \rho)y\}.$$

Thus we get

$$|f(z)\phi(z)| \leq C'_{\delta, \rho} \exp[-M\{(a - a_0 - \delta)x\} + \mathcal{Q}\{(b + b_0 + \rho)y\}]$$

where $C'_{\delta, \rho} = C \cdot C_{\delta, \rho}$.

Hence, the product $f(z)\phi(z)$ belongs to $W_{M, a-a_0}^{\mathcal{Q}, b+b_0}$ and this proves the lemma.

If functions $\mu(\xi)$ and $\omega(\eta)$ in Definition 1 are mutually inverse, namely, $\mu[\omega(\eta)] = \eta$, $\omega[\mu(\xi)] = \xi$, then the corresponding functions $M(x)$, $\mathcal{Q}(y)$ will be said to be “dual in the sense of Young” and they satisfy the wellknown young inequality

$$|xy| \leq M(x) + \mathcal{Q}(y).$$

We find the following three lemmas in [1], Chap. I, 3.

LEMMA 2. Let $\mathcal{Q}_1(y)$ and $M_1(x)$ be functions which are dual in the sense of Young to functions $M(x)$ and $\mathcal{Q}(y)$, respectively. Then the Fourier transform is a homeomorphism of the space $W_M^{\mathcal{Q}, b}$ to the space $W_{M_1}^{\mathcal{Q}_1, \frac{1}{b}}$.

LEMMA 3. All nontrivial spaces type W are sufficiently “rich” in functions. i.e. If for some locally integrable function $f(x)$ and any test function $\phi(x)$

$$\int_{-\infty}^{\infty} f(x)\phi(x)dx \equiv 0$$

then $f(x) \equiv 0$ a.e.x.

LEMMA 4. Any nontrivial space of type W is densely embedded in any normed function space E with the norm defined as

$$\|\phi\| = \int_{-\infty}^{\infty} P(x)|\phi(x)|dx$$

where $P(x)$ is a positive function.

We now formulate our theorem.

We first define functions $\mu_P(x)$ and $\omega_P(y)$ as follows;

$$\mu_P(x) = e^{x^P} - 1 \quad \omega_P(y) = {}^P\sqrt{\log(y+1)}$$

where x, y are non-negative arguments. Set

$$M_P(x) = \int_0^x \mu_P(\xi) d\xi, \quad \Omega_P(y) = \int_0^y \omega_P(\eta) d\eta.$$

Then $M_P(x)$ and $\Omega_P(y)$ satisfy the assumptions of Definition 1 and they are dual in the sense of Young.

§3. Uniqueness classes for single differential equations.

We first consider the Cauchy problem for single equation

$$(3.1) \quad \frac{\partial u(x, t)}{\partial t} = P\left(i \frac{\partial}{\partial x}\right) u(x, t), \quad 0 \leq t \leq T,$$

under the initial condition

$$(3.2) \quad u(x, 0) = u_0(x)$$

where $P(s)$ is an entire function, $u_0(x)$ is some given function and T is some positive number. The next theorem holds.

THEOREM 3. *If the entire function $P(s)$ satisfies the inequality*

$$(3.3) \quad |P(s)| \leq C \{M_P(a_0\sigma) + M_P(b_0\tau)\}$$

where $s = \sigma + i\tau$, then the uniqueness class for the problem is given by the class of functions $f(x)$ which satisfy the inequality,

$$|f(x)| \leq C' \exp[\Omega_P(\alpha x)], \quad \frac{1}{b_0} < \alpha.$$

where the constant C' depends only on $f(x)$.

PROOF. We must find a specified fundamental space Φ in which adjoint Cauchy problem

$$(3.4) \quad \frac{\partial \phi(x, t)}{\partial t} = -P\left(-i \frac{\partial}{\partial x}\right) \phi(x, t)$$

$$(3.5) \quad \phi(x, t_0) = \phi_0(x)$$

is always solvable for any $\phi_0(x) \in \mathcal{O}$, and $t_0, 0 \leq t_0 \leq T$.

We Fourier-transform (3.4), (3.5), formally. Then we get another Cauchy problem

$$(3.6) \quad \frac{d\phi(s, t)}{dt} = \tilde{P}(s) \phi(s, t)$$

$$(3.7) \quad \phi(s, t_0) = \hat{\phi}_0(x) \in \hat{\mathcal{O}} = \mathcal{P}$$

where

$$\begin{aligned} \tilde{P}(s) &= -P(-s), \\ \phi(s, t) &= \int_{-\infty}^{\infty} e^{-isx} \phi(x, t) dx, \\ \phi(s, t_0) &= \int_{-\infty}^{\infty} e^{-isx} \phi_0(x) dx. \end{aligned}$$

Here we neglected the convergence of the series, but in later, we must check it.

According to the equation (3.6), we can formally describe the solution as follows

$$(3.8) \quad \phi(s, t) = \exp(t - t_0) \tilde{P}(s) \phi_0(s),$$

therefore, we have to construct the fundamental space $\mathcal{P}$ so that the operator

$$Q(s, t, t_0) = \exp(t - t_0) \tilde{P}(s)$$

has meaning.

Now, we estimate the operator $Q(s, t, t_0)$ which is an entire function in s . We can assume parameter t, t_0 lies

$$0 \leq t \leq t_0 \leq T$$

where we choose T , such that

$$CT \leq 1$$

Then we have

$$\begin{aligned} |Q(s, t, t_0)| &\leq \exp\{|t - t_0| \tilde{P}(s)\} \\ &\leq \exp\{CT[M_p(a_0\sigma) + M_p(b_0\tau)]\}. \end{aligned}$$

Consequently

$$|Q(s, t, t_0)| \leq \exp[M_p(a_0\sigma) + M_p(b_0\tau)].$$

From Lemma 1, $Q(s, t, t_0)$ is a multiplier of the space $W_{M_p a}^{M_p b}$ for $a > a_0$ and takes the space $W_{M_p a}^{M_p b}$ into the space $W_{M_p a - a_0}^{M_p b + b_0}$. Therefore, we choose the test function space $\Psi = W_{M_p a}^{M_p b}$ where a is any number such that $a > a_0$ and b is an arbitrary positive number.

$$\text{Put } \Psi_1 = W_{M_p a - a_0}^{M_p b + b_0}.$$

According to Lemma 2, we obtain the relations

$$\tilde{\Psi} = \tilde{W}_{M_p a}^{M_p b} = W_{\Omega_P \frac{1}{b}}^{\Omega_P \frac{1}{a}},$$

$$\tilde{\Psi}_1 = \tilde{W}_{M_p a - a_0}^{M_p b + b_0} = W_{\Omega_P \frac{1}{b + b_0}}^{\Omega_P \frac{1}{a - a_0}}.$$

Finally, if we set

$$\Phi = W_{\Omega_P \frac{1}{b}}^{\Omega_P \frac{1}{a}} \quad \Phi_1 = W_{\Omega_P \frac{1}{b + b_0}}^{\Omega_P \frac{1}{a - a_0}},$$

then the operator $Q\left(i\frac{\partial}{\partial x}, t, t_0\right)$ takes Φ into Φ_1 and obviously we have $\Phi_1 \supset \Phi$.

Now, we must examine the next relations;

$$(3.9) \quad \frac{Q(s, t + \Delta t, t_0) - Q(s, t, t_0)}{\Delta t} \phi_0(s) \rightarrow \tilde{P}(s) Q(s, t, t_0) \phi_0(s)$$

as $\Delta t \rightarrow 0$, namely,

$$\lim_{\Delta t \rightarrow 0} \frac{Q(s, t + \Delta t, t_0) - Q(s, t, t_0)}{\Delta t} = \tilde{P}(s) Q(s, t, t_0),$$

and

$$(3.10) \quad Q(s, t, t_0) \psi_0(s) \rightarrow \psi_0(s) \\ \text{as } t \rightarrow t_0. \text{ i.e. } \lim_{t \rightarrow t_0} Q(s, t, t_0) \psi_0(s) = \psi_0(s)$$

in the topology of the space $\mathcal{V}_1$.

According to the definition of convergence in the space $\mathcal{W}_{M,a}^{g,b}$ we must show that the functions under the lim sign converge regularly to their limits and are uniformly bounded with respect to t (in the sense of the topology of $\mathcal{V}_1$). The regular convergence is obvious from the definition of the function $Q(s, t, t_0) = \exp[t - t_0] \tilde{P}(s)$. The uniform boundedness in t is derived from the inequalities

$$|Q(s, t, t_0)| \leq \exp[M(a_0\sigma) + M(b_0\tau)]$$

and

$$\left| \frac{\Delta Q(s, t, t_0)}{\Delta t} \right| \leq C \exp[M(a_0\sigma) + M(b_0\tau)].$$

Thus, for any test function $\phi_0(s) \in \mathcal{V}$, the function

$$\phi(s, t) = Q(s, t, t_0) \phi_0(s)$$

solves the Cauchy problem

$$\frac{d\phi(s, t)}{dt} = \tilde{P}(s) \phi(s, t),$$

$$\phi(s, t_0) = \phi_0(s),$$

and belongs for $0 \leq t \leq T$ to the space $\mathcal{V}_1 = \mathcal{W}_{M,a-a_0}^{M_P b+b_0}$ as we remarked previously. The isomorphism of the space $\mathcal{V}$ and $\mathcal{O}$ implies that there exists an operator on the space $\mathcal{O}$ which solves the Cauchy problem (3.4), (3.5) for any initial test function, that is,

$$\frac{\partial \phi(x, t)}{\partial t} = \tilde{P}\left(i \frac{\partial}{\partial x}\right) \phi(x, t),$$

$$\phi(x, t_0) = \phi_0(x).$$

Hence therefore, according to Theorem 1, the solution of the Cauchy problem

$$\frac{\partial u(x, t)}{\partial t} = P\left(i \frac{\partial}{\partial x}\right) u(x, t) \quad u(x, 0) = u_0(x)$$

is unique in the dual space of Φ . More precisely, the solution $u(x, t)$ is unique, if for every t in the interval $0 \leq t < T$, $u(\cdot, t)$ belongs to a certain E where E is a normed space which contains the spaces $\Phi = W_{\Omega_P \frac{1}{b}}^{\Omega_P \frac{1}{a}}$ and $\Phi_1 = W_{\Omega_P \frac{1}{b+b_0}}^{\Omega_P \frac{1}{a-a_0}}$ as dense subspaces.

By the definition of the spaces $W_{\Omega_P \frac{1}{b}}^{\Omega_P \frac{1}{a}}$ and $W_{\Omega_P \frac{1}{b+b_0}}^{\Omega_P \frac{1}{a-a_0}}$, Φ and Φ_1 consist of functions $\phi(x)$ which decrease along the x -axis as

$$|\phi(x)| \leq C \exp\left[-\Omega_P\left(\frac{1}{b} x\right)\right],$$

$$|\phi(x)| \leq C \exp\left[-\Omega_P\left(\frac{1}{b+b_0} x\right)\right],$$

respectively.

We define the space E by means of the norm;

$$\|\phi\|_E = \int_{-\infty}^{\infty} \exp[\Omega_P(\alpha x)] |\phi(x)| dx$$

where α is any positive number such that $\alpha < \frac{1}{b_0}$. Then obviously we have $\Phi \subset \Phi_1 \subset E$. Since Φ is sufficiently rich in functions, from Lemma 3 and Lemma 4 it is dense in E . Thus we can apply Theorem 2 and arrive at the conclusion that the Cauchy problem (3.1), (3.2) has a unique solution in the space E' .

[Note]. Since $\alpha < \frac{1}{b_0}$, we can find the positive numbers b, δ such that

$$\alpha + \delta = \frac{1}{b+b_0}.$$

Then by the convexity of $\Omega_P(x)$ we obtain

$$\mathcal{Q}_P(\alpha x) - \mathcal{Q}_P\left(\frac{1}{b+b_0}x\right) \leq -\mathcal{Q}_P(\delta x).$$

Thus the integral which defines the norm of E converges for all elements of $\mathcal{O}$ and $\mathcal{O}_1$.

§4. Uniqueness classes for systems of partial differential equation.

Next, we consider the Cauchy problem for the system of partial differential equations;

$$(4.1) \quad \frac{\partial u_j(x, t)}{\partial t} = \sum_{k=1}^m P_{j,k} \left(i \frac{\partial}{\partial x} \right) u_k(x, t) \quad j=1, 2, \dots m.$$

and initial conditions,

$$(4.2) \quad u_j(x, 0) = u_j(x) \quad j=1, 2, \dots m.$$

We use vector notation for systems (4.1), (4.2), that is,

$$(4.3) \quad \frac{\partial u(x, t)}{\partial t} = P \left(i \frac{\partial}{\partial x} \right) u(x, t)$$

$$(4.4) \quad u(x, 0) = u_0(x)$$

where $u(x, t) = (u_1(x, t), u_2(x, t), \dots, u_m(x, t))$

$$u_0(x) = (u_1(x), u_2(x), \dots, u_m(x))$$

and $P(s) = (P_{j,k}(s))$ is a $m \times m$ matrix.

The adjoint of the operator P represented by the matrix $(P_{j,k}(s))$ is the operator P^* which is defined by the matrix $(P^*_{j,k}(s))$, that is obtained from $(P_{j,k}(s))$ by replacing the matrix elements by their adjoints and transposing the matrix. We consider the unknown function $u(x, t)$ as a generalized function, namely, a linear functional over a vector test function space $\mathcal{O}$ depending on the parameter t .

The problem is reduced to the construction of such a test function space $\mathcal{O} = \{\phi(x)\}$ in which the solution of the Cauchy problem for

$$(4.5) \quad \frac{\partial \phi(x, t)}{\partial t} = \tilde{P} \left(i \frac{\partial}{\partial x} \right) \phi(x, t)$$

exists for any Cauchy data

$$(4.6) \quad \phi(x, t_0) = \phi_0(x) \in \mathcal{D}.$$

Here the matrix $\tilde{P}$ is related to the matrix P by means of

$$\tilde{P}_{j,k} = -\bar{P}_{k,j} \quad (j, k = 1, 2, \dots, m).$$

THEOREM 4. *If the coefficient functions $P_{j,k}(s)$ of the system (4.1) are entire functions in s which satisfy the following inequality*

$$(4.6) \quad |P_{j,k}(s)| \leq C \{M_P(a_0\sigma) + M_P(b_0\tau)\}, \quad (j, k = 1, 2, \dots, m),$$

then, the Cauchy problem (4.1), (4.2) has a unique solution in the class of functions $f(x)$ which have the property

$$|f(x)| \leq C \exp[\Omega_P(\alpha x)]$$

where $\alpha < \frac{1}{b_0}$ and constant C depends only on $f(x)$.

PROOF. The essence of the proof is the same as Theorem 3.

In Euclidean space of vectors $\xi = (\xi_1, \xi_2, \dots, \xi_m)$ with the scalar product $(\xi^{(1)}, \xi^{(2)}) = \sum_{j=1}^m \xi_j^{(1)} \xi_j^{(2)}$, we have

$$\max_k \sum_{j=1}^m |P_{j,k}|^2 \leq \|P\|^2 \leq \sum_{j=1}^m \sum_{k=1}^m |P_{j,k}|^2$$

where $\|P\|$ denotes the operator norm. From our hypothesis (4.6), operator $\tilde{P}$ satisfies

$$\|\tilde{P}(s)\| \leq C' \{M_P(a_0\sigma) + M_P(b_0\tau)\}$$

where $C' = mC$. Then formally our solution of the system (4.5) is also described as

$$\phi(s, t) = \exp[(t - t_0) \tilde{P}(s)] \phi_0(s)$$

where

$$\phi_0(s) = \int_{-\infty}^{\infty} e^{-ixs} \phi_0(x) dx \in \hat{\Phi} = \mathcal{P}.$$

Let us set $Q(s, t, t_0) = \exp[(t - t_0)\tilde{P}(s)]$. Then Q satisfies the inequalities

$$\begin{aligned} \|Q(s, t, t_0)\| &= \|\exp[(t - t_0)\tilde{P}(s)]\| \\ &= \left\| \sum_{k=0}^{\infty} \frac{(t - t_0)^k \tilde{P}^k(s)}{k!} \right\| \\ &\leq \sum_{k=0}^{\infty} \frac{|t - t_0|^k}{k!} \|\tilde{P}^k(s)\| \\ &\leq \sum_{k=0}^{\infty} \frac{|t - t_0|^k}{k!} [C'M_P(a_0\sigma) + M_P(b_0\tau)]^k \\ &= \exp[|t - t_0| \{C'M_P(a_0\sigma) + M_P(b_0\tau)\}]. \end{aligned}$$

For time interval $t: 0 \leq t \leq T$.

We choose T , such that

$$TC' \leq 1$$

then we have

$$|Q(s, t, t_0)| \leq \exp[M_P(a_0\sigma) + M_P(b_0\tau)]$$

for every t in $[0, T]$.

The remaining part of the proof is quite same as theorem 3.

§5. Example $\frac{\partial u}{\partial t} = e^{i\frac{\partial}{\partial x}} u.$

Let us consider the equation

$$(5.1) \quad \frac{\partial u(x, t)}{\partial t} = e^{i\frac{\partial}{\partial x}} u(x, t).$$

Formally we have the adjoint equation of (5.1)

$$\frac{\partial \phi(x, t)}{\partial t} = -e^{-i\frac{\partial}{\partial x}} \phi(x, t).$$

Then the resolvent operator is given by

$$Q(s, t, t_0) = \exp[(t - t_0)(-e^{-s})]$$

and we estimate $Q(s, t, t_0)$.

Since

$$\begin{aligned} |e^s| &\leq e^{|s|} \leq e^{|\sigma|} \cdot e^{|\tau|} \\ &\leq \frac{e^{2|\sigma|} + e^{2|\tau|}}{2} \\ &\leq C_0(e^{2|\sigma|} + e^{2|\tau|} - 2|\sigma| - 2|\tau| - 2) + 1 \end{aligned}$$

where $C_0 > \frac{1}{2}$, then we have

$$\begin{aligned} |Q(s, t, t_0)| &\leq \exp[|t - t_0| e^{|s|}] \\ &\leq \exp[|t - t_0| \cdot C_0(e^{2|\sigma|} - 2|\sigma| - 1 + e^{2|\tau|} - 2|\tau| - 1)] \cdot \exp|t - t_0|. \end{aligned}$$

Assuming

$$0 \leq t_0 \leq t \leq \frac{1}{C_0},$$

and observing

$$e^x - x - 1 = \int_0^x (e^\xi - 1) d\xi = M_1(x),$$

then we have

$$\begin{aligned} |Q(s, t, t_0)| &\leq \exp[|t - t_0| C_0 \{M_1(2\sigma) + M_1(2\tau)\}] \cdot \exp|t - t_0| \\ &\leq C' \exp[M_1(2\sigma) + M_1(2\tau)], \end{aligned}$$

where $C' = \exp \frac{1}{C_0}$.

This estimation gives the uniqueness class for the equation (3.7) by the class of functions $f(x)$.

$$|f(x)| \leq C \exp[\mathcal{Q}_1(\alpha x)]$$

where $0 < \alpha < \frac{1}{2}$

and more explicitly

$$|f(x)| \leq C \exp [(\alpha |x| + 1) \log (\alpha |x| + 1) - \alpha |x|].$$

Reference

- [1] I.M. GEL'FAND and G.E. SHILOV: *Generalized functions, Vol. III Theory of Differential Equations*. (1967) (English translation).