A REMARK ON THE TOPOLOGICAL ENTROPY

By

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(Received Apr. 30, 1971)

R. Adler, A. Konheim and M. McAndrew [1] defined the topological entropy of topological dynamical systems and proposed some conjectures on it. In this note we will prove that their last conjecture "5" is true. We will discuss the problem in a little more general form.

Let (X, φ) be a dynamical system where X is a compact Hausdorff space and φ is a continuous mapping from X onto itself. If A is a closed invariant subset of X , we can define a subsystem (A, φ) as the restriction of φ to A . We are concerned with a dynamical system which satisfy the following two assumptions.

Assumption (I). There exists a partition $\{X_\lambda; \lambda \in A\}$ of X consisting of mutually disjoint closed invariant subsets X_λ , $\lambda \in A$.

Let π be the canonical map from X onto A such that $\pi(x) = \lambda$ if $x \in X_\lambda$ and $X/\sim = A$ be the quotient space induced by π .

Assumption (II). The quotient space A is a Hausdorff space.

We define the topological entropy of a dynamical system (X, φ) according to [1]. Let A be a closed invariant subset of X and $\mathfrak{A}$ be an open cover of A . Let $N_A(\mathfrak{A})$ denote the minimal cardinals of open subcovers of $\mathfrak{A}$ enveloping A . We set open covers $\bigvee_{i=0}^{n-1} \varphi^{-i} \mathfrak{A} = \{\bigcap_{i=0}^{n-1} \varphi^{-i} A_{j_i}; A_{j_i} \in \mathfrak{A}, 0 \leq i \leq n-1\}$, $n \geq 1$, of A . It is known that the limit

$$h(\varphi, A; \mathfrak{A}) = \lim_{n \rightarrow \infty} \frac{1}{n} \log N_A\left(\bigvee_{i=0}^{n-1} \varphi^{-i} \mathfrak{A}\right)$$

exists. Then we define the topological entropy of the dynamical system (A, φ) by

$$h(\varphi, A) = \sup h(\varphi, A; \mathfrak{A})$$

where the supremum is taken over all open covers $\mathfrak{A}$ of A .

THEOREM. Under the assumption (I) and (II), we have

$$h(\varphi, X) = \sup_{\lambda \in A} h(\varphi, X_\lambda).$$

For proving this theorem we will use the following

LEMMA ([1]). Let $X_i, i=1, 2$, be closed invariant subsets of a dynamical system (X, φ) such that $X = X_1 \cup X_2$. Then, for any open cover $\mathfrak{U}$ of X , we have

$$h(\varphi, X; \mathfrak{U}) = \max\{h(\varphi, X_1; \mathfrak{U}), h(\varphi, X_2; \mathfrak{U})\}.$$

PROOF OF THEOREM. Given $\varepsilon > 0$, there exists an open cover $\mathfrak{U}$ of X such that

$$h(\varphi, X) - \varepsilon \leq h(\varphi, X; \mathfrak{U}).$$

Fix any $\lambda \in A$. There exists an integer $k > 0$ such that

$$\frac{1}{k} \log N_{X_\lambda}(\bigvee_{i=0}^{k-1} \varphi^{-i} \mathfrak{U}) \leq h(\varphi, X_\lambda; \mathfrak{U}) + \varepsilon.$$

Let $\{A_1, \dots, A_n\}$ be an open subcover of $\mathfrak{U}$ envelopping X_λ such that $n = N_{X_\lambda}(\bigvee_{i=0}^{k-1} \varphi^{-i} \mathfrak{U})$ and let $0_\lambda = \bigcup_{j=1}^n A_j$. We put $\mathcal{F} = \{F \subset A; F \text{ is a closed neighborhood in } A\}$. Since $0_\lambda \cap \bigcap_{F \in \mathcal{F}} \pi^{-1}(F) = 0_\lambda \cap X_\lambda = \emptyset$, there exists a finite collection $\{F_1, \dots, F_m\} \subset \mathcal{F}$ such that $0_\lambda \cap \bigcap_{j=1}^m \pi^{-1}(F_j) = \emptyset$ appealing to the finite intersection property of $\mathcal{F}$. Let U_λ be the set of all interior points of $\bigcap_{j=1}^m \pi^{-1}(F_j)$, then $\bar{U}_\lambda = \bigcap_{j=1}^m \pi^{-1}(F_j)$ and

$$N_{\bar{U}_\lambda}(\bigvee_{i=0}^{k-1} \varphi^{-i} \mathfrak{U}) = N_{X_\lambda}(\bigvee_{i=0}^{k-1} \varphi^{-i} \mathfrak{U}).$$

For any integer $r = pk + q$ where p and q are integers such that $p > 0$ and $0 \leq q < k$, we have

$$\frac{1}{r} \log N_{\bar{U}_\lambda}(\bigvee_{i=0}^{r-1} \varphi^{-i} \mathfrak{U}) \leq \frac{p+1}{p} \cdot \frac{1}{(p+1)k} \log N_{\bar{U}_\lambda}(\bigvee_{i=0}^{k(p+1)-1} \varphi^{-i} \mathfrak{U})$$

$$\begin{aligned}
&\leq \frac{p+1}{p} \cdot \frac{1}{k} \log N_{\bar{U}_\lambda} \left(\bigvee_{i=0}^{k-1} \varphi^{-i} \mathfrak{U} \right) \\
&= \frac{p+1}{p} \cdot \frac{1}{k} \log N_{X_\lambda} \left(\bigvee_{i=0}^{k-1} \varphi^{-i} \mathfrak{U} \right) \\
&< \frac{p+1}{p} (h(\varphi, X_\lambda; \mathfrak{U}) + \varepsilon),
\end{aligned}$$

hence

$$h(\varphi, \bar{U}_\lambda; \mathfrak{U}) \leq h(\varphi, X_\lambda; \mathfrak{U}) + \varepsilon.$$

Since X is compact, there exists a finite set $\{\lambda_1, \dots, \lambda_s\}$ such that $\bigcup_{j=1}^s U_{\lambda_j} = X$. By the lemma,

$$h(\varphi, X; \mathfrak{U}) = \max_{1 \leq j \leq s} h(\varphi, \bar{U}_{\lambda_j}; \mathfrak{U}),$$

and hence

$$\begin{aligned}
h(\varphi, X; \mathfrak{U}) &\leq \max_{1 \leq j \leq s} h(\varphi, X_{\lambda_j}; \mathfrak{U}) + \varepsilon \\
&\leq \sup_{\lambda \in A} h(\varphi, X_\lambda) + \varepsilon.
\end{aligned}$$

Therefore we have

$$h(\varphi, X) - \varepsilon \leq \sup_{\lambda \in A} h(\varphi, X_\lambda) + \varepsilon,$$

and so

$$h(\varphi, X) \leq \sup_{\lambda \in A} h(\varphi, X_\lambda)$$

because ε is arbitrary. The converse inequality is trivial. Thus the proof is complete.

COROLLARY (conjecture “5” of [1]). Let X and Y be compact Hausdorff spaces and $\{\varphi_x; x \in X\}$ be a family of homeomorphisms of Y such that $\varphi: (x, y) \rightarrow (x, \varphi_x y)$ is a continuous mapping from $X \times Y$ onto itself. Then we have

$$h(\varphi, X \times Y) = \sup_{x \in X} h(\varphi_x, Y).$$

REMARK. We don't know if the maximum of $h(\varphi, X_\lambda)$, $\lambda \in \mathcal{A}$, (or $h(\varphi_x, Y)$, $x \in X$) is attained in the theorem (or the corollary) and also if the theorem is true without the assumption (II).

Reference

- [1] R. Adler, A. Konheim and M. McAndrew; Topological entropy. Trans. Amer. Math. Soc., 114 (1965), 309-319.