ON THE TYPE OF AN ASSOCIATIVE H -SPACES OF RANK FOUR

By

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§ 1. Introduction.

Let X be an H -space having the same homotopy type of a connected finite CW-complex. By a well known theorem due to Hopf, we have

$$H^*(X, Q) = A_Q(x_1, \dots, x_l), \quad \deg x_i = 2n_i - 1.$$

l is called the *rank* of X , and $(2n_1 - 1, \dots, 2n_l - 1)$, the *type* of X .

We assume, moreover, that X is associative. The determination of the possible types of associative H -spaces has been done by L. Smith [5] and S. Ochiai [4] for rank 2 and 3 respectively.

If N_1 and N_2 are the types of H -spaces X_1 and X_2 respectively, then $N_1 + N_2$ is the type of $X_1 \times X_2$, where $+$ denotes disjoint union. Type N is called *simple* if N is not a disjoint union of two types of H -spaces. We restrict ourselves to simple types.

When X is not simply connected, the universal covering space $\tilde{X}$ over X is a simply connected H -space. Considering the spectral sequence of Leray-Cartan for this covering space, we have that if the type of $\tilde{X}$ is N , possible types of X is N and $(1, \dots, 1) + N$. But since the latter is also the type of $S^1 \times \dots \times S^1 \times \tilde{X}$ and is not simple, it must be excluded. Hence we can assume without loss of generality that X is simply connected.

THEOREM. *Simple types of associative H -spaces of rank 4 are as follows:*

$$(3, 5, 7, 9), (3, 7, 7, 11), (3, 7, 11, 15), (3, 11, 15, 23).$$

They are, for example, the types of the compact Lie groups $SU(5)$, $SO(8)$, $Sp(4)$ and F_4 .

The proof, which will be given in §3, is mainly based on the Theorems of A. Clark [2] and slight generalization of one of them (Proposition 3).

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§2. Polynomial algebra over $\mathcal{A}_p$.

When X is an associative H -space, we can consider the classifying space BX of X (A. Dold and R. Lashof [3]), and if X has no p -torsion, we have

$$H^*(X, Z_p) = \Lambda_{Z_p}(x_1, \dots, x_l), \quad \deg x_i = 2n_i - 1,$$

$$H^*(BX, Z_p) = Z_p[y_1, \dots, y_l], \quad \deg y_i = 2n_i,$$

on which the Steenrod algebra $\mathcal{A}_p$ operates. We denote $P = Z_p[y_1, \dots, y_l]$ and $A = (2n_1, \dots, 2n_l)$.

PROPOSITION 1. (A. Clark) *Suppose X has no p -torsion. For any $2m \in A$, $m \not\equiv 0 \pmod{p}$, there exists $2n \in A$ such that*

$$n \equiv 1 - p \pmod{m}.$$

For the proof, see [2] Theorem 2. We remark that for one of such n 's and some integer r , we have

$$\mathcal{P}_p^1 y_{2n} \equiv y_{2m}^r \pmod{I(y_{2m})}, \quad 1 \leq r \leq p,$$

Where $I(y_{2m})$ is the ideal of P , generated by all the generators other than y_{2m} .

PROPOSITION 2. *For any $2m \in A$ and any k with $1 < k < m$ and $(k, m) = 1$, there is an element $2(tm - k + 1) \in A$ with t an integer.*

PROOF. As $(k, m) = 1$, by a theorem due to Dirichlet, we can find a sufficiently large prime p such that $p = k + sm$ for some s and that X has no p -torsion and $m \not\equiv 0 \pmod{p}$. (Note that since X has finite type and therefore has torsion for only a finite number of primes.) Therefore from Proposition 1, $2n \in A$ where $n \equiv 1 - p \equiv 1 - k \pmod{m}$, that is $n = tm$

$-k+1$ for some t .

As a corollary of Proposition 2, we have:

PROPOSITION 3. *Let $2m \in A$ and k be the same as in Proposition 2 and moreover let $2(2m-k+1)$ be larger than the maximal elements of A . Then $2(m-k+1) \in A$.*

PROOF. In proposition 2, t must be 1.

If $2m$ is the maximal element of A , we have [2] Theorem 4. So Proposition 2 is a generalization of [2] Theorem 4.

Next proposition will be used repeatedly.

PROPOSITION 4. *If $\mathcal{P}_p^1 y_{2n} = y_{2tm} (t \geq 1)$, then for a generator y_{2m} , there must be a generator $y_{2k} (\neq y_{2n})$ such that*

$$\mathcal{P}_p^1 y_{2k} \equiv y_{2m}^t \pmod{I(y_{2m})}.$$

PROOF. Since $\mathcal{P}_p^1 y_{2n} \equiv y_{2m}^t \pmod{I(y_{2m})}$ does not hold, y_{2m} requires existence of another generator y_{2k} .

§3. Proof of Theorem.

By Proposition 3, it is sufficient to consider the case where the maximal element $2m$ of $A = (2n_1, \dots, 2n_l)$ is 4, 6, 8, 10, 12, 16, 20 and 24, the values of which under the Euler function φ are smaller than five.

$2m$	$\{2(m-k+1); 1 < k < m, (k, m) = 1\}$		
4			
6	4		
8	4		
10	8	6	4
12	4		
16	12	8	4
20	16	8	4
24	16	12	4

For the case where the maximal element of A is 10, 16 and 24, the proof

of the theorem is finished and $(4, 8, 16, 20)$ cannot be a type of an H -space because 16 requires 12 by Proposition 3. So let the maximal element of A be equal or smaller than 12. If $10 \in A$, then by Proposition 3, A includes 8, 6 and 4, and this case is already checked. So there are such types $(2n_1, 2n_2, 2n_3, 2n_4)$ that each $2n_i \in \{4, 6, 8, 12\}$ left to check. They are as follows and are classified for the proof. (Recall that we are dealing with only the simple types, so we omit non-simple types such as $(4, 4, 6, 8)$, which is the type of $SU(3) \times Sp(2)$.)

(α)	(β)	(γ)	(δ)
$(4, 8, 8, 12)$	$(4, 6, 6, 12)$	$(4, 6, 8, 8)$	$(4, 6, 6, 6)$
	$(4, 6, 8, 12)$	$(4, 8, 8, 8)$	$(4, 6, 6, 8)$
	$(4, 6, 12, 12)$		
	$(4, 8, 12, 12)$		
	$(4, 12, 12, 12)$		

(α) is the type of $SO(8)$.

We must show that the elements of the groups (β) , (γ) and (δ) can not be the types of H -spaces. Here we also call A the type of X ambiguously.

The next proposition is evident.

PROPOSITION 5 *Let $\mathcal{P}_p^1 y_{2n} \equiv y_{2m} \pmod{D^2}$. Then by suitably choosing generators, we can regard as $\mathcal{P}_p^1 y_{2n} = y_{2m}$ without loss of generality.*

When X has p -torsion, $H^*(X, Z_p)$ is obtained by iterating the reversed simple replasement $A(x_{2np^{r-1}}) \longrightarrow A(x_{2n-1}) \otimes Z_p[x_{2n}]/(x_{2n}^{p^r})$ on the generators $x_{2np^{r-1}}$ (W. Browder [1]). So we have following proposition.

PROPOSITION 6. *If X has p -torsion, then there exist such $2m \in A$ that $m \equiv 0 \pmod{p}$. Moreover we have $2m = 2np$, $n > 1$.*

Since X is 1-connected, we have the last part.

The case (β) : Suppose one of these is the type of an H -space X . From Proposition 6, X has no 5-torsion. If

$$(*) \quad \mathcal{P}_5^1 y_4 = y_{12}$$

holds for some y_{12} , then for another y'_{12} or y_6 , we cannot find y_{2k} in Proposition 4 and this is a contradiction. So it is sufficient to show $(*)$.

For the cases $(4, 6, 6, 12)$ and $(4, 6, 8, 12)$, $\mathcal{P}_5^1(y_4) = y_{12} + S$, where $S \in I(y_{12}) \cap D^2$ for dimensional reason. So $(*)$ holds by Proposition 5.

For the cases $(4, 6, 12, 12)$, $(4, 8, 12, 12)$ and $(4, 12, 12, 12)$, we also have $(*)$ in spite of existence of two or more generators in dimension 12. For example, let $P = Z_5[y_4, y_6, y_{12}, y'_{12}]$ and

$$\mathcal{P}_5^1 y_4 \equiv \alpha y_{12} + \beta y'_{12} \pmod{D^2} \quad \alpha \neq 0,$$

$$\mathcal{P}_5^1 y_6 \equiv 0 \pmod{D^2},$$

$$\mathcal{P}_5^5 y_{12} \equiv \sum_{i=0}^5 a_i y_4 y_{12}^{5-i} y'_{12}^i \pmod{D^6}.$$

Then

$$\begin{aligned} y_{12}^5 &= \mathcal{P}_5^6 y_{12} = \mathcal{P}_5^1 \mathcal{P}_5^5 y_{12} \\ &= (\alpha y_{12} + \beta y'_{12}) \left(\sum_{i=0}^5 a_i y_4 y_{12}^{5-i} y'_{12}^i \right) \pmod{D^6} \end{aligned}$$

Comparing coefficients on both sides, we have

$$a_0 \neq 0, a_5 = a_4 = \cdots = a_1 = 0 \text{ and } \beta = 0.$$

So we have $\mathcal{P}_5^1 y_4 \equiv \alpha y_{12} \pmod{D^2}$ and $(*)$ follows from Proposition 5. We can have the same results for $(4, 8, 12, 12)$ and $(4, 12, 12, 12)$.

The case (γ) : By Propositions 6 X has no 3-torsion and just the same argument as the case (β) leads us to a contradiction to Proposition 3.

The case (δ) : If $(4, 6, 6, 6)$ is the type of X , then X has no 2-torsion by Proposition 6.

If $(4, 6, 6, 8)$ is the type of an H -space X which has 2-torsion, we have $H^*(X, Z_2) = A_{Z_2}(x_3, x'_3, x_4, x_5, x'_5)$ and it is primitively generated by dimensional reason. By the theorem of Borel, $H^*(BX, Z_2) = Z_2[y_4, y'_4, y_5, y_6, y'_6]$, $Sq^1 y'_4 = y_5$. Since $Sq^1 y_5 = Sq^1 Sq^1 y'_4 = 0$ and $Sq^2 y_5 = 0$ by dimensional reason, we have

$$y_5^2 = Sq^5 y_5 = Sq^2 Sq^1 Sq^2 y_5 + Sq^4 Sq^1 y_5 = 0$$

which is a contradiction.

Hence in both cases, X has no 2-torsion and we have $H^*(BX, Z_2) = Z_2[y_4, y_6, y'_6, y''_6]$ or $Z_2[y_4, y_6, y'_6, y_8]$. By Proposition 1 and Proposition

4, $Sq^2 y_4$ is some 6-dimensional generator, say y_6 . Then for another 6-dimensional generator y'_6 , Proposition 1 and Proposition 4 lead us to a contradiction. qed.

In S. Ochiai [4] $(3, 5, 5)$, $(3, 5, 11)$, $(3, 7, 7)$ and $(3, 11, 11)$ are not eliminated though these cannot be the types of H -spaces. We can see it by applying the methods used in the cases (δ) , (β) , (γ) and (β) respectively.

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