TOPOLOGICAL ENTROPY OF A NON-IRREDUCIBLE INTRINSIC MARKOV SHIFT

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§0. Introduction.

In [2], W. Parry has introduced intrinsic Markov shifts and proved that the topological entropy of an irreducible intrinsic Markov shift is equal to the maximum eigen value of its structure Matrix. In case of a non-irreducible Markov shift we show that its topological entropy is the maximum of topological entropies of its irreducible parts.

§1. Definitions and Theorem.

Let $S = \{0, 1, \dots, s-1\}$ be a state space and $s \times s$ -matrix $M = (m(u, v))_{u, v=0, 1, \dots, s-1}$ be a structure matrix whose elements are all 0 or 1.

DEFINITION 1. For a subset C of S and an integer $n > 0$, an ordered pair $(u_0, u_1, \dots, u_{n-1})$ is called to be a M -compatible n -chain on C with the starting point u_0 and the terminal point u_{n-1} if $u_i \in C$ $0 \leq i \leq n-1$ and $m(u_0, u_1) = m(u_1, u_2) = \dots = m(u_{n-2}, u_{n-1}) = 1$. Let $C(u, v; n)$ be the set of all M -compatible n -chains on C with the starting point u and the terminal point v , whose elements are called to be of $C(u, v; n)$ type, $\sigma_C(u, v; n)$ be its cardinal number and $\phi_C^n = \sum_{u \in C} \sum_{v \in C} \sigma_C(u, v; n)$. In case of $C = S$, we omit the index C of $\sigma_C(u, v; n)$ and ϕ_C^n . It is trivial that $M^n = (\sigma(u, v; n))_{u, v=0, 1, \dots, s-1}$.

Let S_i be a copy of S ($-\infty < i < \infty$), $X = \prod_{i=-\infty}^{\infty} S_i$ be two sided infinite countable product space and the mapping T on X be a shift on X . Namely, $(Tx)_n = x_{n+1}$ for $x = (x_n)_{-\infty < n < \infty} \in X$. T is a homeomorphism under the product topology on X where the topology of each S_i is discrete.

DEFINITION 2. $X \ni (x_n)_{-\infty < n < \infty}$ is a M -compatible chain if for each $m (-\infty < m < \infty)$ and $n > 0$, $(x_m, x_{m+1}, \dots, x_{m+n-1})$ is a M -compatible n -chain.

Let X_M be the set of all M -compatible chains and then it is a T -invariant closed subset of X . Dynamical system (T, X_M) is called to be an intrinsic Markov shift generated by M . As is well-known, the topological entropy $h(T, X_M)$ of (T, X_M) is given as follows [1]

$$h(T, X_M) = \lim_{n \rightarrow \infty} \frac{\log \phi_n}{n}.$$

For an intrinsic Markov shift (T, X_M) , its state space S can be divided into two disjoint sets S_0 and S_+ such that $S_0 = \{u \in S; \sigma(u, u; n) = 0 \text{ for all } n \geq 1\}$ and $S_+ = S - S_0$.

DEFINITION 3. For two states u and v of S_+ , u is equivalent to v if $\sigma(u, v; n) > 0$ and $\sigma(v, u; m) > 0$ for some $n > 0$ and $m > 0$.

Let $C_1, C_2, \dots, C_p$ be all equivalent classes in S_+ , their cardinal numbers be $l_1, l_2, \dots, l_p$, respectively, $M_{C_i} = (m(u, v))_{u, v \in C_i}$ and $X_{M_{C_i}}$ be the set of all M_{C_i} -compatible chains. It is easy to show that each $X_{M_{C_i}}$ is a T -invariant closed subset of X_M and each intrinsic Markov shift $(T, X_{M_{C_i}})$ is irreducible.

Then we have the following

THEOREM.

$$h(T, X_M) = \max_{1 \leq i \leq p} h(T, X_{M_{C_i}})$$

PROOF. The set of all M -compatible n -chains is the set

$$\sum_3 \sum_2 \sum_1 \left\{ \begin{array}{l} M\text{-compatible } n\text{-chains consisting of states} \\ \{t_{j_1}, t_{j_2}, \dots, t_{j_k}\} \text{ in } S_0 \text{ and of } C_{i_l}(u_{i_l}, v_{i_l}; r_l) \\ \text{type } M\text{-compatible chains } (1 \leq l \leq h) \end{array} \right\}$$

where $\sum_1$ is a sum of all M -compatible n -chains consisting of fixed $t_{j_m} \in S_0 (1 \leq m \leq k)$ and $C_{i_l}(u_{i_l}, v_{i_l}; r_l)$ type chains with fixed states u_{i_l} and v_{i_l} which satisfies $r_1 + \dots + r_k + h = n$ ($1 \leq l \leq h$), $\sum_2$ is a sum with respect to the choice of starting points u_{i_l} and terminal points v_{i_l} in each fixed C_{i_l} and $\sum_3$ is a sum with respect to the choice of states $t_{j_m} \in S_0$ with order and the sets C_{i_l} . It is easy to show that

$$\begin{aligned}
\text{number of } \Sigma_1 &\leq_{n-h-1} C_{k-1}, \\
\text{number of } \Sigma_2 &\leq l_{i_1}^2, l_{i_2}^2 \dots l_{i_k}^2 \\
\text{and number of } \Sigma_3 &\leq (p+q)^{p+q}.
\end{aligned}$$

Here the last inequality is derived from the fact that each state $t_{j_m} \in S_0$ and $C_i(u_{i_l}, v_{i_l}; r_l)$ -type chain in a M -compatible n -chain belongs to it at most once. We put $H = \max_{1 \leq i \leq p} h(T, X_{M_{C_i}})$. There exists for any $\varepsilon > 0$, an integer N such that

$$\frac{1}{n} \log \phi_{C_i}^n < H + \varepsilon \quad \text{for all } n \geq N \text{ and } 1 \leq i \leq p.$$

We put $B = \max_{\substack{1 \leq n \leq N-1 \\ 1 \leq i \leq p}} \phi_{C_i}^n$. Then it follows that

$$\begin{aligned}
\phi^n &= \Sigma_3 \Sigma_2 \Sigma_1 \sigma_{C_{i_1}}(u_{i_1}, v_{i_1}; r_1) \dots \sigma_{C_{i_k}}(u_{i_k}, v_{i_k}; r_{i_k}) \\
&\leq \Sigma_3 \Sigma_2 \Sigma_1 \phi_{C_{i_1}}^{r_{i_1}} \dots \phi_{C_{i_k}}^{r_{i_k}} \\
&\leq \Sigma_3 \Sigma_2 \Sigma_1 C_{k-1} B^k e^{(r_1 + \dots + r_k)(H + \varepsilon)} \\
&\leq \Sigma_3 (l_{i_1}^2 \dots l_{i_k}^2) n^{k-1} B^k e^{(n-h)(H + \varepsilon)} \\
&\leq (p+q)^{p+q} (l_1 l_2 \dots l_p)^2 n^p B^p e^{n(H + \varepsilon)}, \\
\frac{\log \phi^n}{n} &\leq \frac{\log(p+q)^{p+q} (l_1 l_2 \dots l_p)^2 B^p}{n} + \frac{p \log n}{n} + H + \varepsilon
\end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \phi^n \leq H + \varepsilon.$$

Since ε is arbitrary,

$$h(T, X_M) = \lim_n \frac{1}{n} \log \phi^n \leq H.$$

The converse inequality is trivial.

REMARK. In another W. Parry's paper [3], he showed that there exists for a regionally transitive symbolic dynamics, an invariant

measure defined on it which its metrical entropy attains its topological entropy using Theorem (10) [2]. But we remark that the above result holds without the assumption of regional transitivity from our Theorem.

References

- [1] R. Adler, A. Konheim and M. McAndrew; Topological entropy. Trans. Amer. Math. Soc., Vol. 114 (1965), 309-319.
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- [3] ———; Symbolic dynamics and transformations of the unit interval. Trans. Amer. Math. Soc., Vol. 122 (1966), 368-378.