PERIODIC SOLUTIONS OF SOME SYSTEMS OF SEMILINEAR PARTIAL DIFFERENTIAL EQUATIONS IN COMPLEX DOMAINS

By

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Introduction

In this paper, we consider a system of semi-linear partial differential equations;

$$(0.1) \quad \begin{cases} \frac{\partial u_j}{\partial x_j} = F_j(x, u) \\ u_j = g_j \quad \text{where } x_j = 0 \\ (j=1, 2, \dots, n) \end{cases}$$

where u_j , F_j , and g_j ($j=1, 2, \dots, n$) are, respectively, different dimensional vector-valued holomorphic functions defined on appropriate domains in complex Euclidean spaces.

Our purpose is to give a sufficient condition that (0.1) has a periodic solution (with respect to x_n), and extend in part the results of [1] [2] in which J. Kajiwara treated periodic solutions of linear systems and studied singularity of solution of following semilinear and singular system;

$$(0.2) \quad \begin{cases} \frac{\partial u_j}{\partial x_j} = F_j(x, u) \\ x_n \frac{\partial u_n}{\partial x_n} = F_n(x, u) \\ u_j = 0 \quad \text{at } x_j = 0 \\ (j=1, 2, \dots, n-1). \end{cases}$$

As an application of the results for (0.1), we also give a sufficient condition that the following semi-linear and singular system has a single valued holomorphic solution in a neighborhood of the origin:

$$(0.3) \quad \begin{cases} \frac{\partial u_j}{\partial x_j} = F_j(x, u) \\ x_n^m \frac{\partial u_n}{\partial x_n} = F_n(x, u) \\ u_j = g_j \quad \text{at } x_j = 0 \\ (j=1, 2, \dots, n-1) \end{cases}$$

where m is a positive integer.

§1. Results of the linear case.

About the Riemann's problem of the system of linear partial differential equations, J. Kajiwara [1] [2] showed the following theorems and others. Our argument is based on his these results.

THEOREM 1.1. *Let Ω_j be a simply connected domain in the complex plane $\mathbb{C}$ which contains the origin, for $j=1, 2, \dots, n$. Let A_{jk} and A_j be, respectively, (p_j, p_k) and $(p_j, 1)$ matrix-valued holomorphic functions in $\Omega = \Omega_1 \times \Omega_2 \times \dots \times \Omega_n$, for $j, k=1, 2, \dots, n$, and $g_j(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ be a $(p_j, 1)$ matrix-valued holomorphic function in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{j-1} \times \Omega_{j+1} \times \dots \times \Omega_n$, for $j=1, 2, \dots, n$.*

Then, the Riemann's problem

$$(1.1) \quad \begin{cases} \frac{\partial u_j}{\partial x_j}(x) = \sum_{k=1}^n A_{jk}(x) u_k(x) + A_j(x) \\ u_j(x) = g_j \quad \text{at } x_j = 0 \\ (j=1, 2, \dots, n) \end{cases}$$

has a unique holomorphic solution $u(x) = (u_1(x), \dots, u_n(x))$ in Ω , where $u_j(x)$ is a $(p_j, 1)$ matrix-valued holomorphic function in Ω .

Furthermore, the solution is represented in such a way that

$$(1.2) \quad u_j(x) = \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ k \in \{\alpha_1, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} \int_0^{x_{\alpha_1}} \dots \int_0^{x_{\alpha_l}} K_{k; \alpha_1 \alpha_2 \dots \alpha_l}^j (x; s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) g_k(x_1, x_2, \dots, x_{\alpha_1-1}, s_{\alpha_1}, x_{\alpha_1+1}, \dots, x_{\alpha_l-1}, s_{\alpha_l}, x_{\alpha_l+1}, \dots, x_n) ds_{\alpha_1} \dots ds_{\alpha_l} + P_j(x) + R_j(x) g_j(x),$$

where

$K_{k; \alpha_1, \alpha_2, \dots, \alpha_l}^j$ is a (p_j, p_k) matrix-valued holomorphic function in $\Omega \times \Omega_{\alpha_1} \times \Omega_{\alpha_2} \times \dots \times \Omega_{\alpha_l}$ which is determined only by $\{A_{jk}\}_{j,k=1,2,\dots,n}$, for $k, j=1, 2, \dots, n$,

$$(1.3) \quad R_j(x) = \exp\left(\int_0^{x_j} A_{jj}(x_1, x_2, \dots, x_{j-1}, \lambda_j, x_{j+1}, \dots, x_n) d\lambda_j\right),$$

that is, each R_j is the holomorphic solution of the system

$$\begin{cases} \frac{\partial R_j}{\partial x_j} = A_{jj} R_j \\ R_j = I \text{ at } x_j = 0 \end{cases}$$

where I is the unit (p_j, p_j) matrix and P_j is the unique holomorphic solution of the Riemann's problem in Ω ;

$$(1.4) \quad \begin{cases} \frac{\partial P_j}{\partial x_j} = \sum_{k=1}^n A_{jk} P_k + A_j \\ P_j(x) = 0 \text{ at } x_j = 0 \\ (j=1, 2, \dots, n). \end{cases}$$

THEOREM 1.2. Let Ω_j be a simply connected domain in $\mathbb{C}$, which contains the origin, for $j=1, 2, \dots, n$, and Ω_n be invariant under the translation; $x_n \rightarrow x_n + b$ for $x_n \in \Omega_n$. Let A_{jk} and A_j be, respectively, (p_j, p_k) , $(p_j, 1)$ matrix-valued holomorphic function in $\Omega = \Omega_1 \times \Omega_2 \times \dots \times \Omega_n$ for $j, k=1, 2, \dots, n$, which have a period b with respect to the last variable x_n , and g_j a $(p_j, 1)$ matrix-valued holomorphic function in $\Omega'_j = \Omega_1 \times \Omega_2 \times \dots \times \Omega_{j-1} \times \Omega_{j+1} \times \dots \times \Omega_n$ for $j=1, 2, \dots, n-1$, which has a period b with respect to x_n .

We assume

$$(1.5) \quad \det \left(\exp \int_0^b A_{nn}(x_1, \dots, x_{n-1}, \lambda_n) d\lambda_n - I \right) \neq 0$$

in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{n-1}$ where we denote the unit (p_n, p_n) matrix by I . Then the problem

$$(1.6) \quad \begin{cases} \frac{\partial u_j}{\partial x_j} = \sum_{k=1}^n A_{jk} u_k + A_j & (j=1, 2, \dots, n) \\ u_j(x) = g_j(x) \text{ at } x_j = 0 & (j=1, 2, \dots, n-1) \end{cases}$$

has a unique holomorphic solution $u(x) = (u_1(x), \dots, u_n(x))$ which has a period b with respect to x_n , where $u_j(x)$ is a $(p_j, 1)$ matrix-valued holomorphic function as well as Theorem 1.1.

Especially, $g_n(x) = u_n(x_1, x_2, \dots, x_{n-1}, 0)$ is determined as the unique holomorphic solution of the integral equation of the second Volterra type; For $x_n = b$

$$\begin{aligned} & g_n(x) + \sum_{\substack{1 \leq l \leq n-1 \\ n \notin \{\alpha_1, \dots, \alpha_l\}}} \int_0^{x_{\alpha_1}} \dots \int_0^{x_{\alpha_l}} (R_n(x) - I)^{-1} \\ & K_{n; \alpha_1, \dots, \alpha_l}^n(x; s_{\alpha_1}, \dots, s_{\alpha_l}) g_n(x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_l}, \dots, x_{n-1}) ds_{\alpha_1} \dots ds_{\alpha_l} \\ & = (I - R_n)^{-1} P_n(x) + \sum_{\substack{1 \leq l \leq n-1 \\ k \notin \{\alpha_1, \dots, \alpha_l\} \\ k \neq n}} \int_0^{x_{\alpha_1}} \dots \int_0^{x_{\alpha_l}} (I - R_n)^{-1} \\ & K_{k; \alpha_1, \dots, \alpha_l}^n(x; s_{\alpha_1}, \dots, s_{\alpha_l}) g_k(x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_l}, \dots, x_n) ds_{\alpha_1} \dots ds_{\alpha_l} \end{aligned}$$

where $(I - R_n)^{-1}$ is the inverse matrix of $(I - R_n)$.

Remark. We note that if A_{nn} is a $(1,1)$ matrix-valued function, namely, a scalar-valued function, the condition (1.5) is equivalent to

$$(1.5)' \quad \frac{1}{2\pi i} \int_0^b A_{nn}(x_1, \dots, x_{n-1}, \lambda_n) d\lambda_n \neq \text{integer}.$$

§2. Local solution of the non-linear case.

In this section, we prove the existence and uniqueness of the local

solution of the non-linear case. (cf. Kajiwara [1]) The following Lemma is well-known, so the proof is omitted.

LEMMA 2.1. *Let S be a founded domain in $\mathbb{C}^m$ and Ω be a polydisc in Ω^n . For domains $S' \subseteq S$ and $\Omega' \subseteq \Omega$, there exists a positive number $M = M(\Omega, S, \Omega', S')$ such that for every holomorphic function $f(x, s)$ in $\overline{\Omega} \times \overline{S}$, there exist holomorphic functions $g_j(x, y, s)$ ($j=1, 2, \dots, n$) in $\Omega \times \Omega \times S$ which satisfy;*

$$(2.1) \quad f(x, s) - f(y, s) = \sum_{j=1}^n g_j(x, y, s)(x_j - y_j)$$

in $\Omega \times \Omega \times S$ and

$$|g_j(x, y, s)| \leq M \|f\|$$

in $\Omega' \times \Omega' \times S'$ for $j=1, 2, \dots, n$ where

$$\|f\| = \sup_{\substack{x \in \Omega \\ s \in S}} |f(x, s)|.$$

THEOREM 2.1. *Let $g_j(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ be a $(p_j, 1)$ matrix-valued holomorphic function in $\{(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \in \mathbb{C}^{n-1} \mid |x_i| < R, i \neq j\}$ and $F_j(x, u)$ be $(p_j, 1)$ matrix-valued holomorphic function in $\{(x, u) \in \mathbb{C}^{2n} \mid |x_i| < R, \|u_i - g_i(x)\| < R, i=1, 2, \dots, n\}$ for $j=1, 2, \dots, n$. Then, for sufficiently small $\rho > 0$, the Riemann's problem*

$$(2.2) \quad \begin{cases} \frac{\partial u_j}{\partial x_j} = F_j(x, u) \\ u_j(x) = g_j(x) \text{ at } x_j = 0 \\ (j=1, 2, \dots, n) \end{cases}$$

has a unique holomorphic solution $u(x) = (u_1(x), \dots, u_n(x))$ in $\{x \in \mathbb{C}^n \mid |x_i| < \rho, i=1, \dots, n\}$ where $u_j(x)$ ($j=1, 2, \dots, n$) is a $(p_j, 1)$ matrix-valued holomorphic function in Ω .

PROOF. (2.2) is equivalent to the system of integral equations;

$$(2.3) \quad u_j(x) = \int_0^{x_j} F_j(x_1, \dots, x_{j-1}, \lambda_j, x_{j+1}, \dots,$$

$$x_n, u(x_1, \dots, x_{j-1}, \lambda_j, x_{j+1}, \dots, x_n) d\lambda_j + g_j \\ (j=1, 2, \dots, n).$$

We define $\{u^{(\nu)}\}_{\nu=1,2,\dots}$, if possible, by the method of successive approximation;

$$(2.4) \quad \begin{cases} u_j^{(\nu+1)}(x) = \int_0^{x_j} F_j(x_1, x_2, \dots, x_{j-1}, \lambda_j, x_{j+1}, \dots, x_n, \\ u^{(\nu)}(x_1, \dots, x_{j-1}, \lambda_j, x_{j+1}, \dots, x_n) d\lambda_j \\ + g_j(x), \quad (j=1, 2, \dots, n) \\ u_j^{(0)}(x) = g_j(x). \end{cases}$$

From Lemma 2.1, there exist (p_j, p_k) matrix-valued holomorphic functions $F_{jk}(x, u, v)$ in $\{(x, u, v) \in \mathbb{C}^{n+2+\sum_{j=1}^n p_j} \mid |x_i| \leq r, \|u_i - g_i(x)\| \leq r, \|v_i - g_i(x)\| \leq r, i=1, 2, \dots, n\}$ such that

$$(2.5) \quad F_j(x, u) - F_j(x, v) = \sum_{k=1}^n F_{jk}(x, u, v)(u_k - v_k)$$

for $j=1, 2, \dots, n$ and $r; 0 < r < R$. Set

$$W^{(\nu)}(\delta_1, \delta_2, \dots, \delta_n) = \max_{\substack{|x_k| \leq \delta_k \\ k=1,2,\dots,n}} \max_{1 \leq j \leq n} \|u_j^{(\nu+1)}(x) - u_j^{(\nu)}(x)\|.$$

From (2.4) and (2.5), if u_i ($0 \leq i \leq \nu$) is defined in $\{x \mid |x_i| \leq \rho, i=1, 2, \dots, n\}$ for $0 < \rho < \pi$, we have with suitable constants $W^{(0)}$ and M ;

$$(2.6) \quad W^{(\nu)}(\delta_1, \dots, \delta_n) \\ = M \sum_{j=1}^n \int_0^{\delta_j} W^{(\nu-1)}(\delta_1, \dots, \delta_{j-1}, t_j, \delta_{j+1}, \dots, \delta_n) dt_j$$

and

$$W^{(0)}(\delta_1, \delta_2, \dots, \delta_n) \leq W^{(0)} \quad (\nu=1, 2, \dots, n).$$

From this, we have

$$(2.7) \quad W^{(\nu)}(\delta_1, \delta_2, \dots, \delta_n) \leq W^{(0)} \frac{(nM(\delta_1 + \dots + \delta_n))^\nu}{\nu!}.$$

It follows that

$$(2.8) \quad \begin{cases} \|u_j^{(\nu+1)}(x) - u_j^{(\nu)}(x)\| \leq W^{(0)} \frac{(n^2 \rho M)^\nu}{\nu!} \\ \|u_j^{(\nu+1)}(x)\| \leq W^{(0)}(e^{n^2 \rho M} - 1). \end{cases}$$

If ρ is chosen so small that

$$W^{(0)}(e^{n^2 \rho M} - 1) < r$$

$u^{(\nu)}(x)$ is to be defined for all ν in $\{x \in C^n \mid \|x\| < \rho\}$ and $\{u^{(\nu)}\}$ converges uniformly in this ball to a holomorphic matrix-valued function $u(x)$, which is a holomorphic solution of (2.2). Uniqueness follows clearly by (2.8) and the theorem of identity. q.e.d.

Now, we consider a singular system

$$(2.9) \quad \begin{cases} \frac{\partial u_j}{\partial x_j} = F_j(x, u) & j=1, 2, \dots, n-1 \\ x_n^{m+1} \frac{\partial u_n}{\partial x_n} = F_n(x, u) \\ u_j(x) = g_j(x) \text{ at } x_j=0; j=1, 2, \dots, n. \end{cases}$$

in $\mathcal{Q}$. If $\mathcal{Q}_n'' (\subset \mathcal{Q}_n)$ does not contain the origin and $\mathcal{Q}' = \mathcal{Q}_1 \times \dots \times \mathcal{Q}_n''$ is simply connected, the theorem above guarantees the unique existence of holomorphic solution $u(x)$ of (2.9) as far as $\mathcal{Q}_i$ and $\mathcal{Q}_n''$ are sufficiently small for $i=1, 2, \dots, n-1$. But, even if the solution $u(x)$ is analytically continued in x_n over a ring domain around the origin, the obtained function might not be single-valued. Thus it is natural that we should seek the condition which allows (2.9) have a single-valued holomorphic solution in a neighborhood of 0. For this purpose, it is useful, as is shown later, to study a periodic solution of non singular and non linear systems.

§3. Periodic solutions.

In this section, we consider the existence and the uniqueness of the periodic (with respect to x_n) solution of the system

$$(3.1) \quad \begin{cases} \frac{\partial u_j}{\partial x_j} = F_j(x, u) & (j=1, 2, \dots, n) \\ u_j(x) = g_j(x) \text{ at } x_j=0 & (j=1, 2, \dots, n-1) \end{cases}$$

where F_j is a holomorphic $(p_j, 1)$ matrix-valued holomorphic function in $\Omega = \Omega_1 \times \dots \times \Omega_n \times \mathbb{C}^{\sum_{j=1}^n p_j}$, g_j is a holomorphic $(p_j, 1)$ matrix-valued function in $\Omega'_j = \Omega_1 \times \Omega_2 \times \dots \times \Omega_{j-1} \times \Omega_{j+1} \times \dots \times \Omega_n$ for $j=1, 2, \dots, n$, and Ω_j is a simply connected domain in $\mathbb{C}$ for $j=1, 2, \dots, n$, furthermore Ω_n is invariant under translation, $x_n \rightarrow x_n + b$ i.e. has a period b , and we assume that F_j and g_k have a period b with respect to the last variable x_n , for $j=1, 2, \dots, n$, $k=1, \dots, n-1$.

We start from estimation of solutions of linear systems.

LEMMA 3.1. *Let $u = (u_1, \dots, u_n)$ be the holomorphic solution in Ω of the problem*

$$(3.2) \quad \begin{cases} \frac{\partial u_j}{\partial x_j} = \sum_{k=1}^n A_{jk} u_k + A_j \\ u_j = 0 \text{ at } x_j = 0 \\ (j=1, 2, \dots, n) \end{cases}$$

where A_{jk} , A_j and u_j are, respectively, (p_j, p_k) , $(p_j, 1)$ and $(p_j, 1)$ matrix-valued holomorphic functions in Ω . Assume that

$A_n = 0$ for $x_1 = x_2 = \dots = x_{n-1} = 0$, that is,

$$A_n = \sum_{k=1}^{n-1} A_n^k x_k$$

for some $(p_n, 1)$ matrix-valued holomorphic function A_n^k for $k=1, 2, \dots, n-1$. Then, for any simply connected compact set $K \subset \Omega_n$ which contains 0 and b , there exists a positive number $\delta_0(K) > 0$ such that

$$(3.3) \quad \|u\|_{\delta, K} \leq C(\delta_0) \delta \sum_{k=1}^{n-1} (\|A_k\|_{\delta, K} + \|A_n^k\|_{\delta, K})$$

for any $0 < \delta \leq \delta_0(K)$ where

$$\|\cdot\|_{\delta, K} = \max_{\substack{|x_i| \leq \delta \\ x_n \in K}} \|\cdot\|$$

and $\|\cdot\|$ denotes a usual complex Euclidean space norm.

PROOF. In the following, $C(\delta, K)$ denotes various constants dependent on δ and K .

(3.2) is equivalent to the system of integral equations

$$(3.4) \quad \begin{aligned} u_j = & \int_0^{x_j} \left(\sum_{k=1}^n A_{jk} u_k(x_1, \dots, x_{j-1}, s_j, x_{j+1}, \dots, x_n) \right. \\ & \left. + A_j(x_1, \dots, x_{j-1}, s_j, x_{j+1}, \dots, x_n) \right) ds_j \\ & (j=1, 2, \dots, n). \end{aligned}$$

We fix $\delta_1 > 0$ arbitrarily so small that $\{x_j \mid |x_j| \leq \delta\} \subset \mathcal{Q}_j$ for $j=1, 2, \dots, n-1$, and from (2.5) we have for any $0 < \delta \leq \delta_1$,

$$\|u_j\|_{\delta, K} \leq \frac{1}{n-1} C(\delta_1, K) \sum_{k=1}^n \delta (\|u_k\|_{\delta, K} + \|A_j\|_{\delta, K})$$

for $j=1, 2, \dots, n-1$. Hence

$$\sum_{j=1}^{n-1} \|u_j\|_{\delta, K} \leq C(\delta_1, K) \delta \left(\sum_{k=1}^{n-1} \|u_k\|_{\delta, K} + \|u_n\|_{\delta, K} + \sum_{j=1}^{n-1} \|A_j\|_{\delta, K} \right).$$

Choose a positive number $\delta_2 (\leq \delta_1)$, if necessary, so small that $C(\delta_1, K) \delta_2 \leq \frac{1}{2}$, and we have;

$$(3.5) \quad \begin{aligned} \sum_{j=1}^{n-1} \|u_j\|_{\delta, K} & \leq \frac{C(\delta_1, K) \delta}{1 - C(\delta_1, K) \delta} (\|u_n\|_{\delta, K} + \sum_{j=1}^{n-1} \|A_j\|_{\delta, K}) \\ & \leq C(\delta_2, K) \delta (\|u_n\|_{\delta, K} + \sum_{j=1}^{n-1} \|A_j\|_{\delta, K}) \end{aligned}$$

for any $0 < \delta \leq \delta_2$. On the other hand, from that

$$\frac{\partial u_n}{\partial x_n} = \sum_{k=1}^n A_{nk} u_k + A_n = A_{nn} u_n + \sum_{k=1}^{n-1} A_{nk} u_k + A_n$$

and

$$u_n = 0 \quad \text{at} \quad x_n = 0,$$

we have;

$$\|u_n\|_{\delta, K} \leq C(\delta_2, K) \left(\sum_{k=1}^{n-1} \|u_k\|_{\delta, K} + \|A_n\|_{\delta, K} \right),$$

by hypothesis,

$$\leq C(\delta_2, K) \left(\sum_{k=1}^{n-1} \|u_k\|_{\delta, K} + \sum_{k=1}^{n-1} \|A_n^k\|_{\delta, K} \delta \right),$$

by (3.5),

$$\leq C(\delta_2, K) \delta \left\{ \|u_n\|_{\delta, K} + \sum_{k=1}^{n-1} (\|A_k\|_{\delta, K} + \|A_n^k\|_{\delta, K}) \right\}.$$

Now, choose a positive number $\delta_3 \leq \delta_2$, if necessary, so small that $C(\delta_2, K) \delta_3 \leq \frac{1}{2}$, and we have,

$$\|u_n\|_{\delta, K} \leq C(\delta_3, K) \delta \left\{ \sum_{k=1}^{n-1} (\|A_k\|_{\delta, K} + \|A_n^k\|_{\delta, K}) \right\}$$

for any $0 < \delta \leq \delta_3$ and also by (3.5) (We note that δ_2 is replaced by δ_3 in (3.5))

$$\begin{aligned} \sum_{k=1}^{n-1} \|u_k\|_{\delta, K} &\leq C(\delta_3, K) \delta \left\{ C(\delta_3, K) \sum_{k=1}^{n-1} (\|A_k\|_{\delta, K} \right. \\ &\quad \left. + \|A_n^k\|_{\delta, K}) + \sum_{j=1}^{j-1} \|A_j\|_{\delta, K} \right\} \\ &\leq C(\delta_3, K) \delta \sum_{k=1}^{n-1} (\|A_k\|_{\delta, K} + \|A_n^k\|_{\delta, K}) \end{aligned}$$

for any $0 < \delta \leq \delta_3$. Thus we have:

$$\begin{aligned} \|u\|_{\delta, K} &= \max_{1 \leq k \leq n} \|u_k\|_{\delta, K} \\ &\leq C(\delta_3, K) \delta \sum_{k=1}^{n-1} (\|A_k\|_{\delta, K} + \|A_n^k\|_{\delta, K}). \end{aligned}$$

If the notation is changed, we obtain (3.3). q.e.d.

LEMMA 3.2. We put the same hypothesis as Theorem 1.2, and let u be the holomorphic solution, which has a period b with respect to x_n , of the problem in Ω ;

$$(3.6) \quad \begin{cases} \frac{\partial u_j}{\partial x_j} = \sum_{k=1}^n A_{jk} u_k + A_j & (j=1, 2, \dots, n) \\ u_j = g_j \text{ at } x_j = 0 & (j=1, 2, \dots, n-1). \end{cases}$$

We assume as well as Lemma 3.1 that $A_n = 0$ for $x_1 = x_2 = \dots = x_{n-1} = 0$, that is,

$$A_n = \sum_{k=1}^{n-1} A_n^k x_k.$$

Then, for any simply connected compact set K which contains 0 and b , there exists a positive numbers $\delta_0(K) > 0$ and $M(\delta_0, K) > 0$ such that

$$(3.7) \quad \|u\|_{\delta, K} \leq C(\delta_0, K) \sum_{k=1}^{n-1} (\|A_k\|_{\delta, K} + \|A_n^k\|_{\delta, K}) \\ + M(\delta_0, K) \|g'\|_{\delta, K}$$

for any $0 < \delta \leq \delta_0(K)$ where

$$g = (g_1, \dots, g_{n-1}, g_n) \text{ and } g' = (g_1, \dots, g_{n-1})$$

$$x' = (x_1, \dots, x_{n-1}) \text{ for } x = (x_1, \dots, x_{n-1}, x_n).$$

PROOF. By Theorem 1.1, u_j ($j=1, 2, \dots, n$) is represented as follows

$$u_j = \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ k \in \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} K_{k; \alpha_1, \alpha_2, \dots, \alpha_l}^j \\ (x; s_{\alpha_1}, \dots, s_{\alpha_l}) g_k(x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_l}, \dots) ds_{\alpha_1} \dots ds_{\alpha_l} \\ + p_j(x) + R_j(x) g_j(x)$$

where

$$g_n(x) = u_n(x_1, x_2, \dots, x_{n-1}, 0).$$

Since K and Ω are simply connected, the right hand does not depend on

the choice of integral contours. Hence, If we fix $\delta_1 > 0$ so small that $\{x' \mid \|x'\| \leq \delta_1\} \subset \Omega' = \Omega_1 \times \cdots \times \Omega_{n-1}$, we have

$$\begin{aligned}
 \|u\|_{\delta, K} &\leq C(\delta_1, K) \|g_n\|_{\delta, K} + M(\delta_1, K) \|g'\|_{\delta, K} \\
 &\quad + \|P\|_{\delta, K} + \|R\|_{\delta, K} \|g\|_{\delta, K} \\
 (3.8) \quad &\leq C(\delta_1, K) \|g_n\|_{\delta, K} + M(\delta_1, K) \|g'\|_{\delta, K} \\
 &\quad + \|P\|_{\delta, K} + \|R\|_{\delta, K} \|g'\|_{\delta, K}.
 \end{aligned}$$

Since P is the holomorphic solution of the problem

$$\begin{cases} \frac{\partial P_j}{\partial x_j} = \sum A_{jk} P_k + A_j \\ P_j = 0 \text{ at } x_j = 0 \end{cases} \quad (j=1, 2, \dots, n),$$

by Lemma 3.1 we have for small $\delta_2 > 0$ that

$$(3.9) \quad \|P\|_{\delta, K} \leq C(\delta_2, K) \delta \left(\sum_{k=1}^{n-1} \|A_k\|_{\delta, K} + \|A_n^k\|_{\delta, K} \right)$$

for $0 < \delta \leq \delta_2$.

Since g_n is the solution of (1.7), we have for small $\delta_3 > 0$;

$$\begin{aligned}
 \|g_n\|_{\delta, K} &\leq C(\delta_3, K) \delta \|g_n\|_{\delta, K} + C(\delta_3, K) \|P_n\| \\
 &\quad + M(\delta_3, K) \|g'\|_{\delta, K}.
 \end{aligned}$$

If $\delta_4 > 0$ is chosen so small that $\delta_4 \leq \min(\delta_1, \delta_2, \delta_3)$ and $C(\delta_3, K) \delta_4 \leq \frac{1}{2}$, we have that

$$\|g_n\| \leq C(\delta_4, K) \|P_n\| + M(\delta_4, K) \|g'\|_{\delta, K}$$

now by (3.9)

$$(3.10) \quad \leq C(\delta_4, K) \left(\sum_{k=1}^{n-1} \|A_k\|_{\delta, K} + \|A_n^k\|_{\delta, K} \right) + M(\delta_4, K) \|g'\|_{\delta, K}$$

for $0 < \delta \leq \delta_4$.

It follows from (3.8), (3.9) and (3.10) that

$$\|u\|_{\delta, K} \leq C(\delta_4, K) \delta \sum_{k=1}^{n-1} (\|A_k\|_{\delta, K} + \|A_n^k\|_{\delta, K}) + M(\delta_4, K) \|g'\|_{\delta, K}$$

for $0 < \delta \leq \delta_4$.

If the notation is changed, we obtain (3.7). q.e.d.

Now we return to the non linear system (3.1). Taylor expansion with respect to u enables us to represent F_j as follows;

$$F_j(x, u) = \sum_{k=1}^n A_{jk}(x, u) u_k + A_j(x) + f_j(x, u),$$

where each $f_j(x, u)$ consists of terms whose degree with respect to $u = (u_1, \dots, u_n)$ are ≥ 2 . Thus the problem (3.1) becomes

$$(3.1)' \quad \begin{cases} \frac{\partial u_j}{\partial x_j} = \sum_{k=1}^n A_{jk}(x, u) u_k + A_j(x) + f_j(x, u) \\ (j=1, 2, \dots, n) \\ u_j = g_j \text{ at } x_j = 0 \quad (j=1, 2, \dots, n-1). \end{cases}$$

By the assumption that $\det \left(\exp \left(\int_0^b A_{nn}(x', \mu) d\mu \right) - I \right) \neq 0$ in Ω' , we can obtain by Theorem 1.2 a sequence $\{u^{(\nu)}\}$ of matrix-valued holomorphic functions defined in Ω , successively in such a way that

$$(3.11) \quad \begin{cases} \frac{\partial u_j^{(\nu+1)}}{\partial x_j} = \sum_{k=1}^n A_{jk}(x) u_k^{(\nu+1)} + A_j(x) f_j(x, u^{(\nu)}) \\ (j=1, 2, \dots, n) \\ u_j^{(\nu+1)} = g_j \text{ at } x_j = 0 \quad (j=1, 2, \dots, n-1) \end{cases}$$

and $u^{(0)} = g_j$, where $u^{(\nu)}$ has a period b with respect to x_n . We conclude by

the following proposition that some additional conditions make $\{u^{(\nu)}\}_{\nu=1,2,\dots}$ a normal family in $\{x' \mid \|x'\| \leq \delta_0(K)\} \times K$ for any compact set $K \subset \Omega_n$ and a positive number $\delta_0(K)$ which depends on K .

PROPOSITION 3.1. Assume that $f_n(x, u) = 0$ and $A_n(x) = 0$ for $x_1 = x_2 = \dots = x_{n-1} = 0$, that is,

$$f_n(x, u) = \sum_{k=1}^{n-1} f_n^k(x, u) x_k, \quad A_n(x) = \sum_{k=1}^{n-1} A_n^k x_k$$

for some matrix-valued holomorphic functions f_n^k and A_n^k , then for any simply connected compact set $K \subset \Omega_n$ which contains 0 and b there exists a positive number $\delta_0(K)$ such that the above sequence $\{u^\nu\}_{\nu=1,2,\dots}$ is bounded in $\{x' \in \mathbb{C} \mid \|x'\| < \delta_0(K)\} \times K$.

PROOF. For such K , we can choose small $\delta_0(K) > 0$ by Lemma 3.2 so that

$$\begin{aligned} \|u^{\nu+1}\| &\leq C(\delta_0, K) \delta \left(\sum_{k=1}^{n-1} \|A_k + f_k(x, u^\nu)\|_{\delta, K} \right. \\ &\quad \left. + \sum_{k=1}^{n-1} \|A_n^k + f_n^k(x, u^\nu)\|_{\delta, K} \right) + M(\delta_0, K) \|g'\|_{\delta, K} \end{aligned}$$

for any $0 < \delta \leq \delta_0(K)$ thus

$$\begin{aligned} (3.2) \quad &\leq C(\delta_0, K) \delta \left\{ \sum_{k=1}^{n-1} (\|f_k(x, u^\nu)\|_{\delta, K}) + \|f_n^k(x, u^\nu)\|_{\delta, K} \right\} \\ &\quad + \sum_{k=1}^{n-1} (\|A_k\|_{\delta, K} + \|A_n^k\|_{\delta, K}) + M(\delta_0, K) \|g'\|_{\delta, K}. \end{aligned}$$

Now, by Lemma 1.1, we can write;

$$f_k(x, u) = \sum_{i=1}^n f_{k,i}(x, u) u_i, \quad f_n^k(x, u) = \sum_{i=1}^n f_{n,i}^k(x, u) u_i$$

for some holomorphic functions $f_{k,i}$ and $f_{n,i}^k$. We fix a positive number R so that

$$\|u^{(\nu)}\|_{\delta_0, K} + M(\delta_0, K) \|g'\|_{\delta_0, K} < R$$

and

$$2M(\delta_0, K) \|g'\|_{\delta, K} < R$$

and put

$$M \Leftarrow \max_{\substack{\|x'\| \leq \delta_0 \\ x_n \in K \\ \|u\| \leq R \\ 1 \leq i, k \leq n}} \{ \|f_{k,i}\|, \|f_{n,i}^k\| \}$$

we have then,

$$\begin{aligned} \|u^{\nu+1}\|_{\delta, K} &\leq C(\delta_0, K) \delta((n-1)M\|u^\nu\|_{\delta, K} + n(n-1)M\|u^\nu\|_{\delta, K} \\ &\quad + C(\delta_0, K) \delta \sum_{k=1}^{n-1} (\|A_k\|_{\delta, K} + \|A_n^k\|_{\delta, K}) + M(\delta_0, K)\|g'\|_{\delta, K} \\ &\leq MC(\delta_0, K) \delta(\|u^\nu\|_{\delta, K}) + C(\delta_0, K) + M(\delta_0, K)\|g'\|_{\delta, K}. \end{aligned}$$

We choose a positive number $\delta_1 \leq \delta_0$, if necessary, so that

$$MC(\delta_0, K)\delta_1 R + C(\delta_0, K)\delta_2 \leq R - 2M(\delta_0, K)\|g'\|_{\delta_0, K}.$$

It is important that the choice of δ_1 depends on only R and other constants, but does not on ν . Then, we have;

$$\|u^{\nu+1}\|_{\delta_1, K} \leq R - M(\delta_0, K)\|g'\|_{\delta_0, K},$$

and

$$\|u^{\nu+1}\|_{\delta_1, K} + M(\delta_0, K)\|g'\|_{\delta_0, K} \leq R.$$

Therefore, if R is fixed so large that

$$\|u'\|_{\delta_0, K} + M(\delta_0, K)\|g'\|_{\delta_0, K} \leq R$$

and

$$2M\|g'\|_{\delta_0, K} \leq R,$$

we have, by induction,

$$\|u^\nu\|_{\delta_1, K} + M(\delta_0, K)\|g'\|_{\delta_1, K} < R$$

for all ν . Hence $\{u^\nu\}_{\nu=1,2,\dots}$ is bounded in $\{x' \mid \|x'\| \leq \delta_1\} \times K$. q.e.d.

Thus the sequence $\{u^\nu\}$ is a normal family, and a suitable subsequence of it is uniformly convergent in $\{x' \mid \|x'\| \leq \delta_1\} \times K$. But it is shown indeed that the sequence $\{u^\nu\}$ itself is convergent.

PROPOSITION 3.2. *Under the same conditions as proposition 3.1, we conclude that the sequence $\{u^\nu\}$ is uniformly convergent in $\{x' \mid \|x'\| \leq \delta_0\} \times K$ for some $\delta_0 = \delta_0(K) > 0$.*

PROOF. Put

$$W^\nu = u^{\nu+1} - u^\nu,$$

then W^ν is the periodic solution of the problem;

$$\begin{cases} \frac{\partial W_j^\nu}{\partial x_j} = \sum_{k=1}^n A_{jk} W_k^\nu + f_j(x, u^\nu) - f_j(x, u^{\nu-1}) \\ (j=1, 2, \dots, n) \\ W_j = 0 \text{ at } x_j = 0 \quad (j=1, 2, \dots, n-1). \end{cases}$$

Hence, by Lemma 3.1, for small $\delta_0(K) > 0$

$$\begin{aligned} \|W^\nu\|_{\delta, K} &\leq C(\delta_0, K) \delta \sum_{k=1}^{n-1} (\|f_k(x, u^\nu) - f_k(x, u^{\nu-1})\|_{\delta, K} \\ &\quad + \|f_n^k(x, u^\nu) - f_n^k(x, u^{\nu-1})\|_{\delta, K}). \end{aligned}$$

From Lemma 1.1

$$f_k(x, u^\nu) - f_k(x, u^{\nu-1}) = \sum_{i=1}^n f_{k,i}(x, u^\nu, u^{\nu-1})(u_i^\nu - u_i^{\nu-1})$$

and

$$f_n^k(x, u^\nu) - f_n^k(x, u^{\nu-1}) = \sum_{i=1}^n f_{n,i}^k(x, u^\nu, u^{\nu-1})(u_i^\nu - u_i^{\nu-1})$$

for some matrix-valued holomorphic functions $f_{k,i}$ and $f_{n,i}^k$. Also, by proposition 1.1, $\{u^\nu\}$ is bounded. Thus we obtain,

$$\|W^\nu\|_{\delta, K} \leq C(\delta_1, K) \delta \|W^{\nu-1}\|_{\delta, K}.$$

Choosing a positive number δ_1 so small that $C(\delta_1, K) \delta_2 < 1$ and $\delta_2 \leq \delta_1$, we obtain;

$$\|W^\nu\|_{\delta_2, K} \rightarrow 0 \quad (\nu \rightarrow \infty).$$

Changing the notation, the proof is complete. q.e.d.

Since Ω_n has a period b , $\{u^\nu\}$ is convergent uniformly also in $\{x' \mid \|x'\| < \delta_1\} \times \bigcup_{m=-\infty}^{\infty} (K + mb)$ and the limit u has a period b with respect to x_n .

Furthermore, it is clear that u satisfies (3.1). Thus we can state our

THEOREM 3.1. *Let Ω_i ($i=1, 2, \dots, n$) be simply connected domains containing the origin, and Ω_n be invariant under the translation; $x_n \rightarrow x_n + b$. Let g_k and F_j ($k=1, 2, \dots, n-1$; $j=1, \dots, n$) have a period b with respect to x_n . Assume that $A_n(x) \equiv 0$ and $f_n(x, u) \equiv 0$ for $x_1 = x_2 = \dots = x_{n-1} = 0$, that is,*

$$A_n = \left| \sum_{k=1}^{n-1} A_n^k x_k, f_n(x, u) = \sum_{k=1}^{n-1} f_n^k(x, u) x_k, \right.$$

and $\det \left(\exp \left(\int_0^b A_{nn}(x', \mu) d\mu \right) - I \right) \neq 0$. Then, for any simply connected compact set $K \subset \Omega_n$ which contains 0 and b there exists a positive number $\eta_0(K)$ such that the system (3.1) has a unique matrix-valued holomorphic solution which has a period b with respect to x_n in $\{x' \mid \|x'\| < \eta_0(K)\} \times \bigcup_{m=-\infty}^{+\infty} (K + mb)$ and also $\bigcup_K [\{x' \mid \|x'\| < \eta_0(K)\} \times \bigcup_{m=-\infty}^{+\infty} (K + mb)]$ where the union $\bigcup_K$ is to be taken over the all simply connected compact set $K \subset \Omega_n$.

PROOF. Existence is already proved. Uniqueness and the latter assertion follows from the proof of proposition 3.2 and the Theorem of identity. q.e.d.

§4. Single-valued solutions of singular systems.

In this section we prove a theorem about singular system;

$$(4.1) \quad \begin{cases} \frac{\partial u_j}{\partial x_j} = F_j(x, u) \\ x_n^{m+1} \frac{\partial u_n}{\partial x_n} = F_n(x, u) \\ u_j = g_j(x) \text{ at } x_j = 0 \\ (j=1, 2, \dots, n-1) \end{cases}$$

where $\Omega_j \ni 0$ is a simply connected domain in $\mathbb{C}$ for $j=1, 2, \dots, n$, $\Omega = \Omega_1 \times \dots \times \Omega_n$, $\Omega' = \Omega_1 \times \dots \times \Omega_{n-1}$, f_j is a $(p_j, 1)$ matrix-valued holomorphic function in $\Omega \times \mathbb{C}^{\sum_{k=1}^n p_k}$ for $j=1, 2, \dots, n$, and $g_j(x) = g_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots,$

x_n) is a $(p_j, 1)$ matrix-valued holomorphic function in $\Omega' \times (\Omega_n - \{0\})$ for $j=1, 2, \dots, n-1$.

THEOREM 4.1. *If the Taylor expansion of $F_n(x, u)$ with respect to u is written;*

$$F_n(x, u) = \sum_{k=1}^n A_{nk}(x) u_k(x) + A_n(x) + f_n(x, u),$$

where $f_n(x, u)$ consists of terms with degree ≥ 2 with respect to $u = (u_1, \dots, u_n)$, we assume that $A_n(x) = 0$ and $f_n(x, u) = 0$ for $x_1 = x_2 = \dots = x_{n-1} = 0$, that is,

$$A_n = \sum_{k=1}^{n-1} A_n^k \cdot x_k, \quad f_n(x, u) = \sum_{k=1}^{n-1} f_n^k(x, u) x_k, \text{ and}$$

and

$$(4.2) \quad \det \left(\exp \left(\int_0^{2\pi i} e^{-m(z+\alpha)} A_{nn}(x', e^{z+\alpha}) dz \right) - I \right) \neq 0$$

in Ω' . Then (4.1) has a unique single-valued matrix-valued holomorphic solution in $\bigcup_K [\{x' \mid \|x'\| \leq \delta_0(K)\} \times K]$ where K is a double connected compact set in Ω_n which does not contain 0, and $\delta_0(K)$ is a positive number depending on K .

Remark. If A_{nn} is a scalar-valued function, the condition (4.2) is equivalent to

$$\frac{1}{m!} \frac{\partial^m A_{nn}(x', 0)}{\partial x_n^m} \neq \text{integer}$$

in Ω' .

PROOF. Fix a real number α so that $\{x_n \mid \|x_n\| \leq \alpha\} \subset \Omega_n$ and put;

$$x' = x', \quad x_n = e^{z+\alpha}, \quad U(x', z) = u(x', e^{z+\alpha}).$$

Then (4.1) becomes

$$\left| \frac{\partial U_j}{\partial x_j}(x, z) = f_j(x; e^{z+\alpha}, U) \right|$$

$$(4.3) \quad \begin{cases} \frac{\partial U_n}{\partial x_n}(x, z) = e^{-m(z+\alpha)} f_n(x, e^{z+\alpha}, U) \\ U_j = g_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n-1}, e^{z+\alpha}) \\ (j=1, 2, \dots, n-1) \end{cases}$$

in $\mathcal{Q}_1 \times \dots \times \mathcal{Q}_{n-1} \times \mathcal{A}$ where $\mathcal{A} = \{z \in \mathbb{C} \mid e^{z+\alpha} \in \mathcal{Q}_{n+1} - \{0\}\}$ which is simply connected and invariant under $z \rightarrow z + 2\pi i$. (4.1) has a unique single-valued holomorphic solution in $\{x' \mid \|x'\| \leq \delta_0\} \times K$ if and only if (4.3) has a unique holomorphic solution with a period $2\pi i$ with respect to z in $\{x' \mid \|x'\| \leq \delta_0\} \times \mathcal{A}_K$ where $\mathcal{A}_K = \{z \in \mathbb{C} \mid e^{z+\alpha} \in K = \bigcup_m (\mathcal{A}_0 K + 2m\pi i)\}$ $\mathcal{A}_0 K = \{z \in \mathbb{C} \mid e^{z+\alpha} \in K, 0 \leq \arg z \leq 2\pi i\}$. Under the assumption of Theorem, this follows from the condition;

$$\det \left(\exp \int_0^{2\pi i} e^{-m(z+\alpha)} A_{nn}(x; e^{z+\alpha}) dz - I \right) \neq 0.$$

Thus the proof is complete. q.e.d.

References

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