q -COMPLETE COMPLEX LIE GROUPS AND HOLOMORPHIC FIBRE BUNDLES OVER A STEIN MANIFOLD

By

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Introduction

In 1960, Y. Matsushima and A. Morimoto [4] established a necessary and sufficient condition for a complex Lie group to be a Stein manifold. Furthermore using this result, they solved affirmatively the problem proposed by J. P. Serre [8].

On the other hand, in 1962, A. Andreotti and H. Grauert [1] introduced the conception of q -completeness to complex analysis, which is a generalization of pseudoconvexity.

From the Narasimhan's result [6], a complex manifold is a Stein manifold if and only if it is 0-complete. Accordingly Y. Matsushima and Morimoto's result gives rise to some problems in the case of q -complete complex manifolds instead of Stein manifolds.

V. Villani [10] proved the any holomorphic vector bundle over a q -complete complex manifold is also q -complete.

In this paper we shall try to solve the above problem slightly.

In §3 we shall prove the following theorem: *Let $P(X, G, \pi)$ be a complex analytic principal fibre bundle over a Stein manifold X , whose structure group is a connected complex Lie group G . And, let G^0 be the complex Lie subgroup $\{a \in G: f(a) = f(e) \text{ for each } f \in H^0(G, {}_c\mathfrak{D})\}$.*

If G^0 is of dimension q , then the total space $P(X, G, \pi)$ is q -complete.

As a consequence of this theorem, we have the fact: *If G^0 is of dimension q , then G is q -complete.* Particularly the inverse of the above fact shall be proved in §2, if G is holomorphically convex.

§1. Preliminary definitions and theorems

All results in this section owe to Y. Matsushima and A. Morimoto

[4] [5], therefore we only state almost part without proof.

1.1. DEFINITION. A complex manifold X is called a Stein manifold if the following three conditions are satisfied.

(S.1) For $p, q \in X, p \neq q$ there exists $f \in H^0(X, {}_X\mathfrak{D})$ such that $f(p) \neq f(q)$, where ${}_X\mathfrak{D}$ denotes the sheaf of germs of holomorphic functions on X .

(S.2) For any $p_0 \in X$, there exists a system of functions $(f_1, \dots, f_n) \in H^0(X, {}_X\mathfrak{D}^n)$ such that $(f_1, \dots, f_n)$ is a system of coordinates in some neighborhood of p_0 .

(S.3) X is holomorphically convex, i.e. for any compact subset K of X , the holomorphically convex hull

$$\hat{K} = \{p \in X: |f(p)| \leq \sup_K |f| \text{ for each } f \in H^0(X, {}_X\mathfrak{D})\}$$

of K is also compact in X .

Particularly a complex Lie group is called a Stein group if it is a Stein manifold as a complex manifold.

Matsushima and Morimoto proved the following deep theorem with respect to Stein groups.

1.2. THEOREM ([4]). For a connected complex Lie group G , the following conditions are mutually equivalent:

- (1) G is a Stein groups.
- (2) The centre of G is a Stein group.
- (3) The connected component of the centre of G is analitically isomorphic onto $\mathbf{C}^l \times \mathbf{C}^{*m}$ for some positive integers l and m .
- (4) For some maximal compact subgroup K , let $\mathfrak{k}$ be the Lie algebra of K , then $\mathfrak{k}$ satisfies the condition

$$\mathfrak{k} \cap \sqrt{-1}\mathfrak{k} = \{0\}$$

(5) (S.1) of Definition 1.1 is satisfied for G .

(6) G satisfies the condition (S.2) of Def. 1.1.

Remark, $\mathfrak{k} \cap \sqrt{-1}\mathfrak{k}$ in (4) is independent of the choice of K .

Using a consequence of this theorem, Matsushima and Morimoto solved, affirmatively, the problem raised by J. P. Serre.

(Is a complex analytic fibre bundle a Stein manifold, whose base and fibre are Stein manifolds ?)

We recall the definition of fibre bundles in this place.

1.3. DEFINITION. $E(X, F, G, \pi)$ is called a complex analytic (differentiable) fibre bundle with base space X , fibre F , structure group G , projection π and total space E if the following four conditions are satisfied.

(1) E , X and F are complex (differentiable) manifolds, and complex (real) Lie group G acts on F effectively and complex analytically (differentiably).

(2) π is a complex analytic (differentiable) mapping of E onto X .

(3) There exist an open covering $\{U_\alpha\}$ of X and biholomorphic (diffeomorphic) mappings ϕ_α of $U_\alpha \times F$ onto $\pi^{-1}(U_\alpha)$ such that

$$\pi \circ \phi_\alpha(x, f) = x \text{ for each } (x, f) \in U_\alpha \times F$$

(4) If $U_\alpha \cap U_\beta \neq \emptyset$, there exist complex analytic (differentiable) mappings $\Phi_{\alpha\beta}$ of $U_\alpha \cap U_\beta$ into G such that

$$\phi_\beta(x, \Phi_{\alpha\beta}(x) \circ f) = \phi_\alpha(x, f)$$

$$\text{for each } (x, f) \in (U_\alpha \cap U_\beta) \times F$$

1.4. DEFINITION. In Definition 1.3, if $F=G$ and the action of G on itself is left translation, then fibre bundles with structure group and fibre G are called complex analytic (differentiable) principal bundles.

A complex analytic (differentiable) principal bundle $P(X, G, \pi)$ with base X and structure group G is determined uniquely by open covering $\{U_\alpha\}$ of X and transition functions $\Phi_{\alpha\beta}$ of $U_\alpha \cap U_\beta$ into G in Def. 1.3. Then we may write $P(X, G, \pi) = (\{U_\alpha\}, \{\Phi_{\alpha\beta}\})$

1.5. DEFINITION. Two principal bundles

$$P(X, G, \pi) = (\{U_\alpha\}, \{\Phi_{\alpha\beta}\})$$

$$P_1(X, G, \pi_1) = (\{V_\nu\}, \{\Psi_{\nu\mu}\})$$

with base X and structure group G are called to be analytically (differentiably) equivalent, if for each (α, ν) there exists complex analytic (differentiable) mapping $\chi_{\alpha\nu}$ of $U_\alpha \cap V_\nu$ into G such that $(\{U_\alpha, V_\nu\}, \{\Phi_{\alpha\beta}, \chi_{\alpha\nu}, \Psi_{\nu\mu}\})$ is also a complex analytic (differentiable) principal bundle.

The significant following theorem answer the question raised by J. P. Serre [8] almost conclusively.

1.6. THEOREM ([4]). A complex analytic fibre bundle $E(X, F, G, \pi)$

is a Stein manifold in each case:

(1) Base X and fibre F are Stein manifolds, and structure group G is a connected complex Lie group.

(2) $E(X, F, G, \pi)$ is a principal fibre bundle (therefore $F=G$) and, moreover, base X and structure group G are Stein manifolds.

Morimoto [5] proved that every connected complex Lie group G contains the smallest closed complex normal subgroup G^0 such that the factor group G/G^0 is a Stein group.

Then the subgroup G^0 has not non-constant holomorphic functions on G^0 at all. But he gave the example: Non compact complex Lie groups without non-constant holomorphic functions. Therefore, it is not always compact. The following Lemma 1.7 and Theorem 1.8 show the some character of the subgroup G^0 .

1.7. LEMMA ([5]). For a connected complex Lie group G , let G^0 be the subgroup $\{a \in G: f(a) = f(e) \text{ for each } f \in H^0(G, {}_G\mathfrak{D})\}$, where e denotes the unit element of G .

Then G^0 satisfies (1)–(6):

- (1) G^0 is a closed complex normal Lie subgroup.
- (2) G/G^0 is a Stein group.
- (3) If there exists a closed complex normal Lie subgroup N such that G/N is a Stein group, then $G^0 \subset N$.
- (4) G^0 is connected.
- (5) G^0 is contained in the center of G .
- (6) G^0 has not non-constant holomorphic functions on it.

PROOF. (1) We shall prove that $\mathfrak{g}^0$ is a complex Lie subalgebra of $\mathfrak{g}$, where $\mathfrak{g}, \mathfrak{g}^0$ are the Lie algebra of G, G^0 , respectively. We take an element $X \in \mathfrak{g}^0$ and put $Y = \sqrt{-1} X$. Let V be a sufficiently small connected neighborhood of the origin 0 in the complex plane $\mathbb{C}$. Then the mapping of V into a neighborhood of e in G

$$\varphi(x + \sqrt{-1}y) = \exp x X \exp y Y \quad (x + \sqrt{-1}y) \in V$$

is holomorphic on V . Take a holomorphic function $f \in H^0(G, {}_G\mathfrak{D})$, we put

$$f_0(x + \sqrt{-1}y) = f(\exp x X \exp y Y)$$

then f_0 is holomorphic on V such that $f_0(x) = f_0(0)$. From theorem of

identity, $f_0(z) = f_0(0)$ $z \in V$.

Hence $f(\exp Y) = f(e)$, which proves $Y \in \mathfrak{g}^0$.

(2) Let π be the natural projection of G onto G/G^0 . Since π induces the mapping of $H(G, {}_G\mathfrak{D})$ onto $H^0(G/G^0, {}_{G/G^0}\mathfrak{D})$, G/G^0 satisfies (S.1) of Definition 1.1. Hence from Theorem 1.2(5), G/G^0 is a Stein group.

(3) Let $\hat{\pi}$ be the natural projection of G onto G/N . By assumption, if $a \notin N$, there exists $\hat{f} \in H^0(G/N, {}_{G/N}\mathfrak{D})$ such that $\hat{f}(\hat{\pi}(a)) \neq \hat{f}(\hat{\pi}(e))$. We put $f(x) = \hat{f} \circ \hat{\pi}(x)$ then $f \in H^0(G, {}_G\mathfrak{D})$ and $f(a) \neq f(e)$. Therefore $G^0 \subset N$.

(4) Let G_0^0 be the connected component of G^0 . Since G/G_0^0 is a covering group of G/G^0 and since G/G^0 is a Stein group, using the result of K. Stein [9], G/G_0^0 is also a Stein group. Hence from (3) $G^0 = G_0^0$.

(5) Let Z be the centre of G . Then by the adjoint representation of G , G/Z is a complex linear group and so G/Z is a Stein group. (It is not necessary that G/Z is closed in the complex general linear group, by [4].).

(6) Let $(G^0)^0$ be the subgroup

$$\{a \in G^0: f(a) = f(e) \quad \text{for each } f \in H^0(G^0, {}_{G^0}\mathfrak{D})\}$$

By (5), $(G^0)^0$ is a normal subgroup of G . Therefore $(G^0)^0$ is a closed complex normal Lie subgroup of G .

Since we may regard $G/(G^0)^0$ as a complex analytic principal bundle with base G/G^0 and structure group $G^0/(G^0)^0$ (c. f. §2), and since G/G^0 and $G^0/(G^0)^0$ are Stein groups, by theorem 1.6(2), $(G/G^0)^0$ is also a Stein groups. Hence $G^0 = (G^0)^0$ by (3). q.e.d.

1.8. THEOREM ([5] theorem 2). *On the same assumption in Lemma 1.7, the following conditions are mutually equivalent:*

(1) G is holomorphically convex.

(2) G^0 is compact.

(3) Let $\mathfrak{k}$ be a Lie algebra of a maximal compact subgroup K and $\mathfrak{g}^0$ be the Lie algebra of G^0 , then

$$\mathfrak{g}^0 = \mathfrak{k} \cap \sqrt{-1} \mathfrak{k}$$

PROOF. (1) $\Rightarrow$ (2) If G^0 is non-compact, for fixed $a \in G^0$, the holomorphically convex hull of $\{a\}$ is just G^0 . Hence G is not holomorphically convex.

(2) $\Rightarrow$ (1) Since we regard G as a complex analytic principal bundle with base G/G^0 and fibre G^0 and since G/G^0 is a Stein group by lemma 1.7(2), if G^0 is compact, G is holomorphically convex.

(2) $\Rightarrow$ (3) Let Z be the centre of G and $\mathfrak{g}, \mathfrak{z}$ be the Lie algebras of G, Z (respectively). We put $\mathfrak{k}_0 = \mathfrak{k} \cap \sqrt{-1} \mathfrak{k}$ and let K_0 be the complex Lie subgroup of G with the Lie algebra $\mathfrak{k}_0$. Since K_0 is contained in a compact subgroup, any holomorphic function on G is constant on K_0 .

Hence $K_0 \subset G^0$ i.e. $\mathfrak{k} \cap \sqrt{-1} \mathfrak{k} \subset \mathfrak{g}^0$. On the other hand, since G^0 is compact, there exists a maximal compact subgroup K_1 such that $G^0 \subset K_1$, i.e. $\mathfrak{g}^0 \subset \mathfrak{k}_1$ where $\mathfrak{k}_1$ is the Lie algebra of K_1 . Hence $\mathfrak{g}^0 \subset \mathfrak{k}_1 \subset \sqrt{-1} \mathfrak{k}_1$. By the same reason of the above, $\mathfrak{k}_1 \cap \sqrt{-1} \mathfrak{k}_1 \subset \mathfrak{g}^0 \subset \mathfrak{z}$. Moreover since all maximal compact subgroups of G are conjugate, we have $\mathfrak{k}_1 \cap \sqrt{-1} \mathfrak{k}_1 = \mathfrak{k} \cap \sqrt{-1} \mathfrak{k}$. Therefore, $\mathfrak{g}^0 = \mathfrak{k} \cap \sqrt{-1} \mathfrak{k}$.

(3) $\Rightarrow$ (2) If $\mathfrak{g}^0 = \mathfrak{k} \cap \sqrt{-1} \mathfrak{k}$, then $G^0 \subset K$. Since G^0 is closed in G , G^0 is compact in G . q.e.d.

§2. q -Complete and holomorphically convex complex Lie groups.

In this section, when a complex Lie group G is holomorphically convex, (i.e. when G^0 is compact, by theorem 1.8) we investigate the relation between the dimension of G^0 and q -completeness of G .

2.1. DEFINITION. Let M be a complex manifold of dimension n . A real-valued function φ of class C^∞ is said to be strongly q -plurisubharmonic on M , if for each $x \in M$ the Hermitian form in n variables:

$$L(\varphi, x) = \sum_{i,j=1}^n \left(\frac{\partial^2 \varphi(x)}{\partial z_i \partial \bar{z}_j} \right) \xi^i \bar{\xi}^j$$

has at least $n - q$ positive eigenvalues, where $(z_1, \dots, z_n)$ is a local coordinate in a neighborhood of x .

Moreover a complex manifold M is called to be q -complete if there exists a strongly q -plurisubharmonic function φ defined on M such that for every $c \in \mathbf{R}$ the sets

$$B_c = \{x \in M: \varphi(x) < c\}$$

are relatively compact in M .

By Definition 2.1, if M is q -complete, then it is p -complete for each $p \geq q$. To avoid the confusion, we have need of the following definition.

2.2. DEFINITION. A q -complete complex manifold M is said to be purely q -complete, if it is not $(q-1)$ -complete.

An important result in the theory of q -complete manifold expressed by the following theorem.

2.3. THEOREM ([1]). Let M be a q -complete complex manifold, let $\mathfrak{F}$ be any coherent analytic sheaf on M . Then

$$H^i(M, \mathfrak{F}) = 0 \quad i \geq q+1$$

For the proof of the above theorem see Andreotti and Grauert [1]. We recall the following theorem by Narashimhan [6].

2.4. THEOREM. A complex manifold M is 0-complete if and only if it is a Stein manifold.

The following some properties complex Lie groups is well known.

Let G be a complex Lie group of dimension n and H be a closed complex normal Lie subgroup of dimension q . By the property of canonical coordinate of G , there exists a open neighborhood U of e and canonical coordinate $(z_1, \dots, z_n)$ in U such that, if we put

$$U_1 = \{x \in U: z_1(x) = \dots = z_q(x) = 0\}$$

$$U_2 = \{x \in U: z_{q+1}(x) = \dots = z_n(x) = 0\}$$

then $U = U_1 \cdot U_2$, $U \cap H = U_2$, $U_1 \cap U_2 = \{e\}$, $U \cdot H = U_1 \cdot H$ and $U_1^{-1} \cdot U_1 \cap H = \{e\}$. Let π is the natural mapping of G onto G/H . There exists the sequence $\{a_\nu\}$ in G such that $\pi(a_\nu U_1)$ is a locally finite open covering of G/H . For each $\pi(g) \in \pi(a_\nu U_1)$, there exists $x \in U_1$ uniquely, such that $\pi(g) = \pi(a_\nu x)$. Actually if $\pi(a_\nu x) = \pi(a_\nu y)$, then $y^{-1} \cdot x \in H$ and $y^{-1} \cdot x \in U_1^{-1} \cdot U_1$, hence $x = y$.

We put

$$\Phi_{\nu\mu}(\pi(g)) = (a_\nu x_\nu)^{-1} (a_\mu x_\mu) \quad \text{for } \pi(g) \in \pi(a_\nu U_1) \cap \pi(a_\mu U_1)$$

where $x_\nu, x_\mu \in U_1$, then $\Phi_{\nu\mu}$ is a holomorphic mapping of $\pi(a_\nu U_1) \cap \pi(a_\mu U_1)$ into H . Moreover

$$\{\Phi_{\nu\mu}\} \in Z^1(\{\pi(a_\nu U_1)\}, \mathfrak{A}_H)$$

where $\mathfrak{A}_H$ is the sheaf of germs of holomorphic mappings in H over G/H . Therefore the complex analytic principal bundle $P(G/H, H, \pi)$ is uniquely determined by $(\{\pi(a_\nu U_1)\}, \{\Phi_{\nu\mu}\})$. G is biholomorphic onto the above bundle $P(G/H, H, \pi)$ with structure group H and base G/H . Actually, we put

$$j(g) = (\pi(g), h_\nu) \in \pi^{-1}(\pi(a_\nu U_1))$$

where $g = a_\nu x_\nu h_\nu$, $x_\nu \in U_1$, $h_\nu \in H$, then j is a biholomorphic mapping of G onto P .

Similarly if G is a real Lie group and H is a closed subgroup of G , then G is diffeomorphic onto a differentiable principal bundle $P(G/H, H, \pi)$ over G/H with structure group H .

2.5. LEMMA (maximum principle). *Let Δ be the unit polydisc $\{z \in \mathbb{C}^n : |z_i| < 1\}$ in $\mathbb{C}^n$ and φ be a strongly q -pseudoconvex function in Δ . If $q < n$, then φ cannot attain the maximum in Δ .*

PROOF. Suppose that φ attains the maximum at $z^0 = (z_1^0, \dots, z_n^0) \in \Delta$, i.e. $\max \varphi = \varphi(z^0)$.

By the assumption, the Hermitian matrix

$$\left[\frac{\partial^2 \varphi(z^0)}{\partial z_i \partial \bar{z}_j} \right]$$

has at least $n - q$ positive eigenvalues at $z^0 \in \Delta$.

Since $n - q > 0$, after a suitable coordinate transformation, we may have

$$\frac{\partial^2 \varphi(z)}{\partial z_1 \partial \bar{z}_1} \Bigg> 0 \quad \text{in a neigh. of } z^0.$$

We put $\hat{\varphi}(z_1) = \varphi(z_1, z_2^0, \dots, z_n^0)$, then $\hat{\varphi}$ attains the maximum at z_1^0 , and

$$\frac{\partial^2 \hat{\varphi}(z_1^0)}{\partial z_1 \partial \bar{z}_1} \Bigg> 0.$$

It is a contradiction.

q.e.d.

2.6. PROPOSITION. *Let G be a holomorphically convex complex Lie group of dimension n , and let G^0 be the complex Lie subgroup $\{a \in G: f(a) = f(e) \text{ for all } f \in H^0(G, {}_G\mathfrak{D})\}$, where e denotes the unit element of G .*

Then G is purely q -complete if and only if G^0 is of dimension q .

PROOF. Suppose that G^0 is of dimension q . By lemma 1.7(2). G/G^0 is a Stein group, therefore it is 0-complete. Namely there exists a strongly 0-plurisubharmonic function ψ on G/G^0 such that $\{\psi < c\}$ are compact for every $c \in \mathbf{R}$. We put $\varphi = \psi \circ \pi$, where π is the natural projection of G onto G/G^0 . Then φ is strongly q -plurisubharmonic on G . As mentioned above, we may regard G as a principal bundle $P(G/G^0, G^0, \pi)$ over G/G^0 with structure group G^0 . By assumption G^0 is compact, then for every $c \in \mathbf{R}$

$$\{\varphi < c\} = \pi^{-1}(\{\psi < c\})$$

are compact in G , since $\{\psi < c\}$ is compact in G/G^0 .

Therefore G is q -complete. But G is not $(q-1)$ -complete. Actually, suppose that there exists a strongly $(q-1)$ -plurisubharmonic function φ on G . For fixed $\pi(g) \in G/G^0$, there exists a neighborhood $\pi(gU_1)$ of $\pi(g)$ in G/G^0 such that $\pi^{-1}(\pi(gU_1))$ is biholomorphic onto $\pi(gU_1) \times G^0$.

Since $\dim \pi(gU_1) = n - q$, we may regard $\varphi|_{\pi^{-1}(\pi(g))}$ as a strongly $(q-1)$ -plurisubharmonic function on G^0 and since G^0 is compact, then $\varphi|_{\pi^{-1}(\pi(g))}$ attains the maximum in G^0 . It is a contradiction for Lemma 2.5. Therefore G is purely q -complete. Conversely, suppose that $\dim G^0 = p$ and G is purely q -complete. Since G is considered a principal bundle over G/G^0 and G/G^0 is a $(n-p)$ -dimensional Stein group and since G^0 is compact, we have $q \leq p$. Hence by the compactness of G^0 and maximum principle (Lemma 2.5) for q -plurisubharmonic function, we have $q = p$. q.e.d.

Suppose that G is holomorphically convex.

By theorem 1.8, G^0 is compact and $\mathfrak{g}^0 = \mathfrak{k} \cap \sqrt{-1}\mathfrak{k}$, where $\mathfrak{g}^0$ is the complex Lie algebra of G^0 and $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup of G . Then we have the following proposition.

2.7. PROPOSITION. *Let G be a holomorphically convex complex Lie group, and let K be a maximal compact subgroup with Lie algebra $\mathfrak{k}$.*

Then G is purely q -complete if and only if $\dim \mathfrak{c} \mathfrak{k} \cap \sqrt{-1} \mathfrak{k} = q$.

REMARK. In particular if $\dim G^0 = 1$, then G is always compact. Actually if G^0 is non-compact, by Behnke and Stein [2], G^0 is a Stein manifold.

It is a contradiction for Lemma 1.7(6). Therefore if $\dim G^0 = 1$, then proposition 2.6 and 2.7 is always true.

§ 3. Complex analytic principal bundle over a Stein manifold.

In this section we shall effect our purpose, that is to say, we shall prove the theorem stated in the introduction of this paper. To prove it, we prepare the following lemma.

3.1. LEMMA. Let $P(X, G, \pi)$ be a complex analytic (differentiable) principal bundle over a complex (differentiable) manifold X with a complex (real) Lie group G as its structure group. Let H be a closed complex normal Lie subgroup of G . Then we can regard $P(X, G, \pi)$ as a complex analytic (differentiable) principal bundle $P_2(P_1, H, \pi_2)$ over P_1 with structure group H , where $P_1 = P_1(X, G/H, \pi_1)$ is the induced principal bundle over X with structure group G/H by $P(X, G, \pi)$.

PROOF. Using the stated result in §2, we may regard G as the principal bundle $G(G/H, H, \pi_0)$ where π_0 is the natural projection of G onto G/H .
suppose that

$$G(G/H, H, \pi_0) = (\{\pi_0(a_\nu U_1)\}, \{\Phi_{\nu\mu}\})$$

where $\{\pi_0(a_\nu U_1)\}$ is a covering of G/H and $\{\Phi_{\nu\mu}\} \in Z^1(\{\pi_0(a_\nu U_1)\}, \mathfrak{A}_H)$. Moreover let $P(X, G, \pi) = (\{W_\alpha\}, \{F_{\alpha\beta}\})$ where $\{W_\alpha\}$ is a covering of X and $\{F_{\alpha\beta}\} \in Z^1(\{W_\alpha\}, \mathfrak{A}_G)$, then we have

$$P_1(X, G/H, \pi_1) = (\{W_\alpha\}, \{\pi_0 \circ F_{\alpha\beta}\}).$$

By the stated result in §2, $\Phi_{\nu\mu}(\pi_0(g)) = (a_\nu x_\nu)^{-1} (a_\mu x_\mu)$ where $x_\nu, x_\mu \in U_1$, $\pi_0(g) = \pi_0(a_\nu x_\nu) = \pi_0(a_\mu x_\mu)$. We put $P_2(P_1, H, \pi_2) = (\{O_{(\alpha,\nu)}\}, \{\Psi_{(\alpha,\nu)(\beta,\mu)}\})$ where $O_{(\alpha,\nu)} = W_\alpha \times \pi_0(a_\nu U_1) \subset \pi_1^{-1}(W_\alpha)$, and $\Psi_{(\alpha,\nu)(\beta,\mu)} = (a_\nu x_\nu)^{-1} \circ F_{\alpha\beta} \circ (a_\mu x_\mu)$.

Since $\{O_{(\alpha, \beta)}\}$ is a covering of P_1 and $\{\Psi_{(\alpha, \nu), (\beta, \mu)}\} \in Z^1(\{O_{(\alpha, \nu)}\}, \mathfrak{U}_H)$. Then P_2 is a principal bundle over P_1 with structure group H . Moreover $P(X, G, \pi)$ is biholomorphic or diffeomorphic onto $P_2(P_1, H, \pi_2)$, respectively. Acturally for $w \in W_\alpha$, $x_\nu \in U_1$ and $h_\nu \in H$, we put

$$j_{(\alpha, \nu)}(w, a_\nu x_\nu h_\nu) = ((W, \pi_0(a_\nu x_\nu)), h_\nu),$$

then $j_{(\alpha, \nu)}$ is a mapping of $W_\alpha \times a_\nu U_1 H$ onto $(W_\alpha \times \pi_0(a_\nu U_1)) \times H$.

We may regard $j = \{j_{(\alpha, \nu)}\}$ as a well-defined mapping of $P(X, G, \pi)$ onto $P_2(P_1, H, \pi_2)$. It is easy that we verify for j to be biholomorphic or diffeomorphic, respectively. q.e.d.

3.2. THEOREM. *Let $P(X, G, \pi)$ be a complex analytic principal bundle over a Stein manifold X , whose structure group and fibre is a connected complex Lie group G . And let G^0 be the complex Lie subgroup*

$$\{a \in G: f(a) = f(e) \quad \text{for every } f \in H^0(G, {}_G\mathfrak{D})\}.$$

If G^0 is of dimension q , then the total space P is q -complete.

PROOF. Suppose that $\dim G = n$, $\dim X = m$ and $\dim G^0 = q$. We may assume

$$P(X, G, \pi) = (\{W_\alpha\}, \{F_{\alpha\beta}\}),$$

where $\{W_\alpha\}$ is a locally finite covering of X and $(F_{\alpha\beta}) \in Z^1(\{W_\alpha\}, \mathfrak{U}_G)$. Moreover we put $G(G/G^0, G^0, \pi_0) = (\{\pi_0(a_\nu U_1)\}, \{\Phi_{\nu\mu}\})$ where $\{\pi_0(a_\nu U_1)\}$ and $\{\Phi_{\nu\mu}\}$ are the same in the proof of lemma 3.1 when H is regarded as G^0 . We put

$$P_1(X, G/G^0, \pi_1) = (\{W_\alpha\}, \{\pi_0 \cdot F_{\alpha\beta}\})$$

and

$$P_2(P_1, G^0, \pi_2) = (\{O_{(\alpha, \nu)}\}, \Psi_{(\alpha, \nu)(\beta, \mu)})$$

where $O_{(\alpha, \nu)} = W_\alpha \times \pi_0(a_\nu U_1) \subset \pi_1^{-1}(W_\alpha)$, and $\Psi_{(\alpha, \nu)(\beta, \mu)} = (a_\nu x_\nu)^{-1} \circ F_{\alpha\beta} \circ (a_\mu x_\mu)$. Then by lemma 3.1 $P(X, G, \pi)$ is biholomorphic onto $P_2(P_1, G^0, \pi_2)$.

On the other hand, by lemma 1.7(4) and (5) G^0 is connected and commutative. Therefore, when we regard G^0 as a real Lie group of dimension $2q$, G^0 is isomorphic to $\mathbf{T}^l \times \mathbf{R}^s$ for some l, s ($l+s=2q$), where $\mathbf{T}$ is 1-dimensional torus. We may identify $\mathbf{T}^l = \mathbf{T}^l \times 0$ and $\mathbf{R}^s = 0 \times \mathbf{R}^s$, then $\mathbf{T}^l$ and $\mathbf{R}^s$ are closed subgroups of G^0 . By Lemma 3.1, $P_2(P_1, G^0, \pi_2)$ is regard as the differentiable principal bundle

$$P_3(P_4, \mathbf{R}^s, \pi_3)$$

where its base $P_4 = P_4(P_1, \mathbf{T}^l, \pi_4)$ is the differentiable principal bundle over P_1 with structure group $\mathbf{T}^l$ induced by $P_2(P_1, G^0, \pi_2)$. Clearly the bundle $P_3(P_4, \mathbf{R}^s, \pi_3)$ is trivial, since $H^1(P_4, \mathfrak{G}^s) = 0$ where $\mathfrak{G}$ is the sheaf of germs of differentiable functions over P_4 . By the following way, we can verify the assertion.

We may regard $\Psi_{(\alpha, \nu)(\beta, \mu)}$ as

$$\Psi_{(\alpha, \nu)(\beta, \mu)} = (\Omega_{(\alpha, \nu)(\beta, \mu)}, A_{(\alpha, \nu)(\beta, \mu)})$$

where

$$\Omega_{(\alpha, \nu)(\beta, \mu)}: O_{(\alpha, \nu)} \cap O_{(\beta, \mu)} \rightarrow \mathbf{T}^l$$

$$A_{(\alpha, \nu)(\beta, \mu)}: O_{(\alpha, \nu)} \cap O_{(\beta, \mu)} \rightarrow \mathbf{R}^s$$

are of class C^∞ . Since the sheaf $\mathfrak{G}^s$ is soft, then $H^1(\{O_{(\alpha, \nu)}\}, \mathfrak{G}^s) = 0$. Therefore there exist differentiable mappings

$$A_{(\alpha, \nu)}: O_{(\alpha, \nu)} \rightarrow \mathbf{R}^s$$

such that for every (α, ν) and (β, μ) ,

$$A_{(\alpha, \nu)(\beta, \mu)} = A_{(\beta, \mu)} - A_{(\alpha, \nu)} \text{ on } O_{(\alpha, \nu)} \cap O_{(\beta, \mu)}.$$

We put

$$\iota_{(\alpha, \nu)}(p_1, h_{(\alpha, \nu)}) = ((P_1, h_{(\alpha, \nu)}^1), h_{(\alpha, \nu)}^2 + A_{(\alpha, \nu)}(p_1))$$

for $(p_1, h_{(\alpha, \nu)}) \in O_{(\alpha, \nu)} \times G^0$, where $h_{(\alpha, \nu)} = (h_{(\alpha, \nu)}^1, h_{(\alpha, \nu)}^2)$, $h_{(\alpha, \nu)}^1 \in \mathbf{T}^l$ and $h_{(\alpha, \nu)}^2 \in \mathbf{R}^s$. Then $\iota_{(\alpha, \nu)}$ is a mapping of $O_{(\alpha, \nu)} \times G^0$ onto $((O_{(\alpha, \nu)} \times \mathbf{T}^l) \times \mathbf{R}^s)$ and so $\iota_{(\alpha, \nu)}$ is considered as a mapping of $\pi_3^{-1}(O_{(\alpha, \nu)})$ onto $\pi_4^{-1}(O_{(\alpha, \nu)}) \times \mathbf{R}^s$.

Hence, $\iota = \{\iota_{(\alpha, \nu)}\}$ is a mapping of $P_3(P_4, \mathbf{R}^s, \pi_3)$ onto $P_4 \times \mathbf{R}^s$. It is easy to verify that ι is a diffeomorphic bundle isomorphism onto $P_4 \times \mathbf{R}^s$.

We have need to make the following function γ of class C^∞ . Let λ be a C^∞ -function with the compact support such that

$$\lambda(x) = \begin{cases} 1 & \text{on } [-a, a] \\ 0 & \text{on } \mathbf{R} - (-a - \varepsilon, a + \varepsilon) \end{cases}$$

and $0 \leq \lambda \leq 1$ for fixed $a > 0$ and $\varepsilon > 0$. We put

$$\gamma(x) = |x|(1 - \lambda(x)),$$

then we have $\gamma \in C^\infty(\mathbf{R})$, $\gamma(x) \geq 0$ and $\gamma(x) = |x|$ on $\mathbf{R} - (-a - \varepsilon, a + \varepsilon)$. By $\gamma \in C^\infty(\mathbf{R})$, we define $\gamma_s: P_4 \times \mathbf{R}^s \rightarrow [0, \infty)$ such that

$$\gamma_s(p_4, (t^1, \dots, t^s)) = \sum_{k=1}^s \gamma(t^k) \quad \text{for } p_4 \in P_4 \text{ and } (t^1, \dots, t^s) \in \mathbf{R}^s.$$

Then $\{(p_4, t) \in P_4 \times \mathbf{R}^s: \gamma_s(p_4, t) < c\}$ are compact, for every $c \in \mathbf{R}$ and γ_s is a mapping of class C^∞ .

Taking a refinement of the covering $\{O_{(\alpha, \nu)}\}$ of P_1 , we may regard $\Psi_{(\alpha, \nu)(\beta, \mu)}$, $\Omega_{(\alpha, \nu)(\beta, \mu)}$ and $A_{(\alpha, \nu)(\beta, \mu)}$ defined in a neighborhood of $\bar{O}_{(\alpha, \nu)} \cap \bar{O}_{(\beta, \mu)}$, and regard $A_{(\alpha, \nu)}$ defined in a neighborhood of $\bar{O}_{(\alpha, \nu)}$ for each (α, ν) . Moreover we may consider $\{O_{(\alpha, \nu)}\}$ to be a locally finite covering such that $O_{(\alpha, \nu)} \subseteq P_1$ for each (α, ν) and such that there exists a system of coordinates in some neighborhood of $\bar{O}_{(\alpha, \nu)}$. Let $(z_1^{(\alpha, \nu)}, \dots, z_{m+n-q}^{(\alpha, \nu)})$ be one of system of coordinates in a neighborhood of $\bar{O}_{(\alpha, \nu)}$.

We consider the mapping

$$\gamma_s \circ \iota: P_2(P_1, G^0, \pi_2) \rightarrow [0, \infty)$$

of class C^∞ . For any $p_2 \in \pi_2^{-1}(O_{(\alpha, \nu)}) = O_{(\alpha, \nu)} \times G^0$, let S_{p_2} be the $(m+n-q)$ -dimensional submanifold $\{(p, h_{(\alpha, \nu)}): p \in O_{(\alpha, \nu)} \text{ and } p_2 = (p_0, h_{(\alpha, \nu)})\} \cong O_{(\alpha, \nu)}$ of $\pi_2^{-1}(O_{(\alpha, \nu)})$ through p_2 . Then $(z_1^{(\alpha, \nu)}, \dots, z_{m+n-q}^{(\alpha, \nu)})$ is considered a system of coordinate of S_{p_2} for each (α, ν) . We put $h_{(\alpha, \nu)} = (h_{(\alpha, \nu)}^1, h_{(\alpha, \nu)}^2) \in \mathbf{T}^l \times \mathbf{R}^s$ and $A_{(\alpha, \nu)}(p) + h_{(\alpha, \nu)}^2 = (A_{(\alpha, \nu)}^1(p) + h_{(\alpha, \nu)}^{21}, \dots, A_{(\alpha, \nu)}^s(p) + h_{(\alpha, \nu)}^{2s}) \in \mathbf{R}^s$. Then on S_{p_2} we have

$$\frac{\partial^2 \gamma_s \circ \iota}{\partial z_i^{(\alpha, \nu)} \partial z_j^{(\alpha, \nu)}} = \sum_{k=1}^s \frac{\partial^2 \gamma(h_{(\alpha, \nu)}^{2k} + A_{(\alpha, \nu)}^k(p))}{\partial z_i^{(\alpha, \nu)} \partial z_j^{(\alpha, \nu)}}$$

$$\begin{aligned}
 &= \sum_{k=1}^s \gamma'(h_{(\alpha, \nu)}^{2k} + A_{(\alpha, \nu)}^k(p)) \frac{\partial^2 A_{(\alpha, \nu)}^k(p)}{\partial z_i^{(\alpha, \nu)} \partial \bar{z}_j^{(\alpha, \nu)}} \\
 &+ \sum_{k=1}^s \gamma''(h_{(\alpha, \nu)}^{2k} + A_{(\alpha, \nu)}^k(p)) \frac{\partial A_{(\alpha, \nu)}^k(p)}{\partial z_i^{(\alpha, \nu)}} \frac{\partial A_{(\alpha, \nu)}^k(p)}{\partial \bar{z}_j^{(\alpha, \nu)}},
 \end{aligned}$$

where $1 \leq i, j \leq m+n-q$. Since $\gamma(x) = |x|$ on $\mathbf{R} - (-a-\varepsilon, a+\varepsilon)$, there exists a positive constant A such that

$$\max_{\mathbf{R}}(\sup |\gamma'|, \sup |\gamma''|) < A.$$

Moreover since $A_{(\alpha, \nu)}$ is defined on $\bar{O}_{(\alpha, \nu)}$, there exists a positive constant $B_{(\alpha, \nu)}$ such that

$$\sup_{\substack{p \in O_{(\alpha, \nu)} \\ 1 \leq k \leq s}} \left| \frac{\partial^2 A_{(\alpha, \nu)}^k(p)}{\partial z_i^{(\alpha, \nu)} \partial \bar{z}_j^{(\alpha, \nu)}} \right| < B_{(\alpha, \nu)}$$

and

$$\sup_{\substack{p \in O_{(\alpha, \nu)} \\ 1 \leq k \leq s}} \left| \frac{\partial A_{(\alpha, \nu)}^k(p)}{\partial z_i^{(\alpha, \nu)}} \frac{\partial A_{(\alpha, \nu)}^k(p)}{\partial \bar{z}_j^{(\alpha, \nu)}} \right| < B_{(\alpha, \nu)}$$

for $1 \leq i, j \leq m+n-q$. Therefore we have the following estimation,

$$\begin{aligned}
 \sup_{\substack{p_2 \in \pi^{-1}(O_{(\alpha, \nu)}) \\ 1 \leq i, j \leq m+n-q}} \left| \frac{\partial^2 \gamma_s \circ \iota}{\partial z_i^{(\alpha, \nu)} \partial \bar{z}_j^{(\alpha, \nu)}} \right| &< s(AB_{(\alpha, \nu)} + AB_{(\alpha, \nu)}^2) \\
 &= C_{(\alpha, \nu)} \\
 &< +\infty
 \end{aligned}$$

where we put $C_{(\alpha, \nu)} = s(AB_{(\alpha, \nu)} + AB_{(\alpha, \nu)}^2)$.

By the assumption and lemma 1.7(2), X and G/G^0 are Stein manifold. Then $P_1 = P_1(X, G/G^0, \pi_1)$ is also a Stein manifold by theorem 1.6(2), P_1 is a $(m+n-q)$ -dimensional Stein manifold. Therefore there exists a strongly plurisubharmonic function φ on P_1 into $[0, \infty)$ such that for some $\delta_{(\alpha, \nu)} > 0$

$$\sum_{i, j=1}^{m+n-q} \frac{\partial^2 \varphi(p)}{\partial z_i^{(\alpha, \nu)} \partial \bar{z}_j^{(\alpha, \nu)}} \xi^i \bar{\xi}^j \geq \delta_{(\alpha, \nu)} \sum_{i=1}^{m+n-q} |\xi^i|^2$$

where $\xi \in \mathbf{C}^{m+n-q}$ and $p \in O_{(\alpha, \nu)}$, and such that $K_c = \{p \in P_1 : \varphi(p) < c\} \subseteq P_1$. Then the function

$$f(c) = \frac{\max\{C_{(\alpha, \nu)} : O_{(\alpha, \nu)} \cap K_c \neq \emptyset\}}{\min\{\delta_{(\alpha, \nu)} : O_{(\alpha, \nu)} \cap K_c \neq \emptyset\}}$$

is bounded on every compact subsets of $[0, \infty)$.

Hence, there exists a increasing convex function $\chi \in C^\infty([0, \infty))$ such that

$$\chi'(c) > f(c).$$

We put $\emptyset = \chi(\varphi \circ \pi_2) + \gamma_s \circ \iota$, then $\emptyset$ is a strongly q -plurisubharmonic function on P_2 . Actually, for each $p_2 \in P_2$ let S_{p_2} be the $m+n-q$ -dimensional submanifold of $\pi_2^{-1}(O_{(\alpha, \nu)})$ through p_2 , then on S_{p_2} we have

$$\begin{aligned} & \sum_{i,j=1}^{m+n-q} \frac{\partial^2 \emptyset}{\partial z_i^{(\alpha, \nu)} \partial \bar{z}_j^{(\alpha, \nu)}} \xi^i \bar{\xi}^j \\ &= \sum \frac{\partial^2 \chi(\varphi \circ \pi_2)}{\partial z_i^{(\alpha, \nu)} \partial \bar{z}_j^{(\alpha, \nu)}} \xi^i \bar{\xi}^j \\ &+ \sum \frac{\partial^2 \gamma_s \circ \iota}{\partial z_i^{(\alpha, \nu)} \partial \bar{z}_j^{(\alpha, \nu)}} \xi^i \bar{\xi}^j \\ &\geq \sum \frac{\partial^2 \varphi}{\partial z_i^{(\alpha, \nu)} \partial \bar{z}_j^{(\alpha, \nu)}} \chi'(\varphi) \xi^i \bar{\xi}^j \\ &+ \sum \left| \frac{\partial \varphi}{\partial z_i^{(\alpha, \nu)}} \xi^i \right|^2 \chi''(\varphi) \\ &- C_{(\alpha, \nu)} \sum_{i=1}^{m+n-q} |\xi^i|^2 \\ &\geq (\delta_{(\alpha, \nu)} \chi'(\varphi) - C_{(\alpha, \nu)}) \sum_{i=1}^{m+n-q} |\xi^i|^2 \end{aligned}$$

and $(\delta_{(\alpha, \nu)} \chi'(\varphi) - C_{(\alpha, \nu)}) > 0$ by the definition of χ . Moreover for every $c \in \mathbf{R}$ we have

$$\{\emptyset < c\} \subset \iota^{-1}((\pi_4^{-1}(\{\chi(\varphi) < c\}) \times \mathbf{R}^s) \cap \{\gamma_s < c\}),$$

hence $\{\emptyset < c\}$ are compact in P_2 . Since P is biholomorphic onto P_2 , then P is q -complete. q.e.d.

By theorem 3.2, we have the following Corollary at once.

3.3. COROLLARY. *Let G be a complex Lie group and G^0 be the subgroup*

$$\{a \in G : f(a) = f(e) \quad \text{for every } f \in H^0(G, {}_c\mathfrak{D})\}.$$

If G^0 is of dimension q , then G is q -complete.

If G is holomorphically convex, the inverse of the above is true by proposition 2.6.

Grauert ([3] p. 347) conjectured the following proposition: Let X be a complex space with countable topology. If $\dim X \leq n$, then X is strongly $(n-1)$ -pseudoconvex.

In the case of complex Lie group, we have the following result.

3.4. COROLLARY. *Let G be a complex Lie group of dimension n . If there is at least one non-constant holomorphic function on a connected component of G . Then G is $(n-1)$ -complete. Therefore we have*

$$H^n(G, \mathfrak{F}) = 0$$

for every coherent analytic sheaves $\mathfrak{F}$ over G .

PROOF. Since $\dim G^0 < n$, we have the above by theorem 3.2 and 2.3.

REMARK. H. J. Reiffen [7] asserts that if X is a paracompact complex space of dimension n , then

$$H^{n+1}(X, \mathfrak{F}) = 0$$

for any coherent analytic sheaf $\mathfrak{F}$ over X .

References

- [1] A. ANDREOTTI AND H. GRAUERT, *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France **90** (1962), 193-259.

- [2] H. BEHNKE AND K. STEIN, *Entwicklungen analytischer Funktionen auf Riemannschen Flächen*, Math. Ann. **120** (1948), 430-461.
- [3] H. GRAUERT, *Une notion de dimension cohomogique dans la theorie des espaces complexes*, Bull. Soc. Math. France **87** (1959), 341-350.
- [4] Y. MATSUSHIMA AND A. MORIMOTO, *Sur cestains espaces fibrés holomorphes sur une variété de Stein*, Bull. Soc. Math. France, **88** (1960), 137-155.
- [5] A. MORIMOTO, *Non-compact complex Lie groups without non-constant holomorphic functions*, Proc. Conf. Complex Analysis, Minneapolis, 1965, 256-272.
- [6] R. NARASIMHAN, *The Levi problem for complex spaces*, Math. Ann. **142** (1961), 355-365.
- [7] H. J. REIFFEN, *Prolongement de Riemann concernant les classes cohomologie a supports compacts*, C.R. Acad. Sci Paris **259** (1964), 2333-2335.
- [8] J. -P. SERRE, *Quelques problèmes globaux relatifs aux variétés de Stein*, Colloque sur les fonctions de plusieurs variables, C.B.R. M. (1953), 57-68.
- [9] K. STEIN, *Überlagerungen holomorphvollständiger komplexen Räume*, Arch. Math. **7** (1956), 355-361.
- [10] V. VILLANI, *Fibrati vettoriali olomofi su una varieta' complessa q-completa*, Annali della Scuola Norm. Sup. -Pisa., **20** (1966), 15-23.