ON ZERO TYPE AND POSITIVE TYPE TRANSFORMATIONS WITH INFINITE INVARIANT MEASURES

By

Motosige OSAKAWA and Toshihiro HAMACHI

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§1. Introduction.

We shall consider a measure preserving invertible transformation φ defined on a sigma finite infinite measure space $(X, \mathcal{B}_X, \mu)$. As a classification of transformations which have infinite invariant measures, A.B. Hajian and S. Kakutani [5] introduced the following

DEFINITION 1. A transformation φ with an infinite invariant measure μ is of zero type if $\lim_{n \rightarrow \infty} \mu(A \cap \varphi^n A) = 0$ for every set A of positive finite measure; of positive type if $\overline{\lim}_{n \rightarrow \infty} \mu(A \cap \varphi^n A) > 0$ for every set A of positive finite measure. In [5] there is an assertion that, if a transformation with an infinite invariant measure is ergodic it is of zero type or otherwise, of positive type. U. Krengel and L. Sucheston [6] have proved that a transformation with an infinite invariant measure is of zero type if and only if it is mixing in their sense. The following definition of α -type is orally communicated to us by Prof. S. Kakutani.

DEFINITION 2. A transformation φ with an invariant measure μ is of α -type, $0 \leq \alpha \leq 1$ if $\overline{\lim}_{n \rightarrow \infty} \mu(A \cap \varphi^n A) = \alpha \mu(A)$ for every set A of finite measure.

In §2 we construct ergodic transformations with rational spectrums defined on the unit interval and show that every one of them is a homomorphic image of a transformation induced by any aperiodic transformation with a finite invariant measure. In §3 we construct an example of a transformation of α -type, $0 \leq \alpha \leq 1$, and also an example of a transformation of not any α -type. In §4 we show that on any aperiodic measure preserving transformation there exists a sky scraper (which is a transformation with infinite invariant measure) of zero type

and one of positive type. This result means that zero type property and positive type property are not invariant under a time change, which is a generalization of Chacon's change of velocity [1] in finite measure case (see [7]). The last section is devoted to another proof of a part of theorem 2 in §4.

§2. Ergodic transformation with rational spectrums.

We shall say $\{A^1, A^2, \dots, A^p; \mu; \varphi\}$ is a φ -building with height p if $\{A^1, A^2, \dots, A^p\}$ is an ordered class of mutually disjoint sets, if φ is a one to one mapping from $A^1 + A^2 + \dots + A^{p-1}$ onto $A^2 + A^3 + \dots + A^p$ such as $\varphi A^{i-1} = A^i$, $i=2, 3, \dots, p$, and if μ is a φ -invariant measure defined on $A^1 + A^2 + \dots + A^p$.

LEMMA 1 ([2]). Let φ be an aperiodic invertible measure preserving transformation of a probability space $(X, \mathcal{B}_X, \mu)$. Then, for any positive integer k , X can be represented by a disjoint sum of φ -buildings whose heights are not smaller than 2^k nor larger than 3^k .

PROOF. Let $\mathcal{C}$ be a family of all measurable subsets C with $\mu(C \cap \varphi C) = 0$. There exists a maximal element X_0 of $\mathcal{C}$ and the maximality of X_0 implies $\varphi(X - X_0 - \varphi X_0) \subset X_0$ and $\varphi^{-1}(X - X_0 - \varphi X_0) \subset \varphi X_0$. We prove this lemma by an induction. We denote $\varphi^{-2}(X - X_0 - \varphi X_0)$ by A_1 and $X_0 - A_1$ by A_2 . Then $\{A_1, \varphi A_1, \varphi^2 A_1; \mu; \varphi\}$ and $\{A_2, \varphi A_2; \mu; \varphi\}$ are φ -buildings with height 3 and 2, respectively, and $X = A_1 + \varphi A_1 + \varphi^2 A_1 + A_2 + \varphi A_2$. Let $\{A_{k\lambda}^1, A_{k\lambda}^2, \dots, A_{k\lambda}^{l_\lambda}; \mu; \varphi\}$, $\lambda \in A$ be φ -buildings, $2^k \leq l_\lambda \leq 3^k$, $k \geq 1$, and $X = \sum_{\lambda \in A} \sum_{i=1}^{l_\lambda} A_{k\lambda}^i$ (disjoint sum), where A is a finite set. We denote a transformation induced by φ on a set $X_k = \sum_{\lambda \in A} A_{k\lambda}^1$ by φ_{X_k} and apply the same method in the first part of this proof to φ_{X_k} to obtain a subset X_{k+1} of X_k such that $\mu(X_{k+1} \cap \varphi_{X_k} X_{k+1}) = 0$ and $\varphi_{X_k}(X_k - X_{k+1} - \varphi_{X_k} X_{k+1}) \subset X_{k+1}$.

Let
$$A_{k+1\lambda\mu\nu}^1 = X_{k+1} \cap A_{k\lambda}^1 \cap \varphi^{-l_\lambda}(\varphi_{X_k} X_{k+1} \cap A_{k\mu}^1) \cap \varphi^{-l_\lambda - l_\mu}(X_k - X_{k+1} - \varphi_{X_k} X_{k+1}) \cap \varphi^{-l_\lambda - l_\mu} A_{k\nu}^1,$$

and $A_{k+1\lambda\mu\nu}^i = \varphi^{i-1} A_{k+1\lambda\mu\nu}^1$, $i=2, 3, \dots, l_\lambda + l_\mu + l_\nu$, $\lambda, \nu, \mu \in A$, then $\{A_{k+1\lambda\mu\nu}^1, A_{k+1\lambda\mu\nu}^2, \dots, A_{k+1\lambda\mu\nu}^{l_\lambda + l_\mu + l_\nu}; \mu; \varphi\}$ is a φ -building with height $l_\lambda + l_\mu + l_\nu$, 2^{k+1}

$$\leq l_\lambda + l_\mu + l_\nu \leq 3^{k+1}, \lambda, \mu, \nu \in A.$$

Let
$$A_{k+1\lambda\mu}^1 = X_{k+1} \cap A_{k\lambda}^1 \cap \varphi^{-l_\lambda} [A_{k\mu}^1 \cap \{\varphi_{X_k} X_{k+1} - \varphi_{X_k}^{-1}(X_k - X_{k+1} - \varphi_{X_k} X_{k+1})\}]$$

and $A_{k+1\lambda\mu}^i = \varphi^{i-1} A_{k+1\lambda\mu}^1$, $i=2, 3, \dots$, $l_\lambda + l_\mu$, $\lambda, \mu \in A$, then $\{A_{k+1\lambda\mu}^1, A_{k+1\lambda\mu}^2, \dots, A_{k+1\lambda\mu}^{l_\lambda+l_\mu}; \mu; \varphi\}$ is a building with height $l_\lambda + l_\mu$, $2^{k+1} \leq l_\lambda + l_\mu \leq 3^k$, $\lambda, \mu \in A$. It is easy to see

$$X = \sum_{\lambda, \mu, \nu \in A} \sum_{i=1}^{l_\lambda + l_\mu + l_\nu} A_{k+1\lambda\mu\nu}^i + \sum_{\lambda, \mu \in A} \sum_{i=1}^{l_\lambda + l_\mu} A_{k+1\lambda\mu}^i.$$

We complete the proof.

LEMMA 2 ([2]). Let φ be an aperiodic invertible measure preserving transformation defined on a probability space $(X, \mathcal{B}_X, \mu)$. Then, for any $\varepsilon > 0$ and for any positive integer n there exists a set F such that $F, \varphi F, \dots, \varphi^{n-1}F$ are mutually disjoint with $\mu(X - F - \varphi F - \dots - \varphi^{n-1}F) < \varepsilon$.

PROOF. Fix a positive integer k such as $\frac{n}{2^k} < \varepsilon$ and let $\{A_{k\lambda}^1, A_{k\lambda}^2, \dots, A_{k\lambda}^{l_\lambda}; \mu; \varphi\}$, $\lambda \in A$ be a φ -building from lemma 1, which satisfy $X = \sum_{\lambda \in A} \sum_{i=1}^{l_\lambda} A_{k\lambda}^i$, $2^k \leq l_\lambda \leq 3^k$, $\lambda \in A$. Let $F = \sum_{\lambda \in A} \sum_{j=1}^{\lfloor \frac{l_\lambda}{n} \rfloor - 1} A_{k\lambda}^{jn+1}$, where $\lfloor \cdot \rfloor$ is the Gauss symbol.

Then $F, \varphi F, \dots, \varphi^{n-1}F$ are mutually disjoint and

$$\begin{aligned} \mu(X - F - \dots - \varphi^{n-1}F) &= \mu\left(\sum_{\lambda \in A} \sum_{i=\lfloor \frac{l_\lambda}{n} \rfloor + 1}^{l_\lambda} A_{k\lambda}^i\right) \\ &\leq \sum_{\lambda \in A} \left(l_\lambda - \left\lfloor \frac{l_\lambda}{n} \right\rfloor n\right) \mu(A_{k\lambda}^1) \\ &\leq n \mu\left(\sum_{\lambda \in A} A_{k\lambda}^1\right) \\ &\leq n \cdot \frac{1}{2^k} < \varepsilon. \end{aligned}$$

We shall construct an ergodic transformation T on the unit interval $I=[0, 1)$ with the Lebesgue measure m onto itself by an induction.

Let $\{n_1, n_2, \dots, n_k, \dots\}$ be an infinite sequence of positive integers and let $p_0=1$, $p_k=p_{k-1}n_k$, $k=1, 2, \dots$. Let $I_0^1=I$, $\{I_0^1; m; T_0\}$ can be seen as a building with height 1 even if T_0 has non-sense. Let $\{I_{k-1}^1, I_{k-1}^2, \dots, I_{k-1}^{p_{k-1}}; m; T_{k-1}\}$ be a building with height p_{k-1} , where $I_{k-1}^1, I_{k-1}^2, \dots, I_{k-1}^{p_{k-1}}$ are mutually disjoint intervals with equal length, $I = \sum_{i=1}^{p_{k-1}} I_{k-1}^i$ and T_{k-1} is a motion such as

$$T_{k-1}I_{k-1}^{i-1}=I_{k-1}^i, i=2, 3, \dots, p_{k-1}, k \geq 1.$$

Then, we construct a building $\{I_k^1, I_k^2, \dots, I_k^{p_k}; m; T_k\}$ with height p_k , which satisfies the following property such that T_k is a motion and

$$I_k^{p_{k-1}(j-1)+i}=T_{k-1}^{i-1}\tilde{I}_j, j=1, 2, \dots, n_k, i=1, 2, \dots, p_{k-1},$$

where $\{\tilde{I}_1, \tilde{I}_2, \dots, \tilde{I}_{n_k}\}$ is a partition of subintervals of the interval I_{k-1}^1 with equal length. It is easy to see $I = \sum_{i=1}^{p_k} I_k^i$ and $T_k x = T_{k-1} x$, for $x \in I - I_{k-1}^{p_{k-1}}$, $k \geq 1$. Define Tx by $T_k x$ for $x \in I - I_k^{p_k}$, $k \geq 1$. Since $m(I_k^{p_k}) = 1/p_k \rightarrow 0$ as $k \rightarrow \infty$, T is defined on I neglecting a Lebesgue measure zero set and the Lebesgue measure m is invariant under the transformation T .

We shall prove the ergodicity of T . Assume that T is not ergodic, then there exists a T -invariant set A such as $0 < m(A) < 1$. Let ε be a positive number smaller than $m(A)$ and $1-m(A)$. Since $\{I_k^i; i=1, 2, \dots, p_k, k=1, 2, \dots\}$ is a base of the unit interval I , there exists a set $B = \sum_{i \in A} I_k^i$ such as $m(A \triangle B) < \varepsilon$, where A is a subset of $\{1, 2, \dots, p_k\}$ for some $k \geq 1$. From $TA = A$, we have

$$A = \sum_{i=1}^{p_k} (I_k^i \cap A) = \sum_{i=1}^{p_k} T_k^{i-1}(I_k^1 \cap A)$$

and

$$m(I_k^i \cap A) = \frac{m(A)}{p_k}, \quad i=1, 2, \dots, p_k.$$

Accordingly we have

$$m(A \triangle B) = \sum_{i \in A} m(I_k^i - I_k^i \cap A) + \sum_{i \notin A} m(I_k^i \cap A)$$

$$\begin{aligned}
&= \sum_{i \in A} \frac{1-m(A)}{p_k} + \sum_{i \notin A} \frac{m(A)}{p_k} \\
&\geq \frac{1}{p_k} \sum_{i=1}^{p_k} \varepsilon = \varepsilon.
\end{aligned}$$

This is a contradiction, and so T is ergodic. We shall say T constructed as above is an ergodic transformation with rational spectrums of type $\{n_1, n_2, \dots, n_k, \dots\}$.

Let φ and ψ be invertible measure preserving transformations on sigma finite measure spaces $\{X, \mathcal{B}_X, \mu\}$, $\{Y, \mathcal{B}_Y, \nu\}$, respectively. We shall say ψ is a homomorphic image of φ if there exists a mapping θ from X onto Y such that $\theta^{-1}\mathcal{B}_Y \subset \mathcal{B}_X$, $\nu(A) = \mu(\theta^{-1}A)$, $A \in \mathcal{B}_Y$ and $\psi\theta = \theta\varphi$. θ is said to be a homomorphic mapping.

THEOREM 1. Let φ be an aperiodic invertible measure preserving transformation on a probability space $(X, \mathcal{B}_X, \mu)$. Then, for any sequence of positive integers $\{n_1, n_2, \dots, n_k, \dots\}$, and for any $\varepsilon > 0$, there exists a subset E of X such that $\mu(X-E) < \varepsilon$ and the ergodic transformation T on the unit interval I with rational spectrums of type $\{n_1, n_2, \dots, n_k, \dots\}$ is a homomorphic image of a transformation φ_E induced by φ on a set E .

PROOF. Let $\varepsilon_k = \frac{\varepsilon}{2^k}$, $k=1, 2, \dots$ and let $p_0=1$, $p_k=p_{k-1}n_k$, $k=1, 2, \dots$. Let $F_0^1=X_0=X$, then $\{F_0^1; \mu; \varphi_{X_0}\}$ can be seen as a building with height 1, even if φ_{X_0} has non-sense. If we have a building $\{F_{k-1}^1, F_{k-1}^2, \dots, F_{k-1}^{p_{k-1}}; \mu; \varphi_{X_{k-1}}\}$ with height p_{k-1} , where $\varphi_{X_{k-1}}$ is a transformation induced by φ on a set $X_{k-1} = \sum_{i=1}^{p_{k-1}} F_{k-1}^i$, $k \geq 1$, then we apply lemma 2 to $\varphi_{F_{k-1}^1}$ to obtain a subset F_k^1 of the set F_{k-1}^1 such that $F_k^1, \varphi_{F_{k-1}^1}F_k^1, \dots, \varphi_{F_{k-1}^1}^{n_k-1}F_k^1$ are mutually disjoint and

$$\mu(F_{k-1}^1 - F_k^1 - \varphi_{F_{k-1}^1}F_k^1 - \dots - \varphi_{F_{k-1}^1}^{n_k-1}F_k^1) < \frac{\varepsilon_k}{p_{k-1}}.$$

Let $F_k^{p_{k-1}(j-1)+i} = \varphi_{X_{k-1}}^{i-1} \varphi_{F_{k-1}^1}^{j-1} F_k^1$, $i=1, 2, \dots, p_{k-1}$, $j=1, 2, \dots, n_k$ and let $X_k = \sum_{i=1}^{p_k} F_k^i$, then $\{F_k^1, F_k^2, \dots, F_k^{p_k}; \mu; \varphi_{X_k}\}$ is a building with height p_k

and

$$\begin{aligned} \mu(X_{k-1} - X_k) = & \mu\left(\sum_{i=1}^{p_{k-1}} \varphi_{X_{k-1}}^{i-1}(F_{k-1}^1 - F_k^1 - \varphi_{F_{k-1}^1} F_k^1 - \dots \right. \\ & \left. - \varphi_{F_{k-1}^1}^{n_{k-1}} F_k^1)\right) \leq p_{k-1} \frac{\varepsilon_k}{p_{k-1}} = \varepsilon_k. \end{aligned}$$

Let $E = \bigcap_{k=0}^{\infty} X_k$ and $E_k^i = F_k^i \cap E$, $i=1, 2, \dots, p_k$, $k=1, 2, \dots$, then we have

$$\mu(X - E) = \mu\left(\sum_{k=1}^{\infty} (X_{k-1} - X_k)\right) \leq \sum_{k=1}^{\infty} \varepsilon_k = \varepsilon$$

and $(E_k^1, E_k^2, \dots, E_k^{p_k}; \mu; \varphi_E)$ is a buildings with height p_k , $k=1, 2, \dots$. Since the structure of the sequence of building $(E_k^1, E_k^2, \dots, E_k^{p_k}; \mu; \varphi_E)$, $k \geq 1$ is as same as the one of the sequence of buildings $(I_k^1, I_k^2, \dots, I_k^{p_k}; m; T)$, $k \geq 1$, and since $\{I_k^i; i=1, 2, \dots, p_k, k=1, 2, \dots\}$ is a base of the unit interval I , we can construct a homomorphic mapping θ from E onto I such as $T\theta = \theta\varphi_E$.

§3. α -type.

We first construct an example of a transformation of α -type, $0 \leq \alpha \leq 1$. The following result is easily verified.

LEMMA 3. For any triple of integers $1 \leq p$, $2 \leq n$, $1 \leq m \leq n-1$ and for any subset A of $\{1, 2, \dots, p\}$, define

$$A' = \sum_{j=1}^{m+1} \{2p(j-1) + i; i \in A\} + \sum_{j=2}^{n-m} \{2p[2^{j-1}(m+2) - 2] + i; i \in A\}.$$

Then the set A' is a subset of $\{1, 2, \dots, p'\}$ where $p' = 2p[2^{n-m-1}(m+2) - 1]$ and $\max_{p \leq u \leq p'} |A' + u \cap A'| = |A' + 2p \cap A'| = m|A|$, where $| \cdot |$ is a number of a set.

For a fixed α , $0 \leq \alpha \leq 1$, we chose a sequence of integers $\{n_1, m_1, n_2, m_2, \dots, n_k, m_k, \dots\}$ such as

$$\lim_k \frac{m_k}{n_k} = \alpha, \quad 1 \leq n_k, 1 \leq m_k \leq n_k - 1, k \geq 1.$$

We define $p_0 = 1$, $p_k = 2p_{k-1}[2^{n_k - m_k - 1}(m_k + 2) - 1]$, $k=1, 2, \dots$. Let $I_0^1 =$

$J_0=I$ (the unit interval), $\{I_0^1; m; T_0\}$ can be seen as a building with height 1 even if T_0 has non-sense. Let $\{I_{k-1}^1, I_{k-1}^2, \dots, I_{k-1}^{2^{p_{k-1}}}; m; T_{k-1}\}$ be a building with height $2^{p_{k-1}}$ where $\{I_{k-1}^1, I_{k-1}^2, \dots, I_{k-1}^{2^{p_{k-1}}}\}$ are mutually disjoint intervals with equal length and T_{k-1} is a motion such as

$$T_{k-1} I_{k-1}^{i-1} = I_{k-1}^i, \quad i=2, 3, \dots, p_{k-1}, k \geq 1.$$

Then we can construct a building $\{I_k^1, I_k^2, \dots, I_k^{2^{p_k}}; m; T_k\}$ with height 2^{p_k} such that, $I_k^1, I_k^2, \dots, I_k^{2^{p_k}}$ are mutually disjoint intervals with equal length,

$$\begin{aligned} I_k^{2^{p_{k-1}}(j-1)+i} &= T_{k-1}^{i-1} \tilde{I}_j, \\ j &= 1, 2, \dots, m_k, m_k+1, \quad i=1, 2, \dots, 2^{p_{k-1}}, \\ I_k^{2^{p_{k-1}}[2^{j-1}(m_k+2)-2]+i} &= T_{k-1}^{i-1} I_{m_k+j}, \\ j &= 2, 3, \dots, n_k - m_k, \quad i=1, 2, \dots, 2^{p_{k-1}}, \end{aligned}$$

where $\{\tilde{I}_1, \tilde{I}_2, \dots, \tilde{I}_{n_k}\}$ is a partition of subintervals of the interval I_{k-1}^1 with equal length, and T_k is a motion such as

$$T_k I_k^{i-1} = I_k^i, \quad i=1, 2, \dots, 2^{p_k}.$$

Define $J_k = \sum_{i=1}^{2^{p_k}} I_k^i$, $k \geq 1$, then $J_{k-1} \subset \sum_{i=1}^{p_k} I_k^i \subset J_k$, $k \geq 1$ and $T_k x = T_{k-1} x$ for $x \in J_{k-1} - I_{2^{p_{k-1}}}$, $k \geq 2$. Define $J = \bigcup_{k=0}^{\infty} J_k$ and define $Tx = T_k x$ for $x \in J_k - I_{2^{p_k}}$, $k \geq 1$, then T is a Lebesgue measure m preserving transformation defined on a set J neglecting a Lebesgue measure zero set, since $J_k \subset J_{k+1} - I_k^{2^{p_k}}$, $k \geq 1$ and $\lim_{k \rightarrow \infty} m(I_k^{2^{p_k}}) = 0$.

A proof of the ergodicity of a transformation T is as same as one of a transformation with rational spectrums in §2. Let us note that a transformation T_I induced by T on a set $I = J_0$ is an ergodic transformation with rational spectrums of type $\{n_1, n_2, \dots, n_k, \dots\}$. For any subset A of $\{1, 2, \dots, p_{k-1}\}$, define

$$\begin{aligned} A' &= \sum_{j=1}^{m_k+1} \{2^{p_{k-1}}(j-1)+i; i \in A\} \\ &\quad + \sum_{j=2}^{n_k-m_k} \{2^{p_{k-1}}[2^{j-1}(m_k+2)-2]+i; i \in A\}, \end{aligned}$$

then we have

$$\begin{aligned}
 \sum_{i \in A} I_{k-1}^i &= \sum_{i \in A} T_{k-1}^{i-1} I_{k-1}^1 \\
 &= \sum_{i \in A} \sum_{j=1}^{n_k} T_{k-1}^{i-1} \tilde{I}_j \\
 &= \sum_{i \in A} \left(\sum_{j=1}^{m_k+1} I_k^{2p_{k-1}(j-1)+i} + \sum_{j=2}^{n_k-m_k} I_k^{2p_{k-1}[2^{j-1}(m_k+2)-2]+i} \right) \\
 &= \sum_{i \in A'} I_k^i.
 \end{aligned}$$

For a set $A = \sum_{i \in A} I_{k-1}^i$, we have

$$\begin{aligned}
 \max_{p_{k-1} \leq u \leq p_k} m(A \cap T^u A) &= \max_{p_{k-1} \leq u \leq p_k} m\left(\sum_{i \in A} I_k^{i+u} \cap \sum_{i \in A} I_k^i\right) \\
 &= \max_{p_{k-1} \leq u \leq p_k} m\left(\sum_{i \in A'+u \cap A'} I_k^i\right) \\
 &= \max_{p_{k-1} \leq u \leq p_k} |A' + u \cap A'| m(I_k^1) \\
 &= m_k |A| m(\tilde{I}_1) \\
 &= \frac{m_k}{n_k} m(A),
 \end{aligned}$$

which follows from lemma 3. Accordingly we have

$$\overline{\lim}_{u \rightarrow \infty} m(T^u A \cap A) = \alpha m(A).$$

Since any Lebesgue measurable subset of J can be approximated by a set of same type as the above set A , we can show a transformation T is of α -type using the following lemma which is easily proven.

LEMMA 4. Let φ be a measure preserving transformation of a measure space $(X, \mathcal{B}_X, \mu)$, where μ is a sigma finite measure. Then we have

$$|\mu(A \cap \varphi^u A) - \mu(B \cap \varphi^u B)| \leq 2\mu(A \triangle B)$$

for any pair of sets A and B with finite measures and for any integer u .

Nextly we construct an example of a transformation of positive type, which is not any α -type, $0 < \alpha \leq 1$. Let T be a transformation of α -type defined on an infinite measure space (J, m) constructed as above in case $0 < \alpha \leq 1$, and ϕ be a mixing invertible measure preserving transformation defined on a probability space $(Y, \mathcal{B}_Y, \nu)$. Define a transformation $\hat{T}$ by

$$\begin{aligned}\hat{T}(x, y) &= (Tx, \phi y), & \text{if } x \in T^{-1}J_0, y \in Y, \\ &= (Tx, y), & \text{if } x \in J - T^{-1}J_0, y \in Y.\end{aligned}$$

Then $\hat{T}$ is an invertible measure preserving transformation defined on a direct product set $J \times Y$ with a direct product measure $m \times \nu$. Since a transformation $\hat{T}_{J_0 \times Y}$ induced by $\hat{T}_{J_0}$ on a set $J_0 \times Y$ is a direct product of an ergodic transformation T_{J_0} and a mixing transformation ϕ , $\hat{T}_{J_0 \times Y}$ and also $\hat{T}$ are ergodic transformations. Let B be a subset of Y with $\nu(B) = \frac{1}{2}$. The mixing property of ϕ implies that there exists a positive integer v_0 such as $\nu(B \cap \phi^v B) < \frac{3}{8}$ for any $v \geq v_0$ and the ergodicity of ϕ implies

$$\max\{\nu(B \cap \phi B), \nu(B \cap \phi^2 B), \dots, \nu(B \cap \phi^{v_0} B)\} = a < \nu(B).$$

We have

$$\hat{T}^u(J_0 \times B) = \{(x, y); x \in T^u J_0 \text{ and } y \in \phi^{l(x)} B\},$$

where $l(x)$ is a number of a set $\{i; 1 \leq i \leq u-1, T^i x \in T^{-1}J_0\}$, $u \geq 1$ and

$$\begin{aligned}(J_0 \times B) \cap \hat{T}^u(J_0 \times B) \\ = \{(x, y); x \in J_0 \cap T^u J_0 \text{ and } y \in B \cap \phi^{l(x)} B\}, u \geq 1.\end{aligned}$$

Therefore

$$\begin{aligned}m \times \nu((J_0 \times B) \cap \hat{T}^u(J_0 \times B)) \\ = \int_{J_0 \cap \hat{T}^u J_0} \nu(B \cap \phi^{l(x)} B) m(dx)\end{aligned}$$

$$\leq am(J_0 \cap T^u J_0), \quad u \geq 1,$$

and

$$\begin{aligned} & \overline{\lim}_{u \rightarrow \infty} m \times \nu((J_0 \times B) \cap T^u(J_0 \times B)) \\ & \leq \overline{\lim}_{u \rightarrow \infty} am(J_0 \times T^u J_0) \\ & = a\alpha m(J_0) < \alpha m \times \nu(J_0 \times B). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \overline{\lim}_{u \rightarrow \infty} m \times \nu((J_0 \times Y) \cap T^u(J_0 \times Y)) \\ & = \overline{\lim}_{u \rightarrow \infty} m \times \nu((J_0 \times Y) \cap (T^u J_0 \times Y)) \\ & = \overline{\lim}_{u \rightarrow \infty} m(J_0 \cap T^u J_0) \nu(Y) \\ & = \alpha m(J_0) \nu(Y) = \alpha m \times \nu(J_0 \times Y). \end{aligned}$$

Hence, T is not of any α -type.

§4. Zero type and positive type.

Let φ be a measure preserving transformation on a probability space $(X, \mathcal{B}_X, \mu)$, $\{X_0, X_1, \dots, X_k, \dots\}$ be a countable partition of X , namely, $X = \sum_{k=0}^{\infty} X_k$ and let $\{p_1, p_2, \dots, p_k, \dots\}$ be a sequence of positive integers. Let us construct a building $\{X_k^1, X_k^2, \dots, X_k^{p_k}; \mu_k; e_k\}$ with height P_k such as $\mu_k = \mu$ on a set $X_k = X_k^1$, for every $k \geq 1$. Define a set $\hat{X} = \sum_{k=1}^{\infty} \sum_{i=1}^{p_k} X_k^i$, a measure $\hat{\mu}$ defined on $\hat{X}$ such as $\hat{\mu} = \mu_k$ on a set $\sum_{i=1}^{p_k} X_k^i$, and a transformation $\hat{\varphi}$ on $\hat{X}$ such that

$$\begin{aligned} \hat{\varphi}x &= e_k x, & \text{if } x \in \sum_{i=1}^{p_k-1} X_k^i, \quad k \geq 1, \\ &= \varphi_{X-X_0} e_k^{p_k-1} x, & \text{if } x \in X_k^{p_k}, \quad k \geq 1. \end{aligned}$$

We note $\hat{\varphi}$ is a measure μ preserving transformation and we shall say

$\hat{\varphi}$ is a skyscraper of φ generated by $\{X_0, X_1, \dots, X_k, \dots\}$ and $\{0, p_1, \dots, p_k, \dots\}$.

We first construct an example of a transformation of zero type, which is a skyscraper of a Bernoulli shift with a countable state space. Let S be a countable set $\{1, 2, \dots, k, \dots\}$, P be a probability measure defined on a set S and let (S_i, P_i) , $i=0, 1, \dots$ be copies of (S, P) . Define a shift transformation φ on a product measure space $(X, \mu) = \prod_{i=-\infty}^{\infty} (S_i, P_i)$ such that $\varphi x = (x_{i+1})_{-\infty < i < \infty}$ for $x = (x_i)_{-\infty < i < \infty} \in X$. Denote $X_k = \{x \in X; x_0 = k\}$, $k \geq 1$, and let $p_1, p_2, \dots, p_k, \dots$ be any sequence of positive integers such as $\sum_{k=1}^{\infty} p_k P(\{k\}) = \infty$. We now consider a skyscraper $\hat{\varphi}$ of φ generated by $(X_1, X_2, \dots, X_k, \dots)$ and $(p_1, p_2, \dots, p_k, \dots)$, which is an infinite measure preserving transformation. We define a function on a set X such as $f(x) = p_k$ for $x \in X_k$, $k \geq 1$. Since $\{f(\varphi^i x); i=1, 2, \dots\}$ are mutually independent random variables with the same distribution, renewal theory [3] implies

$$\begin{aligned} \hat{\mu}(X \cap \hat{\varphi}^u X) &= \mu\left(\sum_{k=1}^{\infty} \{x \in X; \sum_{i=0}^{k-1} f(\varphi^i x) = u\}\right) \\ &\rightarrow 0 \quad \text{as } u \rightarrow \infty. \end{aligned}$$

This means $\hat{\varphi}$ is of zero type.

The following two lemmas are easily verified.

LEMMA 5. Let φ and ψ be measure preserving transformations on probability spaces $(X, \mathcal{B}_X, \mu)$, and $(Y, \mathcal{B}_Y, \nu)$, respectively. If ψ is a homomorphic image of φ by a homomorphic mapping θ and if $\hat{\psi}$ is a skyscraper of ψ generated by $(X_0, X_1, \dots, X_k, \dots)$ and $(0, p_1, \dots, p_k, \dots)$, where $X = \sum_{k=0}^{\infty} X_k$ and $p_k > 0$, $k \geq 1$, then $\hat{\psi}$ is a homomorphic image of a skyscraper $\hat{\varphi}$ of φ generated by $(\theta^{-1}X_0, \theta^{-1}X_1, \dots, \theta^{-1}X_k, \dots)$ and $(0, p_1, \dots, p_k, \dots)$.

LEMMA 6. Let φ and ψ be measure preserving transformations on infinite measure spaces $(X, \mathcal{B}_X, \mu)$, and $(Y, \mathcal{B}_Y, \nu)$, respectively. Assume that ψ is a homomorphic image of φ . Then, φ is of zero type if ψ is of zero type, and ψ is of positive type if φ is of positive type. Especially in case φ is ergodic, φ is of zero type if ψ is of zero type, and φ is of

positive type if ϕ is of positive type.

Lemma 6 follows from

$$\begin{aligned}\nu(B \cap \phi^u B) &= \mu(\theta^{-1}(B \cap \phi^u B)) \\ &= \mu(\theta^{-1} B \cap \phi^u \theta^{-1} B), \quad B \in \mathcal{B}_Y, u = 0, \pm 1, \dots,\end{aligned}$$

where θ is a homomorphic mapping from X onto Y such as $\phi\theta = \theta\phi$.

From Theorem 1, lemma 5, lemma 6 and from a construction of a transformation of α -type in the previous section, we have

THEOREM 2. Let ϕ be an ergodic measure preserving transformation defined on a probability space $(X, \mathcal{B}_X, \mu)$ without atom.

- (1) There exists a skyscraper of ϕ , which is of zero type.
- (2) There exists a skyscraper of ϕ , which is of positive type.

§5. Another proof of Theorem 2, (1).

Let ϕ be an ergodic measure preserving transformation of a probability space $(X, \mathcal{B}_X, \mu)$ without atom. We construct a countable partition $\{X_1, X_2, \dots, X_k, \dots\}$ of a set X by the induction. Let $X_0 = \phi$ and $X'_1 = X$. If we have sets $X_0, X_1, \dots, X_{k-1}, k \geq 1$, then, from lemma 1 or lemma 2 we can choose a subset X_k of a set $X'_k = X - \sum_{n=1}^{k-1} X_n$ such that $\mu(X_k \cap \phi_{X'_k} X_k) = 0$ and $\frac{1}{3}\mu(X'_k) < \mu(X_k) < \frac{1}{2}\mu(X'_k)$, where $\phi_{X'_k}$ a transformation induced by ϕ on a set X'_k . It follows that

$$\begin{aligned}\mu(X'_{k+1}) &= \mu(X'_k - X_k) \\ &< \mu(X'_k) - \frac{1}{3}\mu(X'_k) = \frac{2}{3}\mu(X'_k), \quad k \geq 1,\end{aligned}$$

and therefore $\lim_{k \rightarrow \infty} \mu(X'_k) = 0$ and $X = \sum_{k=1}^{\infty} X_k$.

LEMMA 7. Let $p_1 = 1, p_k = 2p_{k-1} + 1, k = 2, 3, \dots$. Then, for almost every x which belongs to $\sum_{n=1}^k X_n$ there exists an integer $u, 1 \leq u \leq p_k$ such that $\phi^u x \in \sum_{n=k+1}^{\infty} X_n, k \geq 1$.

PROOF. We prove by an induction. It follows that $\varphi x \in \varphi X_1 \subset \sum_{n=2}^{\infty} X_n$ for almost every $x \in X_1$. Assume that for almost every $x \in \sum_{n=1}^{k-1} X_n$ there exists an integer u , $1 \leq u \leq p_{k-1}$ such as $\varphi^u x \in \sum_{n=k}^{\infty} X_n$, $k \geq 2$. In case of $x \in X_k$, φx belongs to $\sum_{n=k+1}^{\infty} X_n$ or $\sum_{n=1}^{k-1} X_n$. If $\varphi x \in \sum_{n=1}^{k-1} X_n$, then, from the assumption and a definition of X_k there exists an integer u , $1 \leq u \leq p_{k-1}$ such that $\varphi^u \varphi x \in X'_k \cap \varphi_{X'_k} X_k \subset X'_{k+1}$, and hence $\varphi^{u+1} x \in \sum_{n=k+1}^{\infty} X_n$, $1 \leq u + 1 \leq p_k$. In case of $x \in \sum_{n=1}^{k-1} X_n$, from the assumption there exists an integer u_1 , $1 \leq u_1 \leq p_{k-1}$ such that $\varphi^{u_1} x \in \sum_{n=k}^{\infty} X_n$, that is, $\varphi^{u_1} x \in \sum_{n=k+1}^{\infty} X_n$ or $\varphi^{u_1} x \in X_k$. If $\varphi^{u_1} x \in X_k$, then from the previous calculation of the case of $x \in X_k$, there exists an integer u_2 , $1 \leq u_2 \leq p_{k-1}$ such that $\varphi^{u_2+1} \varphi^{u_1} x \in \sum_{n=k+1}^{\infty} X_n$, that is, $\varphi^{u_1+u_2+1} x \in \sum_{n=k+1}^{\infty} X_n$, $1 \leq u_1 + u_2 + 1 \leq p_k$.

Let

$$q_1 = 1, \quad q_k = q_{k-1} + q_{k-1}^2 p_{k-1}, \quad k = 2, 3, \dots,$$

and

$$q_{kj} = q_k + (j-1)q_k p_k, \quad j = 1, 2, \dots, q_k, \quad k = 1, 2, \dots.$$

Let $\{X_{k1}, X_{k2}, \dots, X_{kq_k}\}$ be a partition of a set X_k such as $\mu(X_{kj}) = \frac{\mu(X_k)}{q_k}$, $j = 1, 2, \dots, q_k$, $k \geq 1$, and let $\{X_{kj}^1, X_{kj}^2, \dots, X_{kj}^{q_{kj}}; \mu_{kj}; e_{kj}\}$ be a building with height q_{kj} such that $\mu_{kj} = \mu$ on a set $X_{kj}^1 = X_{kj}$ and

$$e_{kj} X_{kj}^{i-1} = X_{kj}^i, \quad i = 2, 3, \dots, q_{kj}, \quad j = 1, 2, \dots, q_k, \quad k = 1, 2, \dots.$$

Define a set $\hat{X} = \sum_{k=1}^{\infty} \sum_{j=1}^{q_k} \sum_{i=1}^{q_{kj}} X_{kj}^i$, a measure $\hat{\mu}$ defined on $\hat{X}$ such that $\hat{\mu} = \mu_{kj}$ on $\sum_{i=1}^{q_{kj}} X_{kj}^i$, $j = 1, 2, \dots, q_k$, $k = 1, 2, \dots$, and a transformation $\hat{\varphi}$ of $\hat{X}$ such that

$$\hat{\varphi} x = e_{kj} x, \quad \text{if } x \in \sum_{i=1}^{q_{kj}-1} X_{kj}^i, \quad j = 1, 2, \dots, q_k, \quad k = 1, 2, \dots,$$

$$= \varphi e_{kj}^{-q_{kj}+1} x, \quad \text{if } x \in X_{kj}^{q_{kj}}, j=1, 2, \dots, p_k, k=1, 2, \dots.$$

LEMMA 8.

$$(1) \quad \hat{\varphi}^u \sum_{i=1}^{q_k} X_{kj}^i = \sum_{i=1+u}^{q_k+u} X_{kj}^i$$

for $0 \leq u \leq (j-1)q_k p_k, j=1, 2, \dots, q_k, k=1, 2, \dots,$

$$(2) \quad \hat{\varphi}^u \sum_{i=1}^{q_k} X_{kj}^i \subset \sum_{n=k+1}^{\infty} \sum_{m=1}^{q_n} \sum_{i=1}^{u-(j-1)q_k p_k} X_{nm}^i$$

for $j q_k p_k \leq u \leq q_{k+1}, j=1, 2, \dots, q_k, k=1, 2, \dots,$

$$(3) \quad \hat{\varphi}^{q_k} \left(\sum_{n=1}^{\infty} \sum_{j=1}^{q_n} X_{nj}^1 \right) \subset \sum_{n=k}^{\infty} \sum_{j=1}^{q_n} \sum_{i=1}^{q_k} X_{nj}^i, \quad k=1, 2, \dots.$$

PROOF. It is easy to see (1) and (2) which follows from lemma 7. We prove (3) by the induction. Assume (3) for k , then we have

$$\begin{aligned} \hat{\varphi}^{q_{k+1}} \left(\sum_{n=1}^{\infty} \sum_{j=1}^{q_n} X_{nj}^1 \right) &= \hat{\varphi}^{q_{k+1}-q_k} \hat{\varphi}^{q_k} \left(\sum_{n=1}^{\infty} \sum_{j=1}^{q_n} X_{nj}^1 \right) \\ &\subset \hat{\varphi}^{q_{k+1}-q_k} \left(\sum_{n=k}^{\infty} \sum_{j=1}^{q_n} \sum_{i=1}^{q_k} X_{nj}^i \right) \\ &= \hat{\varphi}^{q_k^2 p_k} \left(\sum_{j=1}^{q_k} \sum_{i=1}^{q_k} X_{nj}^i \right) + \hat{\varphi}^{q_{k+1}-q_k} \left(\sum_{n=k+1}^{\infty} \sum_{j=1}^{q_n} \sum_{i=1}^{q_k} X_{nj}^i \right) \\ &\subset \sum_{n=k+1}^{\infty} \sum_{j=1}^{q_n} \sum_{i=1}^{q_k^2 p_k} X_{nj}^i + \sum_{n=k+1}^{\infty} \sum_{j=1}^{q_n} \sum_{i=1+q_{k+1}-q_k}^{q_{k+1}} X_{nj}^i \\ &\subset \sum_{n=k+1}^{\infty} \sum_{j=1}^{q_n} \sum_{i=1}^{q_{k+1}} X_{nj}^i, \quad k=1, 2, \dots. \end{aligned}$$

The above lemma implies that, for $q_k + (j-1)q_k p_k \leq u \leq q_k + j q_k p_k, j=1, 2, \dots, q_k, k=1, 2, \dots,$

$$\begin{aligned} \hat{\varphi}^u \left(\sum_{n=1}^{\infty} \sum_{j=1}^{q_n} X_{nj}^1 \right) &\subset \hat{\varphi}^{u-q_k} \left(\sum_{n=k}^{\infty} \sum_{j=1}^{q_n} \sum_{i=1}^{q_k} X_{nj}^i \right) \\ &= \hat{\varphi}^{u-q_k} \left(\sum_{m=1}^{j-1} \sum_{i=1}^{q_k} X_{km}^i + \sum_{i=1}^{q_k} X_{kj}^i + \sum_{m=j+1}^{q_k} \sum_{i=1}^{q_k} X_{km}^i \right) \end{aligned}$$

$$\begin{aligned}
& + \hat{\phi}^{u-q_k} \left(\sum_{n=k+1}^{\infty} \sum_{j=1}^{q_n} \sum_{i=1}^{q_k} X_{nj}^i \right) \\
\subset & \sum_{m=1}^{j-1} \sum_{n=k+1}^{\infty} \sum_{i=1}^{u-q_k-(j-1)q_k p_k} X_{nm}^i + \hat{\phi}^{u-q_k} \sum_{i=1}^{q_k} X_{kj}^i \\
& + \sum_{m=j+1}^{q_k} \sum_{i=1+u-q_k}^u X_{km}^i + \sum_{n=k+1}^{\infty} \sum_{j=1}^{q_n} \sum_{i=1+u-q_k}^u X_{nj}^i.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \hat{\phi}^u \left(\sum_{n=1}^{\infty} \sum_{j=1}^{q_n} X_{nj}^1 \right) \cap \left(\sum_{n=1}^{\infty} \sum_{j=1}^{q_n} X_{nj}^1 \right) \\
\subset & \sum_{n=k+1}^{\infty} \sum_{j=1}^{q_n} X_{nj}^1 + \hat{\phi}^{u-q_k} \left(\sum_{i=1}^{q_k} X_{kj}^i \right)
\end{aligned}$$

for $q_k + (j-1)q_k p_k \leq u \leq q_k + jq_k p_k$, $j=1, 2, \dots, q_k$, $k=1, 2, \dots$,

and

$$\begin{aligned}
& \hat{\mu} \left(\left(\sum_{n=1}^{\infty} \sum_{j=1}^{q_n} X_{nj}^1 \right) \cap \hat{\phi}^u \left(\sum_{n=1}^{\infty} \sum_{j=1}^{q_n} X_{nj}^1 \right) \right) \\
& \leq \hat{\mu} \left(\sum_{n=k+1}^{\infty} X_n \right) + q_k \mu(X_{kj}) \\
& = \mu \left(\sum_{n=k}^{\infty} X_n \right),
\end{aligned}$$

for $q_k \leq u \leq q_{k+1}$, $k=1, 2, \dots$.

Accordingly we have

$$\overline{\lim}_{u \rightarrow \infty} \hat{\mu} \left(\left(\sum_{k=1}^{\infty} \sum_{j=1}^{q_k} X_{kj}^1 \right) \cap \hat{\phi}^u \left(\sum_{k=1}^{\infty} \sum_{j=1}^{q_k} X_{kj}^1 \right) \right) = 0,$$

that is, $\hat{\phi}$ is of zero type.

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