CONVERGENCE OF NONLINEAR SEMIGROUPS AND APPROXIMATION SCHEMES TO WELL- POSED CAUCHY PROBLEM IN BANACH SPACES

By

Ki Sik HA

(Received Apr. 30, 1971)

§1. Introduction.

Throughout this paper X is a Banach space. Let X_0 be a subset of X , and let $\{T(t); t \geq 0\}$ be a family of one-parameter nonlinear operators from X_0 into itself satisfying the following conditions:

- (i) $T(0) = I$ (the identity), $T(t+s) = T(t)T(s)$ for $t, s \geq 0$,
- (ii) $\|T(t)x - T(t)y\| \leq \|x - y\|$ for $t \geq 0$ and $x, y \in X_0$,
- (iii) $\lim_{t \rightarrow 0^+} T(t)x = x$ for $x \in X_0$.

The family $\{T(t); t \geq 0\}$ is called a contraction semigroup on X_0 .

We shall deal with multi-valued operators. Let A be a multi-valued nonlinear operator in X . By this we mean that A assigns to each $x \in X$ a subset Ax of X . Ax may be empty for some $x \in X$. By definition the domain $D(A)$ of A is the set of all $x \in X$ for which Ax is not empty. We define the range $R(A)$ of A as the union of the Ax for all $x \in D(A)$. We define for $x \in D(A)$

$$\|Ax\| = \inf\{\|x'\|; x' \in Ax\}.$$

DEFINITION 1.1. A multi-valued operator A in X is said to be dissipative if for each $\alpha > 0$, $x, y \in D(A)$, $x' \in Ax$ and $y' \in Ay$

$$\|x - y - \alpha(x' - y')\| \geq \|x - y\|.$$

Let A be dissipative in X . Then for $\alpha > 0$ we can define single-valued operators

$$J_\alpha = (I - \alpha A)^{-1}: R(I - \alpha A) (= D(J_\alpha)) \rightarrow D(A)$$

such that $\|J_\alpha x - J_\alpha y\| \leq \|x - y\|$ for $x, y \in D(J_\alpha)$. We set $A_\alpha = \alpha^{-1}(J_\alpha - I)$. The implication

$$(1.1) \quad A_\alpha x \in AJ_\alpha x \text{ for } x \in D(J_\alpha)$$

is well-known. Recently M.G. Crandall and T.M. Liggett [3] proved the following lemma and theorem.

LEMMA 1.2. *Let A be dissipative in X . Then*

(a) *for $\alpha, \beta > 0$ and $x \in D(J_\alpha)$,*

$$\frac{\beta}{\alpha}x + \frac{\alpha - \beta}{\alpha}J_\alpha x \in D(J_\beta), \text{ and}$$

$$J_\alpha x = J_\beta \left(\frac{\beta}{\alpha}x + \frac{\alpha - \beta}{\alpha}J_\alpha x \right),$$

(b) *for $\alpha \geq \beta > 0$, $n \geq m > 0$ and $x \in D(J_\alpha^m) \cap D(J_\alpha^n) \cap D(A)$*

$$(1.2) \quad \|J_\beta^n x - J_\alpha^m x\| \leq (((n\beta - m\alpha)^2 + n\beta(\alpha - \beta))^{1/2} \\ + (m\alpha(\alpha - \beta) + (m\alpha - n\beta)^2)^{1/2}) \|Ax\|,$$

where m, n are integers.

THEOREM 1.3. *Let A be dissipative in X satisfying*

$$(1.3) \quad R(I - \alpha A) \supset \overline{D(A)} \text{ for } \alpha > 0.$$

Then there exists a contraction semigroup $\{T(t); t \geq 0\}$ on $\overline{D(A)}$ such that for each $x \in \overline{D(A)}$

$$T(t)x = \lim_{\alpha \rightarrow 0^+} J_\alpha^{[t/\alpha]}x$$

uniformly for t in every bounded interval of $[0, \infty)$, and

$$\|T(t)x - T(s)x\| \leq |t - s| \|Ax\|$$

for $t, s \geq 0$ and $x \in D(A)$ where $[\]$ denotes the Gaussian bracket.

The purpose of this paper is to extend the Kato-Trotter Theorem in K. Yosida [9] to the case of nonlinear contraction semigroups given in Theorem 1.3, and to consider the convergence of the approximation scheme

$$\begin{cases} \frac{d}{dt} u_n(t) \in A_n u(t) \\ u_n(0) = x \end{cases}$$

in X to the Cauchy problem

$$\begin{cases} \frac{d}{dt} u(t) \in A u(t) \\ u(0) = x \end{cases}$$

in X , well-posed in the sense of semigroup when A_n approximates A in the sense of (3.3), where A and A_n are nonlinear multi-valued operators.

Let A be dissipative in X satisfying (1.3). Setting $n = \lceil t/\beta \rceil$ and $m = \lceil t/\alpha \rceil$ in (1.2), we have for $x \in D(A)$

$$\begin{aligned} & \|J_{\beta}^{\lceil t/\beta \rceil} x - J_{\alpha}^{\lceil t/\alpha \rceil} x\| \\ & \leq ((\alpha^2 + t(\alpha - \beta))^{1/2} + (t(\alpha - \beta) + \beta^2)^{1/2}) \|Ax\|. \end{aligned}$$

As $\beta \rightarrow 0+$, it follows from Theorem 1.3 that

$$(1.4) \quad \|T(t)x - J_{\alpha}^{\lceil t/\alpha \rceil} x\| \leq ((\alpha^2 + t\alpha)^{1/2} + (t\alpha)^{1/2}) \|Ax\|$$

for $x \in D(A)$.

§2 concerns the convergence of contraction semigroups given in Theorem 1.3.

In §3 we deal with the convergence of the solutions of approximation schemes to Cauchy problem well-posed in the sense of a contraction semigroup.

§2. Convergence of contraction semigroups.

Let $\{A_n\}$ be a sequence of dissipative operators in X satisfying

$$(2.1) \quad R(I - \alpha A_n) \supset \overline{\text{co}D(A_n)} \text{ for } \alpha > 0,$$

and put $J_{\alpha,n} = (I - \alpha A_n)^{-1}$ for $\alpha > 0$.

THEOREM 2.1. *Let $\{A_n\}$ be a sequence of dissipative operators in X satisfying (2.1). If*

$$(2.2) \quad D \subset D(A_n) \text{ for every } n$$

and

$$(2.3) \quad \lim_{n \rightarrow \infty} J_{\alpha_0, n} x (x \in \overline{\text{co}D}) \text{ exists in } D \text{ for some } \alpha_0 > 0,$$

then, denoting by $J_{\alpha_0} x$ the limit value of (2.3), there exists a dissipative operator A in X such that $J_{\alpha_0} = (I - \alpha_0 A)^{-1}$ for $\alpha_0 > 0$. In addition if (1.3) is satisfied then $\lim_{n \rightarrow \infty} T_n(t)x = T(t)x$ for $x \in D(A)$ uniformly for t in every bounded interval of $[0, \infty)$, where $\{T_n(t); t \geq 0\}$, $\{T(t); t \geq 0\}$ are contraction semigroups given in Theorem 1.3 by A_n and A respectively and $\text{co}D$ denotes the convex hull of D .

The following two lemmas make complete the proof of Theorem 2.1.

LEMMA 2.2. Let $\{A_n\}$ be a sequence of dissipative operators in X satisfying (2.1). If the conditions (2.2) and (2.3) are satisfied then

$$(a) \quad \lim_{n \rightarrow \infty} J_{\alpha, n} x, x \in \overline{\text{co}D} \text{ exists in } D \text{ for all } \alpha > 0.$$

We denote by $J_\alpha x$ the limit value of (a).

$$(b) \quad \text{There exists a dissipative operator } A \text{ in } X \text{ such that } J_\alpha = (I - \alpha A)^{-1} \text{ for all } \alpha > 0.$$

PROOF. (a) Since $J_{\alpha_0, n}$ is contraction, so is J_{α_0} . We first prove that for $\alpha > 0$, $\frac{\alpha}{\alpha_0} I + \left(1 - \frac{\alpha}{\alpha_0}\right) J_{\alpha_0}$ maps $\overline{\text{co}D}$ onto itself. If $x \in \overline{\text{co}D}$, then $J_{\alpha_0} x \in \overline{\text{co}D}$, and hence $\frac{\alpha}{\alpha_0} x + \left(1 - \frac{\alpha}{\alpha_0}\right) J_{\alpha_0} x \in \overline{\text{co}D}$. Then taking $y \in \overline{\text{co}D}$, $\frac{\alpha_0}{\alpha} y + \left(1 - \frac{\alpha_0}{\alpha}\right) J_{\alpha_0} x \in \overline{\text{co}D}$. We define an operator K by $Kx = \frac{\alpha_0}{\alpha} y + \left(1 - \frac{\alpha_0}{\alpha}\right) J_{\alpha_0} x$ for $x \in \overline{\text{co}D}$. If $\alpha > \frac{\alpha_0}{2}$, K is a strict contraction operator from $\overline{\text{co}D}$ into itself. In fact for $x_1, x_2 \in \overline{\text{co}D}$,

$$\begin{aligned} & \|Kx_1 - Kx_2\| \\ &= \left\| \frac{\alpha_0}{\alpha} y + \left(1 - \frac{\alpha_0}{\alpha}\right) J_{\alpha_0} x_1 - \left(\frac{\alpha_0}{\alpha} y + \left(1 - \frac{\alpha_0}{\alpha}\right) J_{\alpha_0} x_2 \right) \right\| \\ &= \left| 1 - \frac{\alpha_0}{\alpha} \right| \|J_{\alpha_0} x_1 - J_{\alpha_0} x_2\| \end{aligned}$$

$$\leq \left| 1 - \frac{\alpha_0}{\alpha} \right| \|x_1 - x_2\|.$$

By the fixed point theorem there exists a unique element $x \in \overline{\text{co}D}$ such that $Kx = x$. Thus we have $x = \frac{\alpha_0}{\alpha} y + \left(1 - \frac{\alpha_0}{\alpha}\right) J_{\alpha_0} x$. From this we obtain $y = \frac{\alpha}{\alpha_0} x + \left(1 - \frac{\alpha}{\alpha_0}\right) J_{\alpha_0} x$, and

$$\overline{\text{co}D} = \left\{ y; y = \frac{\alpha}{\alpha_0} x + \left(1 - \frac{\alpha}{\alpha_0}\right) J_{\alpha_0} x, x \in \overline{\text{co}D} \right\}$$

for $\alpha > \frac{\alpha_0}{2}$. Next we show that $\lim_{n \rightarrow \infty} J_{\alpha, n} x, x \in \overline{\text{co}D}$ exists in D for all $\alpha > 0$. For $x, y \in \overline{\text{co}D}$ and $\alpha > \frac{\alpha_0}{2}$, put $y = \frac{\alpha}{\alpha_0} x + \left(1 - \frac{\alpha}{\alpha_0}\right) J_{\alpha_0} x$. Then we obtain

$$\begin{aligned} \|J_{\alpha_0} x - J_{\alpha, n} y\| &\leq \|J_{\alpha_0} x - J_{\alpha_0, n} x\| + \|J_{\alpha_0, n} x - J_{\alpha, n} y\| \\ &\leq \|J_{\alpha_0} x - J_{\alpha_0, n} x\| \\ &\quad + \left\| J_{\alpha, n} \left(\frac{\alpha}{\alpha_0} x + \left(1 - \frac{\alpha}{\alpha_0}\right) J_{\alpha_0, n} x \right) - J_{\alpha, n} \left(\frac{\alpha}{\alpha_0} x + \left(1 - \frac{\alpha}{\alpha_0}\right) J_{\alpha_0} x \right) \right\| \\ &\leq \|J_{\alpha_0} x - J_{\alpha_0, n} x\| + \left| 1 - \frac{\alpha}{\alpha_0} \right| \|J_{\alpha_0} x - J_{\alpha_0, n} x\| \\ &= \left(1 + \left| 1 - \frac{\alpha}{\alpha_0} \right| \right) \|J_{\alpha_0} x - J_{\alpha_0, n} x\| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Here we have used Lemma 1.2 (a). Therefore $\lim_{n \rightarrow \infty} J_{\alpha, n} y, y \in \overline{\text{co}D}$ exists for $\alpha > \frac{\alpha_0}{2}$. Since $\alpha_1 = \frac{2}{3} \alpha_0 > \frac{\alpha_0}{2}$, $\lim_{n \rightarrow \infty} J_{\alpha_1, n} y, y \in \overline{\text{co}D}$ exists. Then in an analogous consideration we find that $\lim_{n \rightarrow \infty} J_{\alpha, n} y, y \in \overline{\text{co}D}$ exists for all $\alpha > \frac{\alpha_0}{3}$. Repeating this process, we have that $\lim_{n \rightarrow \infty} J_{\alpha, n} x, x \in \overline{\text{co}D}$ for all $\alpha > 0$. By Lemma 1.2 (a), $J_{\alpha, n} x = J_{\alpha_0, n} \left(\frac{\alpha_0}{\alpha} x + \left(1 - \frac{\alpha_0}{\alpha}\right) J_{\alpha, n} x \right)$. Since $J_{\alpha, n}$ and J_{α} are contraction, we obtain

$$(2.4) \quad J_{\alpha}x = J_{\alpha_0}\left(\frac{\alpha_0}{\alpha}x + \left(1 - \frac{\alpha_0}{\alpha}\right)J_{\alpha}x\right)$$

for $x \in \overline{\text{co}D}$. This equality means that $R(J_{\alpha}) = R(J_{\alpha_0})$, and hence by (2.3) $\lim_{n \rightarrow \infty} J_{\alpha, n}x = J_{\alpha}x \in D$, $x \in \overline{\text{co}D}$ for all $\alpha > 0$.

(b) Since $R(J_{\alpha})$ is independent of $\alpha > 0$, we define J_{α}^{-1} by $J_{\alpha}^{-1}x = \{y \in \overline{\text{co}D}; J_{\alpha}y = x\}$ for $x \in R(J_{\alpha})$. Then $\frac{1}{\alpha_0}(x - J_{\alpha_0}^{-1}x) = \frac{1}{\alpha}(x - J_{\alpha}^{-1}x)$. Indeed, by (2.4) for $y \in \overline{\text{co}D}$ we have $\left(\frac{\alpha_0}{\alpha}y + \left(1 - \frac{\alpha_0}{\alpha}\right)J_{\alpha}y\right) \in J_{\alpha_0}^{-1}J_{\alpha}y$, and hence $\frac{1}{\alpha}(J_{\alpha}y - y) \in \frac{1}{\alpha_0}(J_{\alpha}y - J_{\alpha_0}^{-1}J_{\alpha}y)$. There exists $x \in R(J_{\alpha})$ such that $x = J_{\alpha}y$. Thus we obtain $\frac{1}{\alpha}(x - J_{\alpha}^{-1}x) \subset \frac{1}{\alpha_0}(x - J_{\alpha_0}^{-1}x)$. Set $D(A) = R(J_{\alpha})$ for $\alpha > 0$ and we define A by $Ax = \frac{1}{\alpha}(x - J_{\alpha}^{-1}x)$ for $x \in D(A)$. Then $J_{\alpha} = (I - \alpha A)^{-1}$ for all $\alpha > 0$. We shall show that A is dissipative. In fact, for $x, y \in D(A)$ and $x' \in Ax$, $y' \in Ay$ we have $J_{\alpha}(x - \alpha x') = x$ and $J_{\alpha}(y - \alpha y') = y$ for $\alpha > 0$. Since J_{α} is contraction we obtain

$$\begin{aligned} \|x - y\| &= \|J_{\alpha}(x - \alpha x') - J_{\alpha}(y - \alpha y')\| \\ &\leq \|x - y - \alpha(x' - y')\|, \end{aligned}$$

which means that A is dissipative in X .

LEMMA 2.3. *Let $\{A_n\}$ be a sequence of dissipative operators in X satisfying (2.1). If the conditions (2.2), (2.3) in Theorem 2.1 and (1.3) are satisfied and if $\{T_n(t); t \geq 0\}$, $\{T(t); t \geq 0\}$ are contraction semigroups given in Theorem 1.3 by A_n and A respectively, then $\lim_{n \rightarrow \infty} T_n(t)x = T(t)x$ for $x \in \overline{D(A)}$ uniformly for t in every bounded interval of $[0, \infty)$. Where A is the dissipative operator given in Lemma 2.2.*

PROOF. Put $A_{\alpha, n} = \alpha^{-1}(J_{\alpha, n} - I)$ and $A_{\alpha} = \alpha^{-1}(J_{\alpha} - I)$ for $\alpha > 0$. Since $\lim_{n \rightarrow \infty} J_{\alpha, n}x = J_{\alpha}x$, $x \in \overline{\text{co}D}$ for all $\alpha > 0$, we have $\lim_{n \rightarrow \infty} A_{\alpha, n}x = A_{\alpha}x$, $x \in \overline{\text{co}D}$ for all $\alpha > 0$. Therefore there exists $M > 0$ such that $\|A_{\alpha, n}x\| \leq M$ and $\|A_{\alpha}x\| \leq M$. By the definition of A , it follows that $D(A) \subset D \subset D(A_n)$. We fix $y \in \overline{D(A)}$ and $\alpha_0 > 0$. Then

$$\|T_n(t)J_{\alpha_0}y - T(t)J_{\alpha_0}y\|$$

$$\begin{aligned} &\leq \|T_n(t)J_{\alpha_0}y - J_{\alpha,n}^{[t/\alpha]}J_{\alpha_0}y\| \\ &+ \|J_{\alpha,n}^{[t/\alpha]}J_{\alpha_0}y - J_{\alpha}^{[t/\alpha]}J_{\alpha_0}y\| \\ &+ \|J_{\alpha}^{[t/\alpha]}J_{\alpha_0}y - T(t)J_{\alpha_0}y\|. \end{aligned}$$

We estimate the first term of the left side in the above inequality as follows.

$$\begin{aligned} &\|T_n(t)J_{\alpha_0}y - J_{\alpha,n}^{[t/\alpha]}J_{\alpha_0}y\| \\ &\leq \|T_n(t)J_{\alpha_0}y - T_n(t)J_{\alpha_0,n}y\| \\ &+ \|T_n(t)J_{\alpha_0,n}y - J_{\alpha,n}^{[t/\alpha]}J_{\alpha_0,n}y\| \\ &+ \|J_{\alpha,n}^{[t/\alpha]}J_{\alpha_0,n}y - J_{\alpha}^{[t/\alpha]}J_{\alpha_0}y\| \\ &\leq 2\|J_{\alpha_0}y - J_{\alpha_0,n}y\| + \|T_n(t)J_{\alpha_0,n}y - J_{\alpha,n}^{[t/\alpha]}J_{\alpha_0,n}y\| \\ &\leq 2\|J_{\alpha_0}y - J_{\alpha_0,n}y\| + ((\alpha^2 + t\alpha)^{1/2} + (t\alpha)^{1/2})\|A_nJ_{\alpha_0,n}y\|. \end{aligned}$$

Since $\|A_nJ_{\alpha_0,n}y\| \leq \|A_{\alpha_0,n}y\| \leq M$, we have

$$\begin{aligned} &\|T_n(t)J_{\alpha_0}y - J_{\alpha,n}^{[t/\alpha]}J_{\alpha_0}y\| \\ &\leq 2\|J_{\alpha_0}y - J_{\alpha_0,n}y\| + ((\alpha^2 + t\alpha)^{1/2} + (t\alpha)^{1/2})M. \end{aligned}$$

Also

$$\begin{aligned} &\|J_{\alpha}^{[t/\alpha]}J_{\alpha_0}y - T(t)J_{\alpha_0}y\| \\ &\leq ((\alpha^2 + t\alpha)^{1/2} + (t\alpha)^{1/2})\|AJ_{\alpha_0}y\| \\ &\leq ((\alpha^2 + t\alpha)^{1/2} + (t\alpha)^{1/2})\|A_{\alpha_0}y\| \\ &\leq ((\alpha^2 + t\alpha)^{1/2} + (t\alpha)^{1/2})M. \end{aligned}$$

Here we have used (1.1) and (1.4). Thus

$$\begin{aligned} &\|T_n(t)J_{\alpha_0}y - T(t)J_{\alpha_0}y\| \\ &\leq 2\|J_{\alpha_0}y - J_{\alpha_0,n}y\| \\ &+ \|J_{\alpha,n}^{[t/\alpha]}J_{\alpha_0}y - J_{\alpha}^{[t/\alpha]}J_{\alpha_0}y\| \\ &+ 2((\alpha^2 + t\alpha)^{1/2} + (t\alpha)^{1/2})M. \end{aligned}$$

Let $t_0 \geq 0$ and choose $\alpha_1 > 0$ such that for each $\varepsilon > 0$,

$$2((\alpha_1^2 + t\alpha_1)^{1/2} + (t\alpha_1)^{1/2})M < \varepsilon, \quad 0 \leq t \leq t_0.$$

Then

$$\begin{aligned} & \|T_n(t)J_{\alpha_0}y - T(t)J_{\alpha_0}y\| \\ & \leq 2\|J_{\alpha_0}y - J_{\alpha_0,n}y\| \\ & \quad + \|J_{\alpha_1,n}^{[t/\alpha]}J_{\alpha_0}y - J_{\alpha_1}^{[t/\alpha]}J_{\alpha_0}y\| \\ & \quad + 2((\alpha_1^2 + t\alpha_1)^{1/2} + (t\alpha_1)^{1/2})M. \end{aligned}$$

An $n \rightarrow \infty$, $\|T_n(t)J_{\alpha_0}y - T(t)J_{\alpha_0}y\| \rightarrow 0$, $0 \leq t \leq t_0$. Since $R(J_{\alpha_0}) = D(A)$, there exists $x \in D(A)$ such that $x = J_{\alpha_0}y$. Therefore it follows that

$$(2.5) \quad \lim_{n \rightarrow \infty} T_n(t)x = T(t)x \quad \text{for } x \in D(A)$$

uniformly for t in every bounded interval of $[0, \infty)$. Since $T_n(t)$, $T(t)$ are contraction, (2.5) holds true for $x \in \overline{D(A)}$.

The proof of Theorem 2.1 is complete.

REMARK 2.4. Recently H. Brezis and A. Pazy [2] proved Theorem 2.1 in the case that $\{A_n\}$ is a sequence of maximal dissipative operators satisfying $R(I - \alpha A_n) \supset \overline{\text{co}D(A_n)}$ in a Hilbert space H , and I. Miyadera [5] proved the Theorem in the case that $\{A_n\}$ is a sequence of m -dissipative operators in X whose dual X^* is uniformly convex.

§3. Convergence of approximation schemes.

We consider the Cauchy problem

$$(3.1) \quad \begin{cases} \frac{d}{dt} u(t) \in Au(t) \\ u(0) = x \end{cases}$$

in X , where A is a given multi-valued operator in X . A single-valued mapping $u(t): [0, \infty) \rightarrow X$ is called a solution of the Cauchy problem if for $t \geq 0$ $u(t)$ is Lipschitz continuous and for a.e. $t \geq 0$ it strongly differentiable belongs to $D(A)$ and satisfies (3.1). If A is a closed operator

and

$$(3.2) \quad R(I - \alpha A) \supset D(A) \quad \text{for } \alpha > 0,$$

then (1.3) holds true because $R(I - \alpha A)$ is closed.

The following theorems are due to H. Brezis and A. Pazy [1] and I. Miyadara [6] respectively.

THEOREM 3.1. *Let A be a dissipative operator in X satisfying (3.2) and let $x \in D(A)$. If $u(t)$ is a solution of the Cauchy problem, then $u(t) = \lim_{\alpha \rightarrow 0+} J_{\alpha}^{[t/\alpha]} x$ uniformly for t in every bounded interval of $[0, \infty)$.*

THEOREM 3.2. *Let A be a closed dissipative operator in X satisfying (3.2) and let $\{T(t); t \geq 0\}$ be a contraction semigroup on $\bar{D}(A)$ given in Theorem 1.3 by A . If $T(t)x$ for $x \in D(A)$ is strongly differentiable at a.e. $t \geq 0$ then it is a unique solution of the Cauchy problem.*

Let $\{A_n\}$ be a sequence of multi-valued operators in X satisfying $D(A) \subset D(A_n)$ which approximates A in the sense that

$$(3.3) \quad \lim_{n \rightarrow \infty} A_n x = Ax \quad \text{for } x \in D(A),$$

i.e. for each $x' \in Ax$ there is a sequence $\{x'_n\}$, $x'_n \in A_n x$, such that $x'_n \rightarrow x'$. Then we obtain the sequence of equations

$$(3.4) \quad \begin{cases} \frac{d}{dt} u_n(t) \in A_n u_n(t) \\ u_n(0) = x \end{cases}$$

in X , which is called an approximation scheme to the Cauchy problem.

Let $\{A_n\}$ be a sequence of dissipative operators in X satisfying

$$(3.5) \quad R(I - \alpha A_n) \supset D(A_n) \quad \text{for } \alpha > 0.$$

By Theorem 3.1 the solution $u_n(t)$ of (3.4) for each n is represented as

$$(3.6) \quad u_n(t) \equiv T_n(t)x = \lim_{\alpha \rightarrow 0+} J_{\alpha, n}^{[t/\alpha]} x \quad \text{for } x \in D(A_n)$$

uniformly for t in every bounded interval of $[0, \infty)$.

DEFINITION 3.3. *The Cauchy problem is said to be well-posed in the*

sense of semigroup if there exists a contraction semigroup on $\overline{D(A)}$ such that, for $x \in D_1 \subset \overline{D(A)}$, $u(t) = T(t)x$ is a unique solution of the Cauchy problem.

We shall consider the problem which the solution $u_n(t)$ of (3.4) converges to the solution $u(t)$ of the Cauchy problem well-posed in the sense of semigroup.

THEOREM 3.4. *Let $\{A_n\}$ be a sequence of dissipative operators in X satisfying (2.1) and let $u_n(t) = T_n(t)x$, $x \in D(A_n)$ be a solution of (3.4). Let A be a closed dissipative operator in X satisfying (3.2). If $D(A) \subset D(A_n)$ for each n and $A_n x \rightarrow Ax$ for $x \in D(A)$ then the Cauchy problem is well-posed in the sense of semigroup and we have*

$$(3.7) \quad \lim_{n \rightarrow \infty} T_n(t)x = T(t)x \quad \text{for } x \in D(A)$$

uniformly for t in every bounded interval of $[0, \infty)$, where $\{T(t); t \geq 0\}$ is a contraction semigroup on $\overline{D(A)}$ given in Theorem 1.3 and $u(t) = T(t)x$, $x \in D(A)$ is strongly differentiable at a.e. $t \geq 0$.

PROOF. We note that $T(t)x = \lim_{\alpha \rightarrow 0+} J_{\alpha}^{[t/\alpha]}x$ for $x \in \overline{D(A)}$ uniformly for t in every bounded interval of $[0, \infty)$. By Theorem 3.2, $T(t)x$ for $x \in D(A)$ is a unique solution of the Cauchy problem. Thus the Cauchy problem is well-posed in the sense of semigroup. Let $x \in D(A)$. There exists $y \in D(A)$ such that for each $y' \in Ay$, there is a sequence $\{y'_n\}$, $y'_n \in A_n y$, $y'_n \rightarrow y'$ and $x = y - \alpha y'$ for $\alpha > 0$. We have

$$\begin{aligned} \|J_{\alpha,n}x - J_{\alpha}x\| &= \|J_{\alpha,n}x - y\| \\ &= \|J_{\alpha,n}x - J_{\alpha,n}(y - \alpha y'_n)\| \\ &\leq \|x - y + \alpha y'_n\| \\ &= \alpha \|y'_n - y'\| \rightarrow 0 \end{aligned}$$

for some $\alpha > 0$ as $n \rightarrow \infty$. It follows that $\lim_{n \rightarrow \infty} J_{\alpha,n}x = J_{\alpha}x$, $x \in D(A)$ for all $\alpha > 0$. Thus, using (3.6) and an analogy of the proof of Lemma 2.3, we have (3.7).

THEOREM 3.5. *Let $\{A_n\}$ be a sequence of dissipative operators in X*

satisfying (2.1) and let $u_n(t) = T_n(t)x$, $x \in D(A_n)$ be a solution of (3.4). Suppose that

$$(3.8) \quad \bar{D} \subset D(A_n) \text{ for each } n,$$

$$(3.9) \quad \lim_{n \rightarrow \infty} A_n x \quad (= Ax) \text{ exists for } x \in D,$$

and

$$(3.10) \quad R(I - \alpha A) \supset \overline{\text{co}D(\bar{A})} \quad \text{for } \alpha > 0.$$

Then the Cauchy problem

$$(3.11) \quad \begin{cases} \frac{d}{dt} u(t) \in \bar{A}u(t) \\ u(0) = x \end{cases}$$

in X , is well-posed in the sense of semigroup, and

$$(3.12) \quad \lim_{n \rightarrow \infty} T_n(t)x = T(t)x \text{ for } x \in D(A)$$

uniformly for t in every bounded interval of $[0, \infty)$, where $\{T(t); t \geq 0\}$ is a contraction semigroup on $\overline{D(\bar{A})}$ and $T(t)x$, $x \in \overline{D(\bar{A})}$ is strongly differentiable a.e. $t \geq 0$.

PROOF. It follows from (3.9) that for $x, y \in D(A)$ and $x' \in Ax$, $y' \in Ay$, there exists $\{x'_n\}$, $\{y'_n\}$, $x'_n \in A_n x$, $y'_n \in A_n y$ such that $x'_n \rightarrow x'$, $y'_n \rightarrow y'$ as $n \rightarrow \infty$. By the dissipativity of A_n ,

$$\|x - y - \alpha(x'_n - y'_n)\| \geq \|x - y\|$$

for $\alpha > 0$. As $n \rightarrow \infty$, we have

$$\|x - y - \alpha(x' - y')\| \geq \|x - y\|$$

for $\alpha > 0$, which means that A is dissipative in X . Also $\bar{A}$ is dissipative in X . Indeed, for $x, y \in D(\bar{A})$ and $x' \in \bar{A}x$, $y' \in \bar{A}y$, there exists $\{x_n\}$, $\{y_n\} \subset D(A)$ such that $x_n \rightarrow x$, $y_n \rightarrow y$ and $x'_n \in Ax_n$, $y'_n \in Ay_n$ imply $x'_n \rightarrow x'$ and $y'_n \rightarrow y'$. Since A is dissipative,

$$\|x_n - y_n - \alpha(x'_n - y'_n)\| \geq \|x_n - y_n\|$$

for $\alpha > 0$. It follows that

$$\|x - y - \alpha(x' - y')\| \geq \|x - y\|$$

for $\alpha > 0$ as $n \rightarrow \infty$, and from (3.10) we obtain $R(I - \alpha \bar{A}) \supset \overline{D(\bar{A})}$. By Theorem 1.3, there exists a contraction semigroup $\{T(t); t \geq 0\}$ on $\overline{D(\bar{A})}$ such that

$$T(t)x = \lim_{\alpha \rightarrow 0+} (I - \alpha \bar{A})^{-[\frac{t}{\alpha}]} x,$$

for $x \in \overline{D(\bar{A})}$ and $\alpha > 0$, and we have

$$\|T(t)x - T(s)x\| \leq |t - s| \|Ax\|$$

for $x \in D(\bar{A})$. If $T(t)x$, then $x \in D(\bar{A})$ is strongly differentiable at a.e. $t \geq 0$, it is a unique solution of the Cauchy problem (3.11). Thus the Cauchy problem is well-posed in the sense of semigroup.

We shall show that

$$(3.13) \quad \lim_{n \rightarrow \infty} J_{\alpha, n} x = (I - \alpha \bar{A})^{-1} x,$$

for $x \in \overline{\text{co}D(\bar{A})}$ and $\alpha > 0$. For $x \in \overline{\text{co}D(\bar{A})} \subset R(I - \alpha \bar{A})$, there exists $y \in D(\bar{A})$, $y' \in A y$ such that $x = y - \alpha y'$. From (3.8) and (3.9), for such y there exists $\{y'_n\}$, $y'_n \in A_n y$ such that $y'_n \rightarrow y'$.

$$\begin{aligned} \|J_{\alpha, n} x - y\| &\leq \|J_{\alpha, n} x - J_{\alpha, n}(y - \alpha y'_n)\| \\ &\leq \|x - y + \alpha y'_n\| \\ &= \|y' - y'_n\| \rightarrow 0 \end{aligned}$$

for some $\alpha > 0$ as $n \rightarrow \infty$. Hence

$$(3.14) \quad \lim_{n \rightarrow \infty} J_{\alpha, n} x = y \in D(A) \quad \text{for some } \alpha > 0,$$

that is,

$$(3.15) \quad \lim_{n \rightarrow \infty} J_{\alpha, n} x \in D(A), \quad x \in \overline{\text{co}D(\bar{A})} \quad \text{for some } \alpha > 0.$$

We denote the limit value of (3.15) by $J_\alpha x$. By Lemma 2.2, there exists

a dissipative operator A_1 in X such that $J_\alpha = (I - \alpha A_1)^{-1}$ for all $\alpha > 0$. From (3.14), $\lim_{n \rightarrow \infty} J_{\alpha, n}(I - \alpha A)y = y$, and $J_\alpha(I - \alpha A)y = y$, $y \in D(A)$ for $\alpha > 0$ by (3.15). Since J_α is contraction, we have

$$(3.16) \quad J_\alpha(I - \alpha \bar{A})y = y, \quad y \in D(\bar{A}) \quad \text{for } \alpha > 0.$$

It follows from (3.16) that $D(\bar{A}) = D(A_1)$. In fact, we obtain $x \in R(J_\alpha) = D(A_1)$ for $x \in D(\bar{A})$, conversely, for $x \in D(A_1) = R(J_\alpha)$, there exists $y \in \overline{\text{co}D(\bar{A})}$ such that

$$\begin{aligned} x &= J_\alpha y \\ &= J_\alpha(I - \alpha \bar{A})(I - \alpha \bar{A})^{-1} y \\ &= (I - \alpha \bar{A})^{-1} y \in D(\bar{A}). \end{aligned}$$

Therefore from (3.16)

$$\begin{aligned} (I - \alpha \bar{A})y &= (I - \alpha A_1)y, \\ \bar{A}y &= A_1y, \text{ that is, } \bar{A} = A_1. \end{aligned}$$

Hence we have $J_\alpha = (I - \alpha \bar{A})^{-1}$ and (3.13). By the same method as in the proof of Lemma 2.3, (3.12) holds true.

REMARK 3.6. Recently H. Sunouchi [8] considered Theorem 3.4 in the case that A, A_n are linear operators.

*Department of Mathematics
Pusan University, Korea*

References

- [1] H. BREZIS and A. PAZY, *Accretive sets and differential equations in Banach spaces*, to appear.
- [2] H. BREZIS and A. PAZY, *Semigroups of nonlinear contractions on convex sets*, Jour. Funct. Anal. 6(1970) 237-281.
- [3] M.G. CRANDALL and T.M. LIGGETT, *Generation of semigroups of nonlinear transformations on general Banach spaces*, to appear.
- [4] T. KATO, *Accretive operators and nonlinear evolution equations in Banach spaces*, Symp. Nonlinear Functional Analysis, Chicago, Amer. Math. Soc. (1968), to appear.
- [5] I. MIYADERA, *On the convergence of a sequence of nonlinear semigroups (in Japanese)*, Gakujyutsu-Kenkyu, Waseda Univ. Faculty of Education 18 (1969) 45-55.

- [6] I. MIYADERA, *Some remarks on semigroups on nonlinear operators*, to appear.
- [7] S. OHARU, *On the generation of semigroups of nonlinear contractions*, Jour. Math. Soc. Jap. 22 (1970) 526-550.
- [8] H. SUNOUCHI, *Convergence of semi-discrete difference schemes of abstract Cauchy problem*, Tôhoku Math. Jour. 22 (1970) 394-408.
- [9] K. YOSHIDA, *Functional Analysis*, Springer Verlag (1968).