A CONSTRUCTION OF THE INVARIANT MEASURE

By

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Let T be a non-singular transformation of a measure space $(X, \mathfrak{B}, m)$ onto itself. The problem of finding necessary and sufficient conditions for the existence of a finite T -invariant measure μ equivalent with m has been discussed by many authors, see [1], [2], [3], [4], and [5].

In [5] E. Hopf introduced boundedness of the transformation T as such a condition and constructed a T -invariant measure μ using the compressible measure. In this paper we construct the invariant measure μ by another method which is more directly related to E. Hopf's condition. If T is an ergodic transformation this method is essentially as same as one in constructing the relative dimension of von Neumann factor algebra [6].

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1. We denote by $(X, \mathfrak{B}, m)$ a measure space where $\mathfrak{B}$ is a σ -field of subsets of an abstract set X , and m is a countably additive finite measure defined on the measurable space $(X, \mathfrak{B})$. One-to-one transformation T of X onto itself is said to be *measurable* if $E \in \mathfrak{B}$ implies $TE \in \mathfrak{B}$ and $T^{-1}E \in \mathfrak{B}$; and be non-singular if $m(E)=0$ implies $m(TE)=m(T^{-1}E)=0$. In the sequel the transformation T is assumed to be non-singular and all subsets of X which are mentioned are assumed to be members of the σ -field $\mathfrak{B}$.

We shall use the notation $A = \sum_{i=1}^{\infty} A_i$ to mean A is the union of mutually disjoint sets A_i , $i=1, 2, \dots$, i.e., $A_i \cap A_j = \emptyset$ for $i \neq j$, $i, j=1, 2, \dots$. A set A is said to be *T -equivalent* to a set B , and we use the notation $A \sim B$, if $A = \sum_{i=1}^{\infty} A_i$, $B = \sum_{i=1}^{\infty} B_i$ and if there exists a sequence of integers n_i , $i=1, 2, \dots$ such that $T^{n_i}A_i = B_i$, $i=1, 2, \dots$. We note the following

properties:

- (1) $A \sim A$.
- (2) $A \sim B$ implies $B \sim A$.
- (3) $A \sim B$ and $B \sim C$ imply $A \sim C$.
- (4) For any sets A and B which are mutually T -equivalent, and for any subset A' of A , there exists a subset B' of B such that $A' \sim B'$ and $A - A' \sim B - B'$.
- (5) $A = \sum_{i=1}^{\infty} A_i$, $B = \sum_{i=1}^{\infty} B_i$ and $A_i \sim B_i$ for $i=1, 2, \dots$ imply $A \sim B$.
- (6) $A \sim B$ and $TC=C$ imply $A \cap C \sim B \cap C$.

A set A is said to be T -infinite if there exists a subset A' of A such that $A' \sim A$ and $A' \neq A$. A set which is not T -infinite is said to be T -finite. For a pair of sets A, B we use the notation $A < B$ if there exists a subset B' of B such that $A \sim B'$. We note the following properties:

- (7) $A < A$.
- (8) $A < B$ and $B < A$ imply $A \sim B$.
- (9) $A < B$ and $B < C$ imply $A < C$.
- (10) $A < B$ implies $A < B$.
- (11) If a set A is T -infinite and $A < B$ then B is T -infinite.

A proof of property (8) is as same as usual Bernstein's method in general theory of sets. Property (11) means that equivalence relation $\sim$ preserves T -finiteness of measurable sets. Ordered relation $<$ is not totally ordered but we have following,

LEMMA 1. For a pair of T -finite sets A, B there uniquely exists a pair of subsets C, D of $\bigcup_{n=-\infty}^{\infty} T^n A$ with following properties;

- (i) Sets C, D are mutually disjoint and T -invariant, i.e.,

$$C \cap D = \emptyset, \quad TC = C \quad \text{and} \quad TD = D.$$

- (ii) $C \cup D = \bigcup_{n=-\infty}^{\infty} T^n A$.
- (iii) For any T -invariant subset C' of C , $B \cap C' < A \cap C'$ but $B \cap C'$ is not T -equivalent to $A \cap C'$.
- (iv) For any T -invariant subset D' of D , $A \cap D' < B \cap D'$.

PROOF. We inductively define sets as follows.

$$\begin{aligned}
A^0 &= A, & B^0 &= B, & A_0 &= A \cap B, & B_0 &= A \cap B, \\
A^i &= A^{i-1} - A_{i-1}, & B^i &= B^{i-1} - B_{i-1}, \\
A_i &= A^i \cap T^i B^i, & B_i &= T^{-i} A^i \cap B^i, & i &= 1, 2, \dots, \\
A^{-1} &= \bigcap_{i=0}^{\infty} A^i = A - \sum_{i=0}^{\infty} A_i, & B^{-1} &= \bigcap_{i=0}^{\infty} B^i = B - \sum_{i=0}^{\infty} B_i, \\
A_{-1} &= A^{-1} \cap T^{-1} B^{-1}, & B_{-1} &= T A^{-1} \cap B^{-1}, \\
A^{-i} &= A^{-i+1} - A_{-i+1}, & B^{-i} &= B^{-i+1} - B_{-i+1}, \\
A_{-i} &= A^{-i} \cap T^{-i} B^{-i}, & B_{-i} &= T^i A^{-i} \cap B^{-i}, & i &= 2, 3, \dots, \\
A^* &= A - \sum_{i=-\infty}^{\infty} A_i, & B^* &= B - \sum_{i=-\infty}^{\infty} B_i, & C &= \bigcup_{n=-\infty}^{\infty} T^n A^*, \\
D &= \bigcup_{n=-\infty}^{\infty} T^n A - C.
\end{aligned}$$

Properties (i) and (ii) are evident. For any non-vacant T -invariant subset C' of C let us assume $B^* \cap C' \neq \emptyset$. Since $B^* \cap C' = \bigcup_{n=-\infty}^{\infty} (B^* \cap T^n A^* \cap C')$ we have $T^n A^* \cap B^* \cap C' \neq \emptyset$ for some n and $T^n A^* \cap B^* \cap C' \subset T^n A^{-n} \cap B^{-n} = B_{-n}$. This contradicts to the definition of B^* , so $B^* \cap C' = \emptyset$.

Since a set $B \cap C' = \sum_{i=-\infty}^{\infty} B_i \cap C'$ is T -equivalent to a set $A \cap C' - A^* \cap C' = \sum_{i=-\infty}^{\infty} A_i \cap C'$, $A^* \cap C' \neq \emptyset$ and since a set A is T -finite, we get property (iii). Since, for any T -invariant subset D' of D a set $A \cap D' = \sum_{i=-\infty}^{\infty} A_i \cap D'$ is equivalent to $\sum_{i=-\infty}^{\infty} B_i \cap D'$ which is a subset of $B \cap D'$, we get property (iv). Uniqueness of lemma is obvious.

Applying Lemma 1 inductively we have following,

LEMMA 2. For a pair of T -finite sets A, B there uniquely exists a countable ordered sequence of T -invariant sets $X_0, X_1, X_2, \dots$ which satisfy following properties:

- (i) $\bigcup_{n=0}^{\infty} X_n = \bigcup_{n=-\infty}^{\infty} T^n A$ and $X_i \cap X_j = \emptyset$, $i \neq j$, $i, j = 1, 2, \dots$.
- (ii) There exist countable disjoint subsets $B_1, B_2, \dots$ of B such that $X_n \cap B_i \sim X_n \cap A$, $i = 1, 2, \dots, n$; $n = 1, 2, \dots$ and for any non-vacant T -invariant subset X'_n of X_n , $B_n^* \cap X'_n \subset A \cap X'_n$ but $B_n^* \cap X'_n$ is not T -equivalent to $A \cap X'_n$ for $n = 0, 1, 2, \dots$, where $B_0^* = B$ and $B_n^* = B - \sum_{i=1}^n B_i$, $n = 1, 2, \dots$.

2. For a pair of T -invariant sets A, B we can define a function which equals n on X_n defined in Lemma 2 and we shall denote such function by $[B/A](x)$, i.e., $[B/A](x) = n$ for $x \in X_n$. It is easy to show that $[B/A](x)$ is a T -invariant measurable function defined on $\bigcup_{n=-\infty}^{\infty} T^n A$ and has following:

LEMMA 3. Let A, B, C , a triple of T -finite sets. Then

- (i) $[A/B](x)[B/C](x) \leq [A/C](x) \leq \{[A/B](x) + 1\} \{[B/C](x) + 1\}$

$$\text{for } x \in \left(\bigcup_{n=-\infty}^{\infty} T^n B \right) \cap \left(\bigcup_{n=-\infty}^{\infty} T^n C \right).$$

- (ii) If A and B are mutually disjoint,

$$\begin{aligned} [A/C](x) + [B/C](x) &\leq [A + B/C](x) \leq [A/C](x) + \\ &+ [B/C](x) + 1 \quad \text{for } x \in \bigcup_{n=-\infty}^{\infty} T^n C. \end{aligned}$$

- (iii) If $A \sim B$, $[A/C](x) = [B/C](x)$ for almost every $x \in \bigcup_{n=-\infty}^{\infty} T^n C$.

3. Let T be a conservative transformation, i.e., for every set E , $E \subset \bigcup_{n=1}^{\infty} T^{-n} E$ (a.e.). For a set E which has positive measure, setting $E_1 = E \cap T^{-1} E$ and $A_n = A \cap T^{-n} A - \bigcup_{i=1}^{n-1} T^{-i} A$, $n = 2, 3, \dots$ we have $A = \sum_{n=1}^{\infty} A_n$ (a.e.). Induced transformation T_E is defined as $T_E x = T^n x$, $x \in E_n$. It is easy to show that transformation T_E defined on a.e. $x \in E$ is non-singular and conservative. In the following lemma we can obtain the partition of the space X to periodic and non-periodic parts having their properties.

LEMMA 4. If the transformation T is conservative there uniquely ex-

ists a pair of T -invariant sets P, A with following properties:

- (i) The space X is a disjoint union of P and A .
- (ii) There exist countable sets $P_1, P_2, \dots$ such that

$$P = \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} T^i P_n \quad (\text{disjoint sum})$$

and for any subset P'_n of P_n , $T^n P'_n = P'_n$, $n=1, 2, \dots$.

- (iii) There exist countable sets $A_1, A_2, \dots$ such that

$$A = \bigcup_{i=-\infty}^{\infty} T^i A_n, \quad n=1, 2, \dots \text{ and } [A_n/A_{n+1}](x) \geq 2 \text{ for } x \in A.$$

PROOF. We set $\mathfrak{A} = \{D \in \mathfrak{B}; m(TD \cap D) = 0\}$ and $\sup_{D \in \mathfrak{A}} (\bigcup_{n=-\infty}^{\infty} T^n D) = p$. Suppose that p is positive, then there exists a sequence of sets $D_1, D_2, \dots$ such that $D_n \in \mathfrak{A}$ and $m(\bigcup_{i=-\infty}^{\infty} T^i D_n) > p - 1/n$, $n=1, 2, \dots$. Setting $D = \bigcup_{n=1}^{\infty} (D_n - \bigcup_{j=1}^{n-1} \bigcup_{i=-\infty}^{\infty} T^i D_j)$, the set D belongs to $\mathfrak{A}$ and $m(\bigcup_{n=-\infty}^{\infty} T^n D) = p$. Setting $B_1 = D$, $C_1 = \bigcup_{n=-\infty}^{\infty} T^n D$ and $Q_1 = X - C_1$ then $[C_1/B_1](x) \geq 2$ for $x \in C_1$ and $TQ'_1 = Q'_1$ for every subset Q'_1 of Q_1 . For $n=2, 3, \dots$ applying the same discussion as above to the induced transformation $T_{B_{n-1}}$ inductively we obtain sets B_n , $C_n = \bigcup_{i=-\infty}^{\infty} T^i_{B_{n-1}} B_n$ and $Q_n = B_{n-1} - C_n$ such that $T_{B_{n-1}} B_n \cap B_n = \emptyset$ and $T_{B_{n-1}} Q'_n = Q'_n$ for every subsets Q'_n of Q_n . It follows that $[C_n/B_n](x) \geq 2$ for $x \in \bigcup_{i=-\infty}^{\infty} T^i B_n$, $n=2, 3, \dots$ and that there exist countable disjoint sets $Q_{n1}, Q_{n2}, \dots$ with following properties;

$$Q_n = \sum_{i=1}^{\infty} Q_{ni}, \quad n=2, 3, \dots,$$

$T^j Q_{ni}$, $j=0, 1, 2, \dots, i-1$ are mutually disjoint,

$$i=1, 2, \dots; n=2, 3, \dots$$

$T^i Q'_{ni} = Q'_{ni}$ for every subset Q'_{ni} of Q_{ni} .

Setting $P = \bigcup_{n=1}^{\infty} \bigcup_{i=-\infty}^{\infty} T^i Q_n$, $P_1 = Q_1 + Q_{21} + Q_{31} + \dots$ and setting $P_n = Q_{2n} + Q_{3n} + \dots$, $n=2, 3, \dots$, then property (ii) is satisfied. Setting $A = \bigcap_{n=1}^{\infty} \bigcup_{i=-\infty}^{\infty} T^i B_n$ and setting $A_n = B_n \cap (\bigcap_{i=1}^{\infty} B_i)$, then properties (i) and (ii) are satisfied. Sets P and A are said to be the *periodic part* and the *non-periodic part* respectively.

4. A measure μ is said to be *equivalent to a measure m* on the measurable space $(X, \mathfrak{B},)$ if $m(E)=0$ implies $\mu(E)=0$ and if, conversely $\mu(E)=0$ implies $m(E)=0$; and to be *T -invariant* if $\mu(E)=\mu(TE)$ for every set $E \in \mathfrak{B}$. It is easy to show that there can not exist a finite T -invariant measure μ equivalent to m if the space X is T -finite. We shall prove the following.

THEOREM. *If the space X is T -finite there exists T -invariant measure μ equivalent to m .*

PROOF. It is easy to see that T -finiteness of X derives conservativity of the transformation T . We can firstly construct finite T -invariant measure μ on the periodid part P such as

$$\mu(E) = \frac{m(T^{-i}E)}{n^3 m(P_n)}, \quad E \subset T^i P_n, \quad i=0, 1, 2, \dots; \quad n=1, 2, \dots$$

Appling Lemma 3 (i), we have

$$\frac{[E/A_k](x)}{[A/A_k](x)} \leq \frac{\{[E/A_h](x)+1\} \{[A_h/A_k](x)+1\}}{[A/A_h](x)[A_h/A_k](x)}$$

for $h \leq k$, $E \subset A$ and $x \in A$.

Because $[A_h/A_k](x) \geq 2^{k-h}$ which follows from Lemma 3 (i) and Lemma 4 (iii), first making $k \uparrow \infty$ and then $h \uparrow \infty$, we have

$$\overline{\lim}_{k \rightarrow \infty} \frac{[E/A_k](x)}{[A/A_k](x)} \leq \lim_{h \rightarrow \infty} \frac{[E/A_h](x)}{[A/A_h](x)}.$$

We set $f_E(x) = \lim_{k \rightarrow \infty} \frac{[E/A_k](x)}{[A/A_k](x)}$, then the function $f_E(x)$ is measurable and $0 \leq f_E(x) \leq 1$. Setting $\mu(E) = \int_x f_E(x) dm(x)$, we now show that μ is T -invariant measure equivalent to m . For any set E which has positive measure, we have

$$[E/A_k](x) \geq \frac{2^{k-1}}{n+1} - 1, \quad x \in C_n,$$

which follows from $\{[A/E](x) + 1\} \{[E/A_k](x) + 1\} \geq [A/A_k](x) \geq 2^{k-1}$, where $C_n = \{x \in \bigcup_{i=-\infty}^{\infty} T^i E; [A/E](x) \leq n\}$, $n = 1, 2, \dots$. For large integer k such as $2^{k-1}/(n+1) \geq 2$ and for $h = k+1, k+2, \dots$ we have

$$\begin{aligned} \frac{[E/A_h](x)}{[A/A_h](x)} &\geq \frac{[E/A_k](x)[A_k/A_h](x)}{\{[A/A_k](x) + 1\} \{[A_k/A_h](x) + 1\}} \\ &\geq \frac{1}{[A/A_k](x) + 1} \times \frac{1}{2} > 0, \quad x \in C_n. \end{aligned}$$

This implies that $f_E(x)$ is positive for $x \in \bigcup_{i=-\infty}^{\infty} T^i E$. And it follows that μ is equivalent, to m .

It is easy to verify that $f_E(x) = f_F(x)$ for a pair of sets E, F which are mutually T -equivalent, and that, if $f_E(x) \leq f_F(x)$, then $E < F$, say, F has a subset F' T -equivalent to E . This implies T -invariance of μ .

Using Lemma 3 (ii), one can see that $f_{E+F}(x) = f_E(x) + f_F(x)$ for a pair of sets E, F , which are mutually disjoint. Therefore, if sets E_n , $n = 1, 2, \dots$ are mutually disjoint we have $\sum_{n=1}^{\infty} f_{E_n}(x) \leq f_{\sum_{n=1}^{\infty} E_n}(x)$. Assume that $m(f_{\sum_{n=1}^{\infty} E_n}(x) - \sum_{n=1}^{\infty} f_{E_n}(x) > \varepsilon) > 0$ for some positive number ε . There exists a integer h such that $0 < f_{A_h}(x) < \varepsilon$ for $x \in A$, which follows from

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{[A_h/A_k](x)}{[A/A_k](x)} &\leq \lim_{k \rightarrow \infty} \frac{[A_h/A_k](x)}{[A/A_h](x)[A_h/A_k](x)} \\ &\leq \frac{1}{2^{h-1}}. \end{aligned}$$

Chose a integer n so that $m(D_n) > 0$, where $D_n = \{x; \sum_{i=n+1}^{\infty} f_{E_i}(x) < f_{A_h}(x)$ and $f_{\sum_{i=1}^{\infty} E_i}(x) - \sum_{i=1}^{\infty} f_{E_i}(x) > \varepsilon\}$. We obtain a subset E'_{n+1} of $A_h \cap D_n$ such that $E'_{n+1} \sim E_{n+1} \cap D_n$, because $f_{E_{n+1} \cap D_n}(x) \leq f_{A_h \cap D_n}(x)$. Inductively we obtain $E'_{n+2}, E'_{n+3}, \dots$, which are mutually disjoint subsets of $A_h \cap D_n$, $E'_i \sim E_i \cap D_n$ $i = n+2, n+3, \dots$. It follows that for $x \in D_n$,

$$\begin{aligned} f_{\sum_{i=1}^{\infty} E_i}(x) - \sum_{i=1}^{\infty} f_{E_i}(x) &= f_{\sum_{i=n+1}^{\infty} E_i}(x) - \sum_{i=n+1}^{\infty} f_{E_i}(x) \\ &\leq f_{\sum_{i=n+1}^{\infty} E_i}(x) \\ &\leq f_{\sum_{i=n+1}^{\infty} E'_i}(x) \\ &\leq f_{A_h}(x) \\ &< \varepsilon. \end{aligned}$$

This is a contradiction, so $f_{\sum_{i=1}^{\infty} E_i}(x) = \sum_{i=1}^{\infty} f_{E_i}(x)$. Hence, the set function μ is σ -additive. We complete the proof.

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